Induction and Recursion

Chapter 5

UWO – November 15, 2021

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Plan for Chapter 5

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Climbing an infinite ladder

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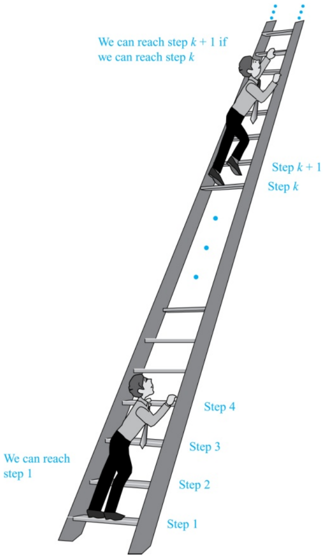

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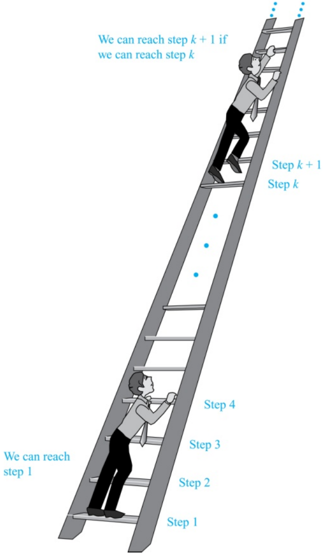

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(2) If we can reach
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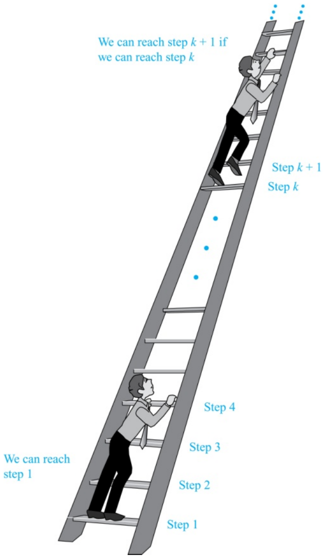
then we can reach the next rung.



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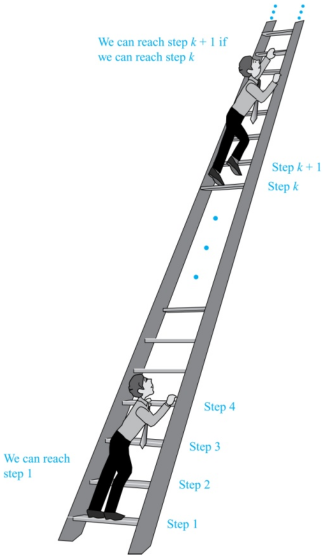
From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on.



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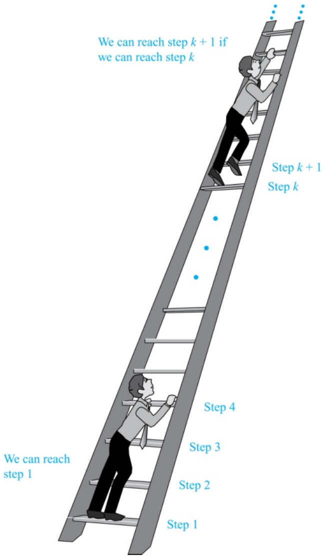
We can apply (2) any number of times so that **we can reach any particular rung, no matter how high up**.



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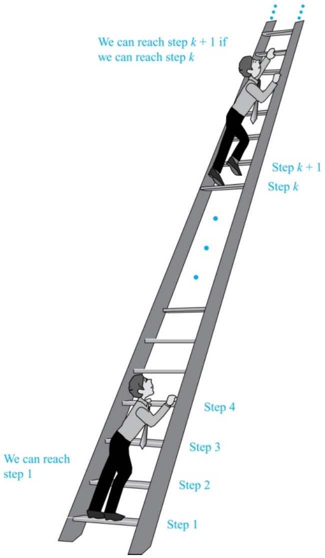
This example motivates the idea of proof by mathematical induction.



