Binary Relations

Chapter 9

UWO – November 25, 2021

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Plan for Chapter 9

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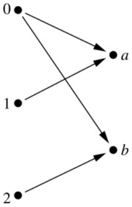
Binary relations

Definition

A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example

1. Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$
2. $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.
3. We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function from A to B is a relation where for each element x of A there is exactly one element y of B related to x.

Binary relation on a set

Definition

A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A.

Example

1. Suppose that $A = \{a, b, c\}$.
2. Then $R = \{(a, a)\ ,\ (a, b), (a, c)\}$ is a binary relation on A.

Example

1. Let $A = \{1, 2, 3, 4\}$.
2. The ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ are $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

Binary relation on a set

Example

How many relations are there on a finite set A?

Solution:

1. Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$.
2. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$.
3. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary relations on a set

Example

Consider these relations on the set of integers:

$$R_1 = \{(a,b) \mid a \leq b\} \qquad R_4 = \{(a,b) \mid a = b\}$$
$$R_2 = \{(a,b) \mid a > b\} \qquad R_5 = \{(a,b) \mid a = b+1\}$$
$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\} \qquad R_6 = \{(a,b) \mid a+b \leq 3\}$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs:

1. (1,1)
2. (1, 2)
3. (2,1)
4. $(1, -1)$
5. (2,2)

Solution: Checking the conditions that define each relation, we see that the pair

1. (1,1) is in R_1, R_3, R_4, and R_6:
2. (1,2) is in R_1 and R_6:
3. (2,1) is in R_2, R_5, and R_6:
4. $(1, -1)$ is in R_2, R_3, and R_6:
5. (2,2) is in R_1, R_3, and R_4.

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Reflexive relations

Definition

The relation R on A is *reflexive* iff $(a, a) \in R$ for every element $a \in A$. That is, R is reflexive if and only if

$$\forall x, \quad x \in A \rightarrow (x, x) \in R$$

Example

The following relations on the integers are reflexive:

$R_1 = \{(a, b) \mid a \leq b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a, b) \mid a = b\}$

If $A = \varnothing$ then the empty relation is reflexive vacuously. That is, the empty relation on an empty set is reflexive!

The following relations are not reflexive:

$R_2 = \{(a, b) \mid a > b\}$ (note that $3 \not> 3$)

$R_5 = \{(a, b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$)

$R_6 = \{(a, b) \mid a + b \leq 3\}$ (note that $4 + 4 \not\leq 3$)

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Symmetric relations

Definition

The relation R on A is *symmetric* iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$. That is, R is symmetric if and only if

$$\forall x,\ \forall y,\ \ (x, y) \in R \ \rightarrow\ (y, x) \in R$$

Example

The following relations on the integers are symmetric:

$R_3 = \{(a, b) \mid |a| = |b|\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$

The following relations are not symmetric:

$R_1 = \{(a, b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$)

$R_2 = \{(a, b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$)

$R_5 = \{(a, b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$)

Antisymmetric relations

Definition

The relation R on A is *antisymmetric* if for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ holds. That is: R is antisymmetric iff

$$\forall x,\ \forall y,\ \ ((x, y) \in R \land (y, x) \in R)\ \rightarrow\ x = y$$

Note: if x and y are distinct ($x \neq y$) then R can not have both (x, y) and (y, x).

Example

The following relations on the integers are antisymmetric:

$R_1 = \{(a, b) \mid a \leq b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$

For any integer, if a $a \leq b$ and $b \leq a$ then $a = b$.

The following relations are not antisymmetric:

$R_3 = \{(a, b) \mid |a| = |b|\}$ (note that both $(1, -1)$ and $(-1, 1)$ belong to R

$R_6 = \{(a, b) \mid a + b \leq 3\}$ (note that both $(1, 2)$ and $(2, 1)$ belong to R_6)

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Transitive relations

Definition

The relation R on A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$. That is, R is transitive if and only if

$$\forall x \forall y \forall z [(x, y) \in R \land (y, z) \in R \to (x, z) \in R]$$

Example

The following relations on the integers are transitive:

$R_1 = \{(a, b) \mid a \leq b\}$
$R_2 = \{(a, b) \mid a > b\}$
$R_3 = \{(a, b) \mid |a| = |b|\}$
$R_4 = \{(a, b) \mid a = b\}$

For any integer, if a $a \leq b$ and $b \leq c$ then $a \leq c$.

