Analysis of Divide and Conquer Algorithms

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CS3101

Plan

1. Review of Complexity Notions

2. Divide-and-Conquer Recurrences

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Orders of magnitude

Let f, g et h be functions from $\mathbb{N}$ to $\mathbb{R}$.

- We say that $g(n)$ is in the **order of magnitude** of $f(n)$ and we write $f(n) \in \Theta(g(n))$ if there exist two strictly positive constants c_1 and c_2 such that for n big enough we have

$$0 \leq c_1 \, g(n) \leq f(n) \leq c_2 \, g(n). \tag{1}$$

- We say that $g(n)$ is an **asymptotic upper bound** of $f(n)$ and we write $f(n) \in \mathcal{O}(g(n))$ if there exists a strictly positive constants c_2 such that for n big enough we have

$$0 \leq f(n) \leq c_2 \, g(n). \tag{2}$$

- We say that $g(n)$ is an **asymptotic lower bound** of $f(n)$ and we write $f(n) \in \Omega(g(n))$ if there exists a strictly positive constants c_1 such that for n big enough we have

$$0 \leq c_1 \, g(n) \leq f(n). \tag{3}$$

Examples

- With $f(n) = \frac{1}{2}n^2 - 3n$ and $g(n) = n^2$ we have $f(n) \in \Theta(g(n))$. Indeed we have
$$c_1\, n^2 \;\; \leq \;\; \frac{1}{2}n^2 - 3n \;\; \leq \;\; c_2\, n^2. \tag{4}$$
for $n \geq 12$ with $c_1 = \frac{1}{4}$ and $c_2 = \frac{1}{2}$.
- Assume that there exists a positive integer n_0 such that $f(n) > 0$ and $g(n) > 0$ for every $n \geq n_0$. Then we have
$$max(f(n), g(n)) \in \Theta(f(n) + g(n)). \tag{5}$$
Indeed we have
$$\frac{1}{2}(f(n) + g(n)) \;\; \leq \;\; max(f(n), g(n)) \;\; \leq \;\; (f(n) + g(n)). \tag{6}$$
- Assume a and b are positive real constants. Then we have
$$(n + a)^b \in \Theta(n^b). \tag{7}$$
Indeed for $n \geq a$ we have
$$0 \;\; \leq \;\; n^b \;\; \leq \;\; (n + a)^b \;\; \leq \;\; (2n)^b. \tag{8}$$
Hence we can choose $c_1 = 1$ and $c_2 = 2^b$.

Properties

- $f(n) \in \Theta(g(n))$ holds iff $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$ hold together.
- Each of the predicates $f(n) \in \Theta(g(n))$, $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$ define a reflexive and transitive binary relation among the $\mathbb{N}$-to-$\mathbb{R}$ functions. Moreover $f(n) \in \Theta(g(n))$ is symmetric.
- We have the following **transposition formula**

$$f(n) \in \mathcal{O}(g(n)) \iff g(n) \in \Omega(f(n)). \tag{9}$$

In practice $\in$ is replaced by $=$ in each of the expressions $f(n) \in \Theta(g(n))$, $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$. Hence, the following

$$f(n) = h(n) + \Theta(g(n)) \tag{10}$$

means

$$f(n) - h(n) \in \Theta(g(n)). \tag{11}$$

Another example

Let us give another fundamental example. Let $p(n)$ be a (univariate) polynomial with degree $d > 0$. Let a_d be its leading coefficient and assume $a_d > 0$. Let k be an integer. Then we have

(1) if $k \geq d$ then $p(n) \in \mathcal{O}(n^k)$,

(2) if $k \leq d$ then $p(n) \in \Omega(n^k)$,

(3) if $k = d$ then $p(n) \in \Theta(n^k)$.

Exercise: Prove the following

$$\Sigma_{k=1}^{k=n} k \ \in \ \Theta(n^2). \tag{12}$$

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Divide-and-Conquer Algorithms

Divide-and-conquer algorithms proceed as follows.

Divide the input problem into sub-problems.

Conquer on the sub-problems by solving them directly if they are small enough or proceed recursively.