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Example (Climbing an infinite ladder)

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1. BASIS STEP: By (1), we can reach Rung 1.
2. INDUCTIVE STEP: Assume the inductive hypothesis that we can reach Rung k. Then by (2), we can reach Rung $(k+1)$.

Hence, $P(k) \to P(k+1)$ is true for all positive integers k. We can reach every rung on the ladder. ■

Important points about using mathematical induction

1. Mathematical induction can be expressed as the rule of inference

$$(P(1) \land \forall k(P(k) \to P(k+1))) \quad \to \quad \forall n P(n)$$

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3. Proofs by mathematical induction do not always start at the integer 1. The basis step may begin at a starting point b where b is an integer. We will see examples of this soon.

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7. Hence, $P(n)$ must be true for every positive integer n.

Remembering how mathematical induction works

Consider an infinite
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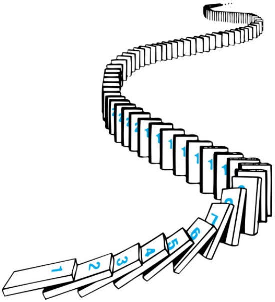
where each domino is standing.



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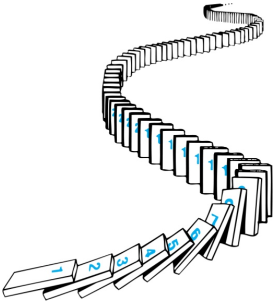

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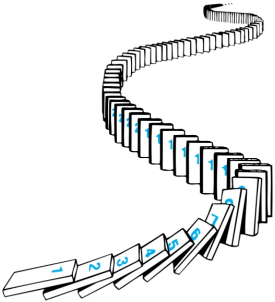

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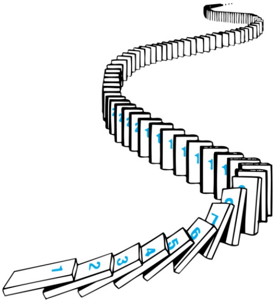

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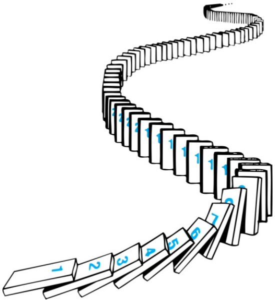

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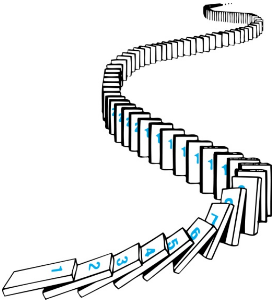

Plan for Chapter 5

Proving a summation formula by mathematical induction

Example

Show that: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Note: Once we have this conjecture, mathematical induction can be used to prove it correct, that is, true for $P(k+1)$.

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Conjecturing and proving correct a summation formula

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Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture.

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   $= k^2 + (2k + 1)$(by the inductive hypothesis)

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   $= k^2 + (2k + 1)$(by the inductive hypothesis)
   $= k^2 + 2k + 1$

Conjecturing and proving correct a summation formula

Example

Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture.

Solution: We have:
$1 = 1, 1 + 3 = 4, 1 + 3 + 5 = 9, 1 + 3 + 5 + 7 = 16, 1 + 3 + 5 + 7 + 9 = 25.$

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   e. Hence, we have shown that $P(k + 1)$ **follows from** $P(k)$.
   f. Therefore the sum of the first n positive odd integers is n^2. ■

Proving inequalities

Example

Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

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   a. $2^{k+1} = 2 \cdot 2^k$
   b. $< 2 \cdot k!$ *by the inductive hypothesis* $P(k)$
   c. $< (k+1)k!$
   d. $= (k+1)!$
4. Therefore, $2^n < n!$ holds, for every integer $n \geq 4$. ■

Note that here the basis step is $P(4)$, since $P(0), P(1), P(2)$, and $P(3)$ are all false.