The following relations are not transitive:

$R_5 = \{(a, b) \mid a = b + 1\}$ (note that both $(3, 2)$ and $(4, 3)$ belong to R_5, but not $(4, 2$
$R_6 = \{(a, b) \mid a + b \leq 3\}$ (note that both $(2, 1)$ and $(1, 2)$ belong to R_6, but not $(2, 2$

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Combining relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

1. $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$
2. $R_1 \cap R_2 = \{(1,1)\}$
3. $R_1 - R_2 = \{(2,2),(3,3)\}$
4. $R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

Combining relations via composition

Definition

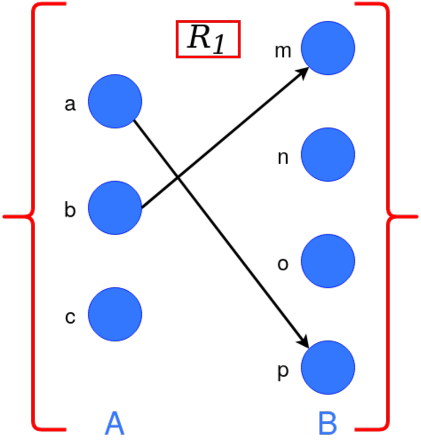
Suppose:

1. R_1 is a relation from a set A to a set B.
2. R_2 is a relation from B to a set C.

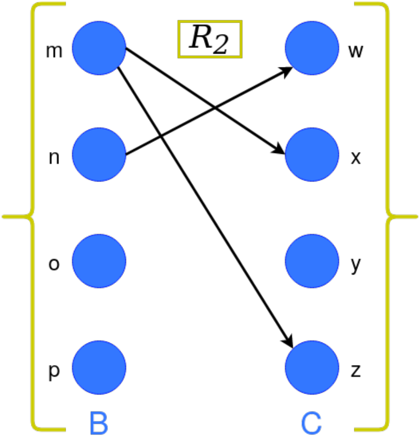
Then the *composition* of R_2 with R_1, is a relation from A to C, denoted $R_2 \circ R_1$, where:

1. if (x, y) is a member of R_1 and (y, z) is a member of R_2 then (x, z) is a member of $R_2 \circ R_1$.
2. also, if $(x, z) \in R_2 \circ R_1$ then there exists some $y \in B$ such that $(x, y) \in R_1$ and $(y, z) \in R_2$