Combine the solutions of the sub-problems to obtain the solution of the input problem.

Equation satisfied by $T(n)$. Assume that the size of the input problem increases with an integer n. Let $T(n)$ be the time complexity of a divide-and-conquer algorithm to solve this problem. Then $T(n)$ satisfies an equation of the form:

$$T(n) = a\,T(n/b) + f(n). \tag{13}$$

where $f(n)$ is the cost of the combine-part, $a \geq 1$ is the number of recursively calls and n/b with $b > 1$ is the size of a sub-problem.

Tree associated with a divide-and-conquer recurrence

Labeled tree associated with the equation. Assume n is a power of b, say $n = b^p$. To *solve* the equation

$$T(n) = a\,T(n/b) + f(n).$$

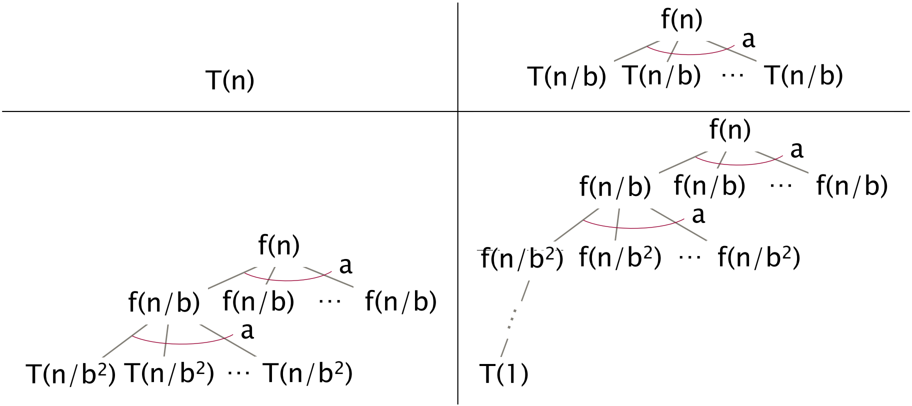
we can associate a labeled tree $\mathcal{A}(n)$ to it as follows.

(1) If $n = 1$, then $\mathcal{A}(n)$ is reduced to a single leaf labeled $T(1)$.

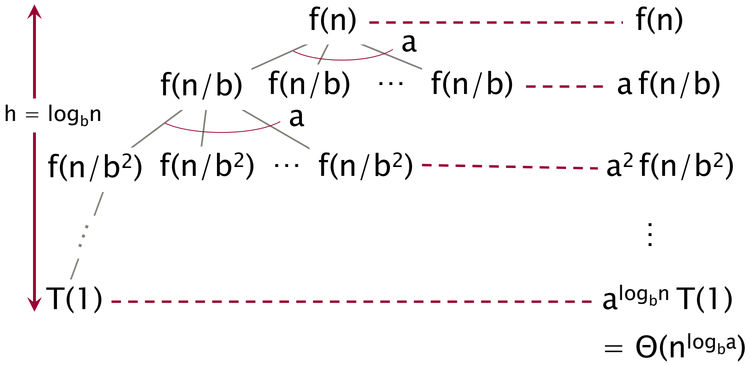
(2) If $n > 1$, then the root of $\mathcal{A}(n)$ is labeled by $f(n)$ and $\mathcal{A}(n)$ possesses a labeled sub-trees all equal to $\mathcal{A}(n/b)$.

The labeled tree $\mathcal{A}(n)$ associated with $T(n) = a\,T(n/b) + f(n)$ has height $p + 1$. Moreover the sum of its labels is $T(n)$.