Proving divisibility results

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4. By the inductive hypothesis, the first term $k^3 - k)$ is divisible by 3 and the second term is divisible by 3 since it is an integer multiplied by 3. By part (i) of Theorem 1 (Sec.4.1), $(k+1)^3 - (k+1)$ is divisible by 3.

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5. Therefore, $n^3 - n$ is divisible by 3, for every integer positive integer n. ■

Number of subsets of a finite set

Example

Use mathematical induction to show that if S is a finite set with n elements, where n is a non-negative integer, then S has 2^n subsets. That is, the cardinality of the *power set* for S is $\mid P(S) \mid = 2^n$

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Solution:

1. Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.
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continued →

Number of subsets of a finite set

Inductive Hypothesis: For an arbitrary non-negative integer k, every set with k elements has 2^k subsets.

Number of subsets of a finite set

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4. Let T be a set with $k+1$ elements. Choose any element $a \in T$. Then $T = S \cup \{a\}$, where $S = T - \{a\}$. Note that $|S| = k$.

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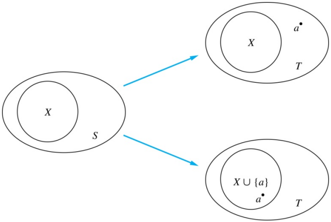
5. For each subset X of S there are exactly two subsets of T, i.e., X and $X \cup \{a\}$.



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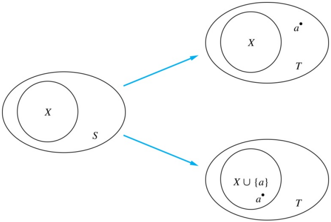

6. By the inductive hypothesis S has 2^k subsets. Since there are two subsets of T for each subset of S, the number of subsets of T is $2 \cdot 2^k = 2^{k+1}$. ∎

Tiling checkerboards

Example

Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right *triominoes*. A right triomino is an L-shaped tile which covers 3 squares at a time.

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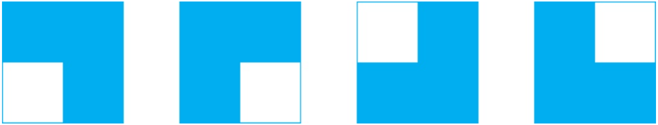
2. BASIS STEP: P(1) is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino.



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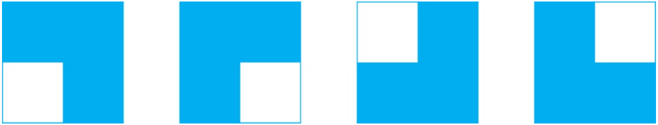

3. INDUCTIVE STEP: Assume that $P(k)$ is true for every $2^k \times 2^k$ checkerboard, for some positive integer k.

continued →

Tiling checkerboards

Inductive Hypothesis: Every $2^k \times 2^k$ checkerboard, for some positive integer k, with one square removed can be tiled using right triominoes.

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4. Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions.

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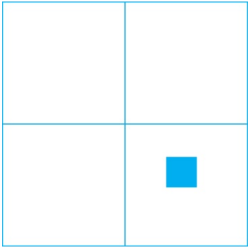

5. Remove a square from one of the four $2^k \times 2^k$ checkerboards. By the inductive hypothesis, this board can be tiled.