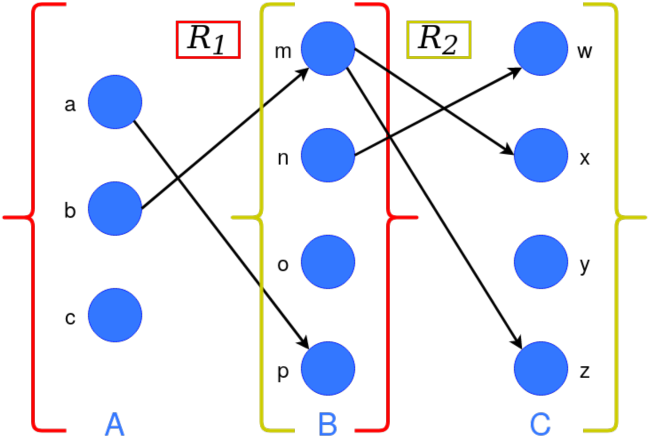
Representing a composition



Representing a composition

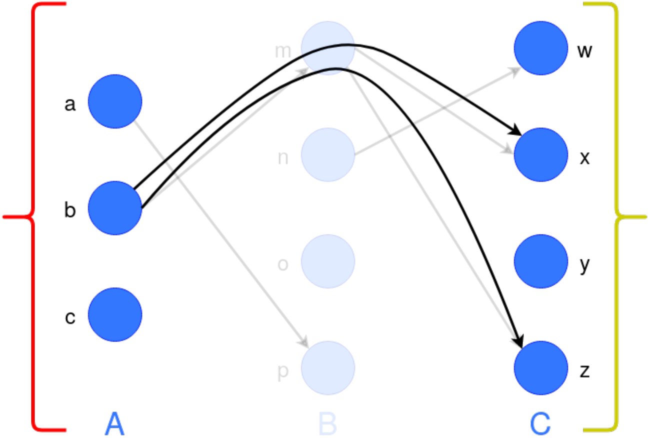


Representing a composition



$$R_2 \circ R_1 = \{(b, z), (b, x)\}$$

Representing a composition



$$R_2 \circ R_1 = \{(b, z), (b, x)\}$$

Composition of a relation with itself

Definition

Let R be a binary relation on a set A. Then the composition of R with R, denoted $R \circ R$, is a relation on A where:

1. if (x, y) is a member of R
2. and (y, z) is a member of R
3. then (x, z) is a member of $R \circ R$.

Example

Let R be a relation on the set of all people such that (a, b) is in R if person a is parent of person b. Then (a, c) is in $R \circ R$ iff there is a person b such that (a, b) is in R and (b, c) is in R. In other words, (a, c) is in $R \circ R$ iff a is a grandparent of c.

Powers of a relation

Definition

Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

1. Basis Step: $R^1 = R$
2. Inductive Step: $R^{n+1} = R^n \circ R$

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem

The relation R on a set A is transitive iff $R^n \subseteq R$ for all positive integers n.

(see Tutorial 11.)

Plan for Chapter 9

1. Relations and Their Properties

1.1 Relations and Functions
1.2 Reflexive Relations
1.3 Symmetric and Antisymmetric Relations
1.4 Transitive Relations
1.5 Combining Relations

2. Representing Relations

2.1 Representing Relations using Matrices
2.2 Representing Relations using Digraphs

3. Equivalence Relations

3.1 Equivalence Relations
3.2 Equivalence Classes
3.3 Equivalence Classes and Partitions

4. Partial Orderings

4.1 Partial orderings and partially-ordered sets
4.2 Lexicographic Orderings

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Representing relations using matrices

1. A relation between finite sets can be represented using a zero-one matrix.
2. Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
   a. The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
3. The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

4. The matrix representing R has a 1 as its (i, j) entry, when a_i is related to b_j, and, a 0 if a_i is not related to b_j.

Examples of matrices representing relations

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B such that:

$$R = \{(a, b) \mid a \in A, b \in B, \; a > b \; \}$$

What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order) ?

Solution:

Because $R = \{(2,1),\ (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of matrices representing relations

Example

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution:

Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

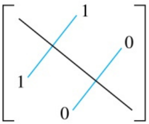
$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}$

Matrices of relations on sets

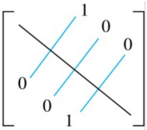
1. If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

2. R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$.



(a) Symmetric



(b) Antisymmetric

3. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

Example of a relation on a set

Example

Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution:

1. Because all the diagonal elements are equal to 1, R is reflexive.
2. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Matrices for combinations of relations

1. The matrix of the union of two relations is the join (Boolean OR) between the matrices of the component relations:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

2. The matrix of the intersection of two relations is the meet (Boolean AND) between the matrices of the component relations:

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

3. The matrix of the composite relation $R_1 \circ R_2$ is the Boolean product of the matrices of the component relations:

$$M_{R_1 \circ R_2} = M_{R_1} \odot M_{R_2}$$

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Representing relations using directed graphs (a.k.a. *digraphs*)

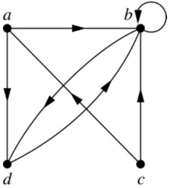
Definition

A *directed graph*, or *digraph*, consists of a set V of *vertices* or *nodes* together with a set E of ordered pairs of elements of V called (directed) *edges* or *arcs*.