Solving divide-and-conquer recurrences (1/2)

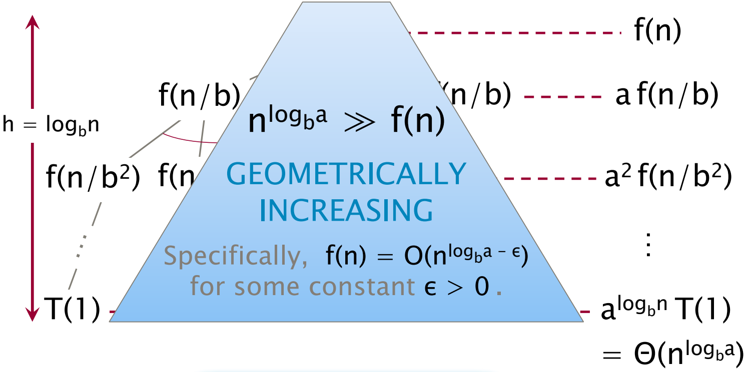


Solving divide-and-conquer recurrences (2/2)



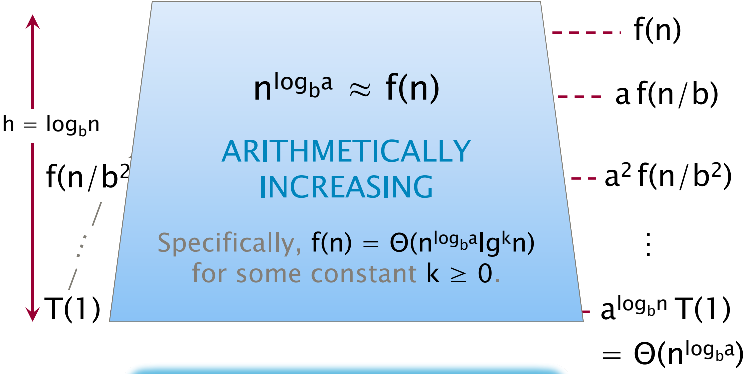
IDEA: Compare $n^{\log_b a}$ with $f(n)$.

Master Theorem: case $n^{\log_b a} \gg f(n)$



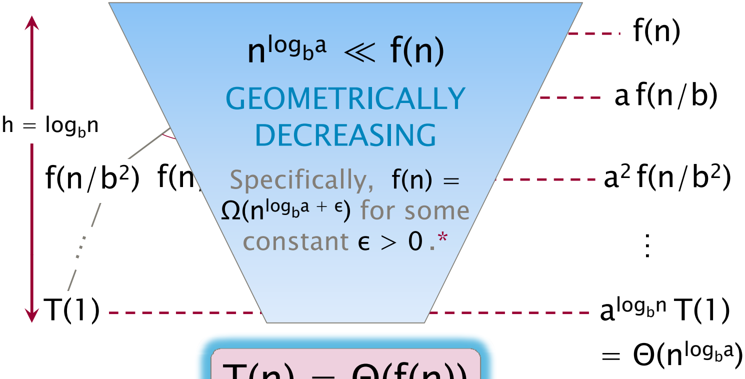
$$T(n) = \Theta(n^{\log_b a})$$

Master Theorem: case $f(n) \in \Theta(n^{\log_b a} \log^k n)$



$$T(n) = \Theta(n^{\log_b a} \lg^{k+1} n))$$

Master Theorem: case where $f(n) \gg n^{\log_b a}$



$$T(n) = \Theta(f(n))$$

*and $f(n)$ satisfies the *regularity condition* that $a\, f(n/b) \le c\, f(n)$ for some constant $c < 1$.

More examples

- Consider the relation:

$$T(n) = 2\,T(n/2) + n^2. \tag{14}$$

We obtain:

$$T(n) = n^2 + \frac{n^2}{2} + \frac{n^2}{4} + \frac{n^2}{8} + \cdots + \frac{n^2}{2^p} + n\,T(1). \tag{15}$$

Hence we have:

$$T(n) \in \Theta(n^2). \tag{16}$$

- Consider the relation:

$$T(n) = 3T(n/3) + n. \tag{17}$$

We obtain:

$$T(n) \in \Theta(\log_3(n)n). \tag{18}$$

Master Theorem when $b = 2$

Let $a > 0$ be an integer and let $f, T : \mathbb{N} \longrightarrow \mathbb{R}_+$ be functions such that

(*i*) $f(2\,n) \geq 2\,f(n)$ and $f(n) \geq n$.

(*ii*) If $n = 2^p$ then $T(n) \leq a\,T(n/2) + f(n)$.