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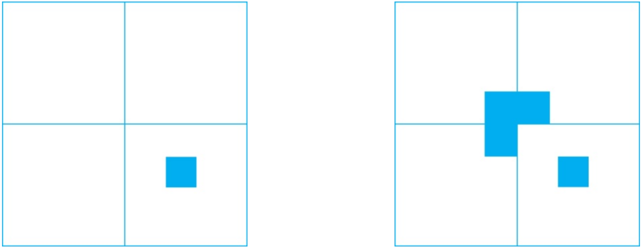

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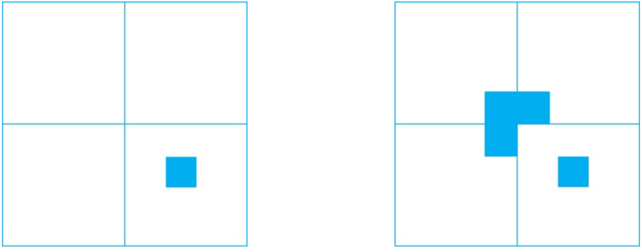

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6. Also by the inductive hypothesis, the other three boards can be tiled with the square from the corner of the center of the original board removed. We can then cover the three adjacent squares with a triomino.
7. Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes. ■

Plan for Chapter 5

An incorrect "proof" by mathematical induction

Example

Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. Here is a "proof" that $P(n)$ is true for all positive integers $n \geq 2$.

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1. BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point.

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1. BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point.
2. INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true for the positive integer $k \geq 2$, i.e., every set of k lines in the plane, no two of which are parallel, meet in a common point.

An incorrect “proof” by mathematical induction

Example

Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. Here is a “proof” that $P(n)$ is true for all positive integers $n \geq 2$.

1. BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point.
2. INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true for the positive integer $k \geq 2$, i.e., every set of k lines in the plane, no two of which are parallel, meet in a common point.
3. We must show that if $P(k)$ holds, then $P(k+1)$ holds, i.e., if every set of k lines in the plane, no two of which are parallel, $k \geq 2$, meet in a common point, then every set of k +1 lines in the plane, no two of which are parallel, meet in a common point.

An incorrect “proof” by mathematical induction

Inductive Hypothesis: Every set of k lines in the plane, where $k \geq 2$, no two of which are parallel, meet in a common point.

An incorrect “proof” by mathematical induction

Inductive Hypothesis: Every set of k lines in the plane, where $k \geq 2$, no two of which are parallel, meet in a common point.

1. Consider a set of $k+1$ distinct lines in the plane, no two parallel. By the inductive hypothesis, the first k of these lines must meet in a common point p_1. By the inductive hypothesis, the last k of these lines meet in a common point p_2.

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1. Consider a set of $k+1$ distinct lines in the plane, no two parallel. By the inductive hypothesis, the first k of these lines must meet in a common point p_1. By the inductive hypothesis, the last k of these lines meet in a common point p_2.
2. If p_1 and p_2 are different points, all lines containing both of them must be the same line since two points determine a line. This contradicts the assumption that the lines are distinct. Hence, point $p_1 = p_2$ lies on all $k+1$ distinct lines, and therefore $P(k+1)$ holds. Assuming that $k \geq 2$, distinct lines meet in a common point, then every $k+1$ lines meet in a common point.

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3. There must be an error in this proof since the conclusion is absurd. But where is the error?

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3. There must be an error in this proof since the conclusion is absurd. But where is the error?
   Answer: $P(k) \rightarrow P(k+1)$ only holds for $k \geq 3$. It is not the case that $P(2)$ implies $P(3)$. The first two lines must meet in a common point p_1 and the second two must meet in a common point p_2. They do not have to be the same point since only the second line is common to both sets of lines.

Plan for Chapter 5

Guidelines: mathematical induction proofs

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

Plan for Chapter 5

1. Mathematical Induction

1.1 Mathematical Induction

1.2 Examples of Proof by Mathematical Induction

1.3 Mistaken Proofs by Mathematical Induction

1.4 Guidelines for Proofs by Mathematical Induction

2. Strong Induction and Well-Ordering

2.1 Strong Induction

2.2 Well-Ordering Property

3. Recursive Definitions and Structural Induction

3.1 Recursively Defined Functions

3.2 Recursively Defined Sets and Structures

3.3 Structural Induction

4. Recursive Algorithms

4.1 Recursive Algorithms

4.2 Proving Correctness of Recursive Algorithms

Plan for Chapter 5

Strong induction

Strong Induction:

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:

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1. *Basis Step*: Verify that the proposition $P(1)$ is true.