1. The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.
2. An edge of the form (a, a) is called a *loop*.

Example

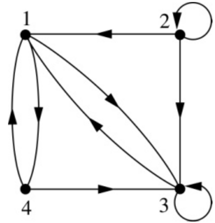
The directed graph with vertices a, b, c, and d, and edges $(a, b), (a, d), (b, b), (b, d), (c, a), (c, b)$, and (d, b) is shown here:



Examples of digraphs representing relations

Example

What are the ordered pairs in
the relation represented by this directed graph?



Solution:

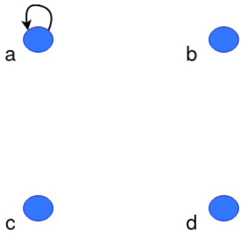
The ordered pairs in the relation are:

1. $(1,3)$
2. $(1,4)$
3. $(2,1)$
4. $(2,2)$
5. $(2,3)$
6. $(3,1)$
7. $(3,3)$
8. $(4,1)$
9. $(4,3)$

Determining which properties a relation has from its digraph

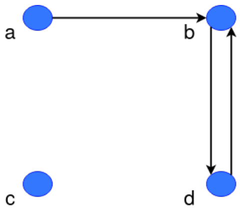
1. *Reflexivity*: A loop must be present at all vertices.
2. *Symmetry*: If (x, y) is an edge, then so is (y, x).
3. *Antisymmetry*: If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.
4. *Transitivity*: If (x, y) and (y, z) are edges, then so is (x, z).

Determining which properties a relation has from its digraph – Example 1



1. *Reflexive?*
   No, not every vertex has a loop
2. *Symmetric?*
   Yes (trivially), there is no edge from one vertex to another
3. *Antisymmetric?*
   Yes (trivially), there is no edge from one vertex to another
4. *Transitive?*
   Yes, (trivially) since there is no edge from one vertex to another

Determining which properties a relation has from its digraph – Example 2



1. *Reflexive?*

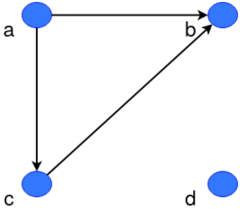
   No, there are no loops
2. *Symmetric?*

   No, there is an edge from a to b, but not from b to a
3. *Antisymmetric?*

   No, there is an edge from d to b and b to d
4. *Transitive?*

   No, there are edges from a to b and from b to d, but there is no edge from a to d

Determining which properties a Relation has from its digraph – Example 3



1. *Reflexive?*

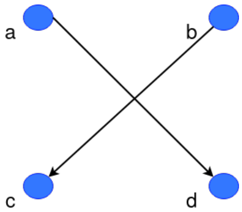
   No, there are no loops
2. *Symmetric?*

   No, for example, there is no edge from c to a
3. *Antisymmetric?*

   Yes, whenever there is an edge from one vertex to another, there is not one going back
4. *Transitive?*

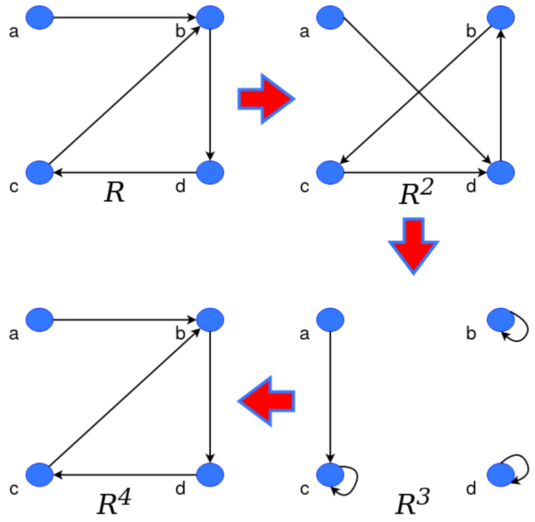
   Yes

Determining which properties a relation has from its digraph – Example 4



1. *Reflexive?*
   No, there are no loops
2. *Symmetric?*
   No, for example, there is no edge from d to a
3. *Antisymmetric?*
   Yes, whenever there is an edge from one vertex to another, there is not one going back
4. *Transitive?*
   Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the powers of a relation



The pair (x, y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

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Equivalence relations

Definition

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

Assume C is the set of all cars and a relation R on C such that $R = \{(a, b) \mid a \in C, b \in C \text{ and, } a \text{ and } b \text{ have the same color } \}$.

