Tutorial #3

Problem 1 Let a, b, q_1, r_1, q_2, r_2 be non-negative integer numbers such that $b \neq 0$ and we have

$$\begin{array}{c|c} a & b \\ \hline q_1 & r_1 \end{array} \quad \text{and} \quad \begin{array}{c|c} a & b \\ \hline q_2 & r_2 \end{array}. \tag{1}$$

Thus we have: $a = bq_1 + r_1 = bq_2 + r_2$ as well as $0 \leq r_1 < b$ and $0 \leq r_2 < b$. Prove that $q_1 = q_2$ and $r_1 = r_2$ necessarily both hold

Solution 1 Let $a = bq_1 + r_1 = bq_2 + r_2$, with $0 \leq r_1 < b$ and $0 \leq r_2 < b$, where a, b, q_1, r_1, q_2, r_2 are non-negative integers. We wish to show that $q_1 = q_2$ and $r_1 = r_2$.

Assume that $r_1 \neq r_2$. Then, without loss of generality, assume that $r_2 > r_1$. We then have that

$$\begin{aligned} bq_1 - bq_2 &= r_2 - r_1 \\ \Rightarrow b(q_1 - q_2) &= r_2 - r_1 \end{aligned} \tag{2}$$

Since $0 \leq r_1 < b$ and $0 \leq r_2 < b$, and $r_2 > r_1$, it must be that

$$0 < (r_2 - r_1) < b, \tag{3}$$

since the largest difference has $r_2 = b - 1$ and $r_1 = 0$, and $r_1 \neq r_2$ by assumption (so $r_2 - r_1 \neq 0$). But equation (2) implies that b divides $r_2 - r_1$, which cannot be given equation (3), because the multiples of b are $0, \pm b, \pm 2b, \ldots$. This is a contradiction, and we conclude that $r_1 = r_2$.

Since we have shown that $r = r_1 = r_2$, it follows that

$$\begin{aligned} bq_1 - bq_2 &= r - r \\ \Rightarrow b(q_1 - q_2) &= 0 \end{aligned} \tag{4}$$

But equation (4) implies either that $b = 0$ or $q_1 - q_2 = 0$. Since $b \neq 0$ by the assumptions of the division theorem, we conclude that it must be that $q_1 - q_2 = 0$, meaning that $q_1 = q_2$, which is what we set out to prove. QED

Problem 2 In the previous exercise, if a, b, q_1, q_2, are non-negative integer numbers satisfying $a = bq_1 + r_1 = bq_2 + r_2$ while r_1, r_2 are integers satisfying $-b < r_1 < b$ and $-b < r_2 < b$. Do we still reach the same conclusion? Justify your answer.

Solution 2 No, we do not. Indeed, with $a = 7$ and $b = 3$, we then have two possible divisions:

$$\begin{array}{r|l} 7 & 3 \\ \cline{2-2} 1 & 2 \end{array} \quad \text{and} \quad \begin{array}{r|l} 7 & 3 \\ \cline{2-2} -2 & 3 \end{array}.$$

Problem 3 Consider the set of ordered pairs $(x,\, y)$ where x are y are real numbers. Such a pair can be seen as a point in the plane equipped with Cartesian coordinates $(x,\, y)$. For each of the following functions determine a (2×2)-matrix A so that the point of coordinates $(x,\, y)$ is sent to the point $(x',\, y')$ when we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6}$$

1. $F_1(x, y) = (x, y)$
2. $F_2(x, y) = (x, 0)$
3. $F_3(x, y) = (0, y)$
4. $F_4(x, y) = (y, x)$

Solution 3

1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
3. $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
4. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Problem 4 Following up on the previous problem, determine which of the above functions F is injective? surjective?

Solution 4

1. F_1 is injective: Indeed, for all (x_1, y_1) and (x_2, y_2) if $F_1(x_1, y_1) = F_1(x_2, y_2)$ holds then we have $(x_1, y_1) = (x_2, y_2)$, which exactly means that F_1 is injective. F_1 is surjective: Indeed, every (x', y') has a pre-image by F_1, namely itself, since $F_1(x', y') = (x', y')$ holds.

2. F_2 is not njective: Indeed, we have $F_2(0, 1) = (0, 0) = F_2(0, 2)$, thus two different points, namely $(0, 1)$ and $(0, 2)$ have the same image by F_2, namely $(0, 0)$. F_2 is not surjective: Indeed, $(1, 1)$ has no pre-image by F_2.

3. For similar reasons as those for F_2, F_3 is neither injective nor surjective.

4. F_4 is injective: Indeed, for all (x_1, y_1) and (x_2, y_2) if $F_4(x_1, y_1) = F_4(x_2, y_2)$ holds then we have $(y_1, x_1) = (y_2, x_2)$ that is, $y_1 = y_2$ and $x_1 = x_2$, thus $(x_1, y_1) = (x_2, y_2)$, which exactly means that F_1 is injective. F_4 is surjective: Indeed, every (x', y') has a pre-image by F_4, namely (y', x'), since $F_1(y', x') = (x', y')$ holds.