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1. *Basis Step*: Verify that the proposition $P(1)$ is true.
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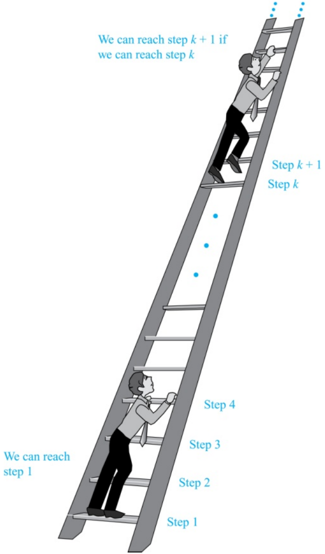
Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

Strong induction and the infinite ladder

Strong induction

tells us that we can reach all rungs if:

1. We can
   reach the first rung of the ladder.



Strong induction and the infinite ladder

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2. For every integer k, if we
   can reach <u>the first k *rungs*</u>, then
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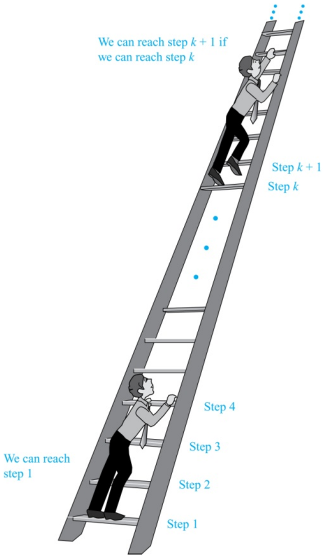

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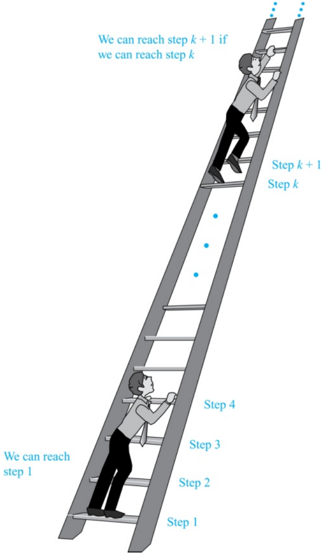

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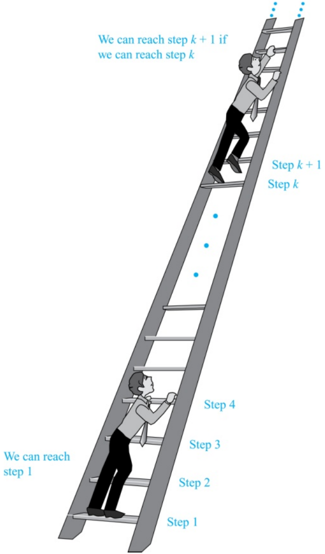

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1. BASIS STEP: $P(1)$ holds
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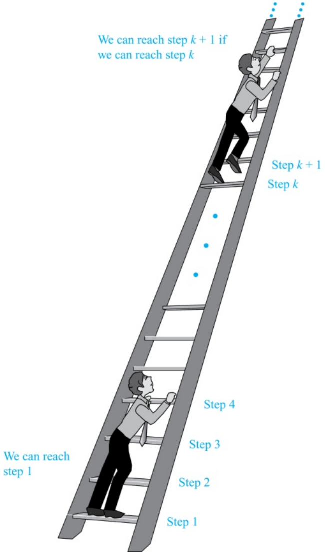