R is an *equivalence relation* on C.

Strings

Example

Suppose that R is the relation on the set of strings of English letters such that $(a, b) \in R$ if and only if $\ell(a) = \ell(b)$, where $\ell(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold:

1. *Reflexivity*: Because $\ell(a) = \ell(a)$, it follows that $(a, a) \in R$ for all strings a.
2. *Symmetry*: Assume $(a, b) \in R$. Since $\ell(a) = \ell(b)$, then $\ell(b) = \ell(a)$ also holds and we have $(b, a) \in R$.
3. *Transitivity*: Suppose that $(a, b) \in R$ and $(b, c) \in R$. Since $\ell(a) = \ell(b)$ and $\ell(b) = \ell(c)$ both hold, then $\ell(a) = \ell(c)$ also holds and we have $(a, c) \in R$.

Yes, R is an equivalence relation.

Congruence modulo m

Example

Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

1. *Reflexivity*:

   $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.

2. *Symmetry*:
   - a. Suppose that $a \equiv b \pmod{m}$.
   - b. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer.
   - c. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.

3. *Transitivity*:
   - a. Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$.
   - b. Then m divides both $a - b$ and $b - c$.
   - c. Hence, there are integers k and ℓ with $a - b = km$ and $b - c = \ell m$.
   - d. We obtain by adding the equations:
     $a - c = (a - b) + (b - c) = km + \ell m = (k + \ell)m$.
   - e. Therefore, $a \equiv c \pmod{m}$.

Divides

Example

Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution:

1. *Reflexive*:
   $a \mid a$ for all a.
2. *Not symmetric*:
   For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
3. *Transitivite*:
   a. Suppose that a divides b and b divides c.
   b. Then there are positive integers k and ℓ such that $b = ak$ and $c = b\ell$.
   c. Hence, we have $c = a(k\ell)$. So, we have: a divides c.
   d. Therefore, the relation is transitive.

The properties of reflexivity, and transitivity do hold, but the relation is not symmetric. Hence, "divides" is not an equivalence relation.

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Equivalence classes

Definition

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by $[a]_R$.

$$[a]_R := \{s \in A \mid (a, s) \in R\} \equiv \{s \in A \mid s \sim a\}$$

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

1. If $b \in [a]_R$, then b is a *representative* of this equivalence class. Any element of a class can be used as representative.
2. The equivalence classes of the relation "congruence modulo m" are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\dots, a-2m, a-m, a+m, a+2m, \dots\}$.

$[0]_4 = \{\dots, -8, -4, 0, 4, 8, \dots\}$ $[1]_4 = \{\dots, -7, -3, 1, 5, 9, \dots\}$

$[2]_4 = \{\dots, -6, -2, 2, 6, 10, \dots\}$ $[3]_4 = \{\dots, -5, -1, 3, 7, 11, \dots\}$

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Equivalence classes and partitions

Theorem

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

1. $(a, b) \in R$,
2. $[a] = [b]$,
3. $[a] \cap [b] \neq \emptyset$.

We show $(i) \to (ii)$. All other implications are proved similarly.

1. Assume that $(a, b) \in R$.
2. Now suppose that c $\in [a]$.
3. Then $(a, c) \in R$.
4. Because $(a, b) \in R$ and R is symmetric, $(b, a) \in R$.
5. Because R is transitive and $(b, a) \in R$ and $(a, c) \in R$, it follows that $(b, c) \in R$.
6. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$.