Then for $n = 2^p$ we have

(1) if $a = 1$ then

$$T(n) \leq (2 - 2/n)\,f(n) + T(1) \in \mathcal{O}(f(n)), \tag{19}$$

(2) if $a = 2$ then

$$T(n) \leq f(n)\log_2(n) + T(1)\,n \in \mathcal{O}(\log_2(n)\,f(n)), \tag{20}$$

(3) if $a \geq 3$ then

$$T(n) \leq \frac{2}{a-2}\left(n^{\log_2(a)-1} - 1\right) f(n) + T(1)\,n^{\log_2(a)} \in \mathcal{O}(f(n)\,n^{\log_2(a)-1}). \tag{21}$$

Master Theorem when $b = 2$

Indeed

$$\begin{array}{rcl} T(2^p) & \leq & a\,T(2^{p-1}) + f(2^p) \\ & \leq & a\left[a\,T(2^{p-2}) + f(2^{p-1})\right] + f(2^p) \\ & = & a^2\,T(2^{p-2}) + a\,f(2^{p-1}) + f(2^p) \\ & \leq & a^2\left[a\,T(2^{p-3}) + f(2^{p-2})\right] + a\,f(2^{p-1}) + f(2^p) \\ & = & a^3\,T(2^{p-3}) + a^2\,f(2^{p-2}) + a\,f(2^{p-1}) + f(2^p) \\ & \leq & a^p\,T(s1) + \sigma_{j=0}^{j=p-1}\,a^j\,f(2^{p-j}) \end{array} \quad (22)$$

Master Theorem when $b = 2$

Moreover

$$\begin{array}{rcl} f(2^p) & \geq & 2\, f(2^{p-1}) \\ f(2^p) & \geq & 2^2\, f(2^{p-2}) \\ \vdots & \vdots & \vdots \\ f(2^p) & \geq & 2^j\, f(2^{p-j}) \end{array} \tag{23}$$

Thus

$$\Sigma_{j=0}^{j=p-1}\, a^j\, f(2^{p-j}) \ \leq \ f(2^p)\, \Sigma_{j=0}^{j=p-1} \left(\frac{a}{2}\right)^j. \tag{24}$$

Master Theorem when $b = 2$

Hence

$$T(2^p) \;\leq\; a^p\, T(1) + f(2^p) \, \Sigma_{j=0}^{j=p-1} \left(\frac{a}{2}\right)^j. \tag{25}$$

For $a = 1$ we obtain

$$\begin{array}{rcl} T(2^p) & \leq & T(1) + f(2^p) \, \Sigma_{j=0}^{j=p-1} \left(\frac{1}{2}\right)^j \\ & = & T(1) + f(2^p) \, \frac{\frac{1}{2^p} - 1}{\frac{1}{2} - 1} \\ & = & T(1) + f(n) \, (2 - 2/n). \end{array} \tag{26}$$

For $a = 2$ we obtain

$$\begin{array}{rcl} T(2^p) & \leq & 2^p \, T(1) + f(2^p) \, p \\ & = & n \, T(1) + f(n) \log_2(n). \end{array} \tag{27}$$

Master Theorem cheat sheet

For $a \geq 1$ and $b > 1$, consider again the equation

$$T(n) = a\,T(n/b) + f(n). \tag{28}$$

- We have:

$$(\exists \varepsilon > 0)\ f(n) \in O(n^{\log_b a - \varepsilon}) \implies T(n) \in \Theta(n^{\log_b a}) \tag{29}$$

- We have:

$$(\exists \varepsilon > 0)\ f(n) \in \Theta(n^{\log_b a} \log^k n) \implies T(n) \in \Theta(n^{\log_b a} \log^{k+1} n) \tag{30}$$

- We have:

$$(\exists \varepsilon > 0)\ f(n) \in \Omega(n^{\log_b a + \varepsilon}) \implies T(n) \in \Theta(f(n)) \tag{31}$$

Master Theorem quizz!

- $T(n) = 4T(n/2) + n$
- $T(n) = 4T(n/2) + n^2$
- $T(n) = 4T(n/2) + n^3$
- $T(n) = 4T(n/2) + n^2/\log n$

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