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1. BASIS STEP: $P(1)$ holds
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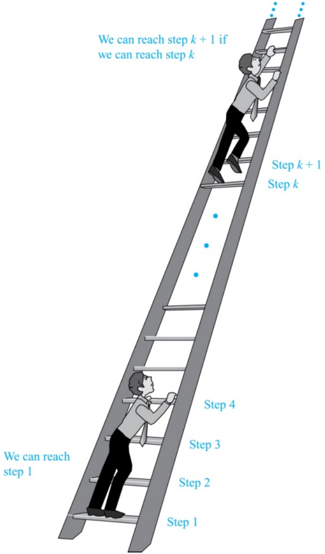
We will have then shown by strong induction that for every integer $n > 0$, the property $P(n)$ holds, that is, we can reach the n-th rung of the ladder.



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Which form of induction should be used?

1. We can always use strong induction instead of mathematical induction. But there is no reason to use it if it is simpler to use mathematical induction.
2. In fact, the principles of mathematical induction, strong induction, and the well-ordering property are all equivalent.
3. Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Complete proof of the fundamental theorem of arithmetic

Example

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution:

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   a. If $k+1$ is prime, then $P(k+1)$ is true.
   b. Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k+1$. By the inductive hypothesis a and b can be written as the product of primes and therefore $k+1$ can also be written as the product of those primes.

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4. Hence, it has been shown that every integer greater than 1 can be written as the product of primes. ■

Proof using strong induction

Example

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

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5. Hence, $P(n)$ holds for all $n \geq 12$. ■

Proof of the same example using mathematical induction

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3. Hence, $P(n)$ holds for all $n \geq 12$. ■

Plan for Chapter 5

Well-ordering property

1. *Well-ordering property*: Every nonempty set of non-negative integers has a least element.

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A set is *well ordered* if every non-empty subset has a least element.

1. $\mathbb{N}$ is well ordered under $\leq$.
2. The set of finite strings over an alphabet using lexicographic ordering is well ordered.

Well-ordering property

Example

Use the well-ordering property to prove the *division algorithm* , which states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$, such that $a = dq + r$.

Solution:

Well-ordering property

Example

Use the well-ordering property to prove the *division algorithm* , which states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$, such that $a = dq + r$.

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$$r^* := a - d(q_0 + 1) = a - dq_0 - d = r - d \geq 0.$$
4. Therefore, there are integers q and r with $0 \leq r < d$. ∎

(*uniqueness of q and r was proved in Tutorial 6*)

Plan for Chapter 5

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Recursively defined functions

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NOTE: a function $f(n)$ is the same as a sequence $a_0, a_1, \ldots$ where $f(n) = a_n$. We previously used recurrence relations to define sequences. The above is essentially the same.

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Suppose f is defined by: $f(0) = 3, \quad f(n+1) = 2f(n) + 3$
Find $f(1), f(2), f(3), f(4)$
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2. The second part is

$$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}$$

Fibonacci Numbers

Fibonacci (1170 - 1250)

Example

The Fibonacci numbers are defined as follows:

$$f_0 = 0$$
$$f_1 = 1$$
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Find f_2, f_3, f_4, f_5.

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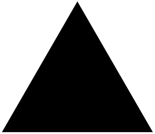
4. $f_5 = f_4 + f_3 = 3 + 2 = 5$

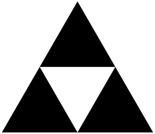
Plan for Chapter 5

Sierpinski triangle

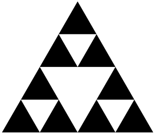
Sierpinski triangles are formed by starting with a triangle and then forming 3 triangles (black) within the original by connecting the midpoints of the sides of the original triangle.



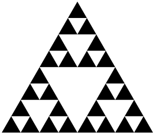
Iteration 0



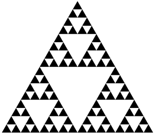
Iteration 1



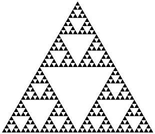
Iteration 2



Iteration 3



Iteration 4



Iteration 5

Recursively defined sets and structures

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3. We will later develop a form of induction, called *structural induction*, to prove results about recursively defined sets.