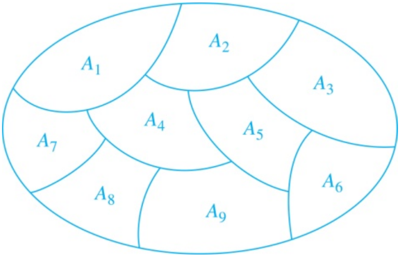
A similar argument proved $[b] \subseteq [a]$. Hence, we have $[b] = [a]$.

Partition of a set

Definition

A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

1. $A_i \neq \emptyset$ for $i \in I$,
2. $A_i \cap A_j = \emptyset$ when $i \neq j$,
3. and $\bigcup_{i \in I} A_i = S$



A partition of a set

An equivalence relation partitions a set

1. Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_r = A$$

2. From the previous theorem, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \varnothing$ when $[a]_R \neq [b]_R$.
3. Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An equivalence relation partitions a set

Theorem

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

Proof.

We have already shown the first part of the theorem. For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

1. *Reflexivity*:

   For every $a \in S, (a, a) \in R$, because a is in the same subset as itself.

2. *Symmetry*:

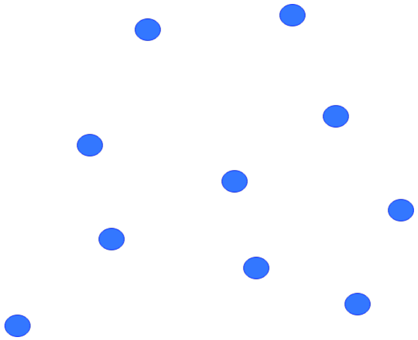
   If $(a, b) \in R$, then b and a are in the same subset of the partition, so $(b, a) \in R$.

3. *Transitivity*:

   If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a, c) \in R$ since a and c belong to the same subset of the partition.

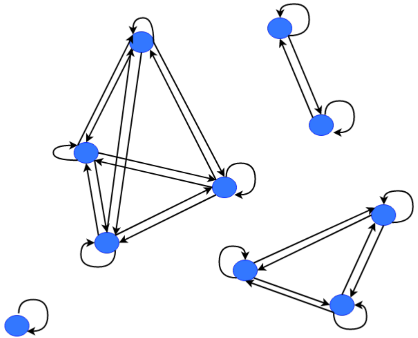
■

An equivalence relation digraph representation



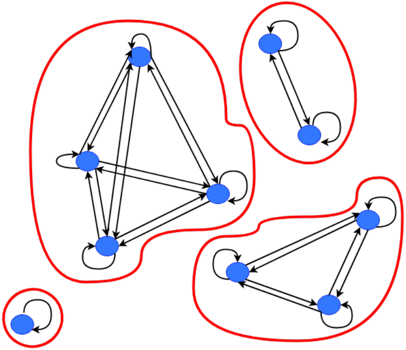
Set A.

An Equivalence Relation digraph representation



Equivalence relation R on set A.

An Equivalence Relation digraph representation



Digraph for equivalence relation R on finite set A is a union of disjoint sub-graphs (representing disjoint equivalent classes). Nodes in each distinct subgraph (equivalence class) are fully interconnected.

Plan for Chapter 9

Plan for Chapter 9

Partial orderings

Definition

A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Partial orderings

Example

Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Solution:

1. *Reflexivity*:

   $a \geq a$ for every integer a.

2. *Antisymmetry*:

   If $a \geq b$ and $b \geq a$, then $a = b$.

3. *Transitivity*:

   If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers.)

Partial orderings

Example

Show that the divisibility relation is a partial ordering on the set of positive integers.

Solution:

1. *Reflexivity*:

   $a \mid a$ for all integers a.

2. *Antisymmetry*:

   If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

3. *Transitivity*:

   a. Suppose that a divides b and that b divides c.
   b. Then, there are positive integers k and ℓ such that $b = ak$ and $c = b\ell$.
   c. Hence, $c = a(k\ell)$, so that a divides c.
   d. Therefore, the relation is transitive.

$(Z^+, |)$ is a poset.

Partial orderings

Example

Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

Solution:

1. *Reflexivity*:

   $A \subseteq A$ whenever A is a subset of S.

