UWO CS2214

Tutorial #2

Problem 1 Let x be a real number. Prove the following identities:

1. $\lceil -x \rceil \; = \; -\lfloor x \rfloor$
2. $\lfloor -x \rfloor \; = \; -\lceil x \rceil$

Solution 1

1. Assume first that x is an integer n. Then we have

$$\lceil -x \rceil \; = \; \lceil -n \rceil \; = -n \; = \; -\lfloor n \rfloor \; = \; -\lfloor x \rfloor.$$

If x is not an integer, then there exist an integer n and $0 < \varepsilon < 1$ so that $x = n + \varepsilon$ holds. In this case, we have

$$\begin{array}{rl} \lceil -x \rceil & = \\ \lceil -(n+\varepsilon) \rceil & = \\ \lceil -n - 1 + (1-\varepsilon) \rceil & = \\ (-n-1)+1 & = \\ -n & = \\ -\lfloor n + +\varepsilon \rfloor & \\ -\lfloor x \rfloor. & \end{array}$$

2. Assume first that x is an integer n. Then we have

$$\lfloor -x \rfloor \; = \; \lfloor -n \rfloor \; = -n \; = \; -\lceil n \rceil \; = \; -\lceil x \rceil.$$

If x is not an integer, then exist an integer n and $0 < \varepsilon < 1$ so that $x = n + \varepsilon$ holds. In this case, we have

$$\begin{array}{rl} \lfloor -x \rfloor & = \\ \lfloor -(n+\varepsilon) \rfloor & = \\ \lfloor -n - 1 + (1-\varepsilon) \rfloor & = \\ -n-1 & = \\ -(n+1) & = \\ -\lceil n + \varepsilon \rceil & = \\ -\lceil x \rceil. & \end{array}$$

Problem 2 Let x be a real number and n be an integer. Prove the following identities:

1. $\lceil x+n \rceil \;=\; \lceil x \rceil + n$
2. $\lfloor x \rfloor \;=\; \lfloor x \rfloor + n$

Solution 2

1. There exist an integer m and $0 \leq \varepsilon < 1$ so that $x = m + \varepsilon$ holds. Then, we have
$$\lceil x+n \rceil \;=\; \lceil (m+n)+\varepsilon \rceil \;=\; m+n \;=\; \lceil m+\varepsilon \rceil + n \;=\; \lceil x \rceil.$$
2. The proof is similar to that of the previous claim.

Problem 3 Which of the functions f below is injective? surjective? When f is bijective, determine its inverse

1. $f_1 : \begin{array}{ccc} \mathbb{Z} & \rightarrow & \mathbb{Z} \\ x & \longmapsto & x+2 \end{array}$
2. $f_2 : \begin{array}{ccc} \mathbb{Z} & \rightarrow & \mathbb{Z} \\ x & \longmapsto & x^2-1 \end{array}$
3. $f_3 : \begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \longmapsto & \frac{x+2}{3} \end{array}$
4. $f_4 : \begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \longmapsto & \lceil x \rceil \end{array}$

Solution 3

1. f_1 is injective since $f_1(x_1) = f_1(x_1)$ implies $x_1 = x_2$. f_1 is surjective and the inverse function of f_1 is: $f_1^{-1} : \begin{array}{ccc} \mathbb{Z} & \rightarrow & \mathbb{Z} \\ y & \longmapsto & y-2 \end{array}$
2. f_2 is not injective since $f_1(1) = 0 = f_1(-1)$. f_2 is not injective since -2 has no pre-image by f_2. Indeed $-2 = x^2 - 1$ has no solution in $\mathbb{Z}$ (and even in $\mathbb{R}$).
3. f_3 is injective since $f_3(x_1) = f_3(x_1)$ implies $x_1 = x_2$. f_3 is surjective as the pre-image of y is $3\,y - 2$.
4. f_4 is not injective since $f_4(\sqrt{2}) = 2 = f_1(2)$. f_4 is not injective since $\sqrt{2}$ has no pre-image by f_4.