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2. If v = "*abra* " and u = "*cadabra* " , the concatenation is vu = "*abracadabra*".

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1. BASIS STEP: A single vertex r is a rooted tree.
2. RECURSIVE STEP: Suppose that $T_1, T_2, \ldots, T_n$ are disjoint rooted trees with roots $r_1, r_2, \ldots, r_n$ respectively. Then the structure formed by starting with a root r (which is not in any of the rooted trees $T_1, T_2, \ldots, T_n$) and adding an edge from r to each of the vertices $r_1, r_2, \ldots, r_n$ is also a rooted tree.

Building up rooted trees

Basis Step

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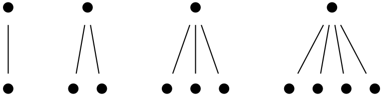
Basis Step

Step 1



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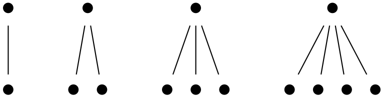
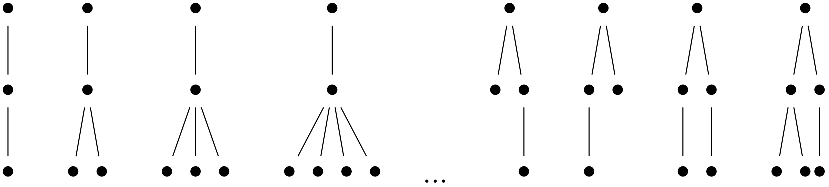


Step 2



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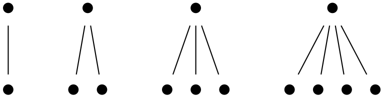
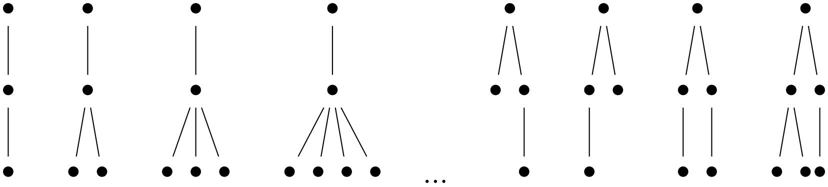
Basis Step

Step 1



Step 2



Next we look at a special type of tree, the full binary tree.

Full binary trees

Definition

The set of *full binary trees* can be defined recursively by these steps.

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1. BASIS STEP: There is a full binary tree consisting of only a single vertex r.
2. RECURSIVE STEP: If T_1 and T_2 are disjoint full binary trees, there is a full binary tree (denoted by $T_1 \cdot T_2$) consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2.

Building up full binary trees

Basis Step

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Basis Step

Step 1

Building up full binary trees

Basis Step

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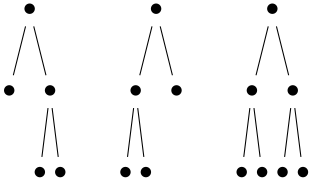
Step 2



Induction and recursively defined sets

Example

Show that the set S (defined by specifying that $3 \in S$ and that if $x \in S$ and $y \in S$, then $x + y$ is in S) is the set of all positive integers that are multiples of 3.

Solution:

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   c. INDUCTIVE STEP: Assume $P(k)$ is true. By the second part of the recursive definition, if $3k \in S$, then since $3 \in S$, $3k + 3 = 3(k + 1) \in S$. Hence, $P(k + 1)$ is true.

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   BASIS STEP: $3 \in S$ by the first part of recursive definition, and $3 = 3 \cdot 1$.
   INDUCTIVE STEP: The second part of the recursive definition adds $x + y$ to S, if both x and y are in S. If x and y are both in A, then both x and y are divisible by 3. It follows that $x + y$ is divisible by 3.

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Part (4) is known as structural induction.