2. *Antisymmetry*:

   If A and B are subsets of S, with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

3. *Transitivity*:

   If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition

The elements a and b of a poset $(S, \preceq)$ are *comparable* if either $a \preceq b$ or $b \preceq a$ holds. When a and b are elements of S so that neither $a \preceq b$ nor $b \preceq a$ holds, then a and b are called *incomparable*.

The symbol $\preceq$ is used to denote the relation in any poset.

Definition

If $(S, \preceq)$ is a poset and any two elements of S are comparable, then S is called a *totally ordered* or *linearly ordered set*, and $\preceq$ is called a *total order* or a *linear order.* (A totally ordered set is also called a *chain*.)

Definition

$(S, \preceq)$ is a well-ordered set if it is a poset such that $\preceq$ is a total ordering and every nonempty subset of S has a least element.

Example

1. $(Z, \leq)$ is a totally ordered set
2. $(Z, |)$ is a partially ordered but not totally ordered set
3. $(N, \leq)$ is a well-ordered set

Plan for Chapter 9

Lexicographic order

Definition

Given two partially ordered sets $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying when (a_1, a_2) is less than (b_1, b_2), written, $(a_1, a_2) \prec (b_1, b_2)$, which holds either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$ holds.

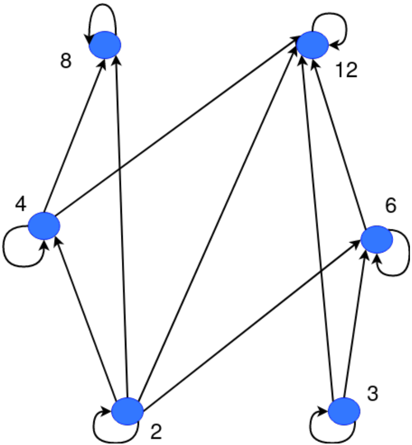
This definition can be easily extended to a lexicographic ordering on strings.

Example

Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

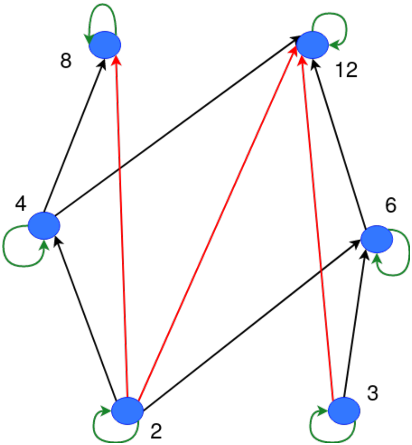
1. *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
2. *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

Partial ordering relation digraph representation



poset $R = (X, |)$ for divisibility $|$ on set $X = \{2, 3, 4, 6, 8, 12\}$

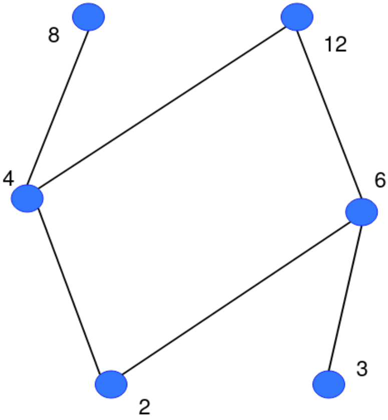
Partial ordering relation Hesse diagram



poset $R = (X, |)$ for divisibility $|$ on set $X = \{2, 3, 4, 6, 8, 12\}$

1. Leave out all edges that are implied by reflexivity (loop)
2. Leave out all edges that are implied by transitivity

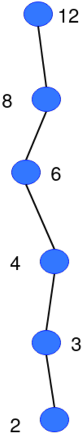
Partial ordering relation Hesse diagram



poset $R = (X, |)$ for divisibility $|$ on set $X = \{2, 3, 4, 6, 8, 12\}$

Can also drop “direction” assuming that (partial) order is **upward**

Partial ordering relation Hesse diagram



poset $R = (X, \leq)$ for
"less than or equal" on set
$X = \{2, 3, 4, 6, 8, 12\}$

Totally ordered sets are also called "chains "