Plan for Chapter 5

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Definition

To prove a property of the elements of a recursively defined set, we use *structural induction*.

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2. RECURSIVE STEP: Show that if the statement is true for each of the (old) elements used to construct new elements in the recursive step of the definition, then the result holds for these new elements.
3. The validity of structural induction can be shown to follow from the principle of mathematical induction.

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Definition

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Structural induction and binary trees

Theorem

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$n(T) = 1 + n(T_1) + n(T_2)$ *by recursive formula of* $n(T)$

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&\leq 2 \cdot max(2^{h(T1)+1}, 2^{h(T2)+1}) - 1 && \\
&= 2 \cdot 2^{max(h(T1),h(T2))+1} - 1 && \textit{since } max(2^x, 2^y) = 2^{max(x,y)}
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Plan for Chapter 5

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An algorithm is called *recursive* if it solves a problem by reducing it to an instance of the same problem with smaller input size.

1. As for any algorithm, proving a recursive algorithm amounts to proving that this algorithm is correct (that is, satisfies its specifications) and proving that this algorithm terminates (that is, any call to that algorithm executes in finitely many steps).
2. For the algorithm to terminate, the instance of the problem must eventually be reduced to some initial case for which the solution is known.

Recursive factorial algorithm

Example

Give a recursive algorithm for computing $n!$, where n is a non-negative integer.

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Algorithm 1 factorial (n)

Require: $n \in \mathbb{Z}^+$
Ensure: $n!$, the factorial of n.

if $n = 0$ **then**
    return 1
else
    return $n \cdot factorial(n - 1)$
end if

Recursive exponentiation algorithm

Example

Give a recursive algorithm for computing a^n , where a is a nonzero real number and n is a non-negative integer.

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Algorithm 2 power (a, n)

Require: $a \in \mathbb{R}, n \in \mathbb{Z}^+, \quad a \neq 0$
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Algorithm 3 $\gcd(a, b)$

Require: $a, b \in \mathbb{Z}^+, \quad a < b$
Ensure: $\gcd(a, b)$, the GCD of a and b.

if $a = 0$ **then**
  return b
else
  return $\gcd(b,\ a \textbf{ mod } b)$
end if

Plan for Chapter 5

Proving recursive algorithms correct

Both mathematical and strong induction are useful techniques to show that recursive algorithms always produce the correct output.

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Prove that the algorithm for computing the powers of real numbers is correct.

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Algorithm 2 power (a, n)

```
Require: a ∈ ℝ, n ∈ ℤ⁺,   a ≠ 0
Ensure: aⁿ, the power of a to n.
if n = 0 then
    return 1
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Prove that the algorithm for computing the powers of real numbers is correct.

Solution: Use mathematical induction on the exponent n.

1. BASIS STEP: $a^0 = 1$ for every nonzero real number a, and $power(a, 0) = 1$.
2. INDUCTIVE STEP: The inductive hypothesis is that $\mathrm{power}(a, k) = a^k$, for all $a \neq 0$.

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Example

Algorithm 2 power (a, n)

Require: $a \in \mathbb{R}, n \in \mathbb{Z}^+, \quad a \neq 0$
Ensure: a^n, the power of a to n.

```
if n = 0 then
    return 1
else
    return a · power(a, n − 1)
end if
```

Prove that the algorithm for computing the powers of real numbers is correct.

Solution: Use mathematical induction on the exponent n.

1. BASIS STEP: $a^0 = 1$ for every nonzero real number a, and $power(a, 0) = 1$.
2. INDUCTIVE STEP: The inductive hypothesis is that $\mathrm{power}(a, k) = a^k$, for all $a \neq 0$.
3. Assuming the inductive hypothesis, the algorithm correctly computes a^{k+1}, since $\mathrm{power}(a, k+1) = a \cdot \mathrm{power}(a, k) = a \cdot a^k = a^{k+1}$. ∎