CS434a/541a: Pattern Recognition
Prof. Olga Veksler

Lecture 2
Outline

- Review of Linear Algebra
  - vectors and matrices
  - products and norms
  - vector spaces and linear transformations
  - eigenvalues and eigenvectors
- Introduction to Matlab
Why Linear Algebra?

- For each data point, we will represent a set of features as feature vector
  - \([\text{length, weight, color, ...}]\)
- Collected data will be represented as collection of (feature) vectors
  - \([l_1, w_1, c_1, ...] [l_2, w_2, c_2, ...] [l_3, w_3, c_3, ...] \ldots\)
- Linear models are simple and computationally feasible
Vectors

- n-dimensional row vector \( \mathbf{x} = [x_1, x_2, \ldots, x_n] \)
- Transpose of row vector is column vector \( \mathbf{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \)
- Vector product (or inner or dot product)
\[
\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{i=1}^{k} x_iy_i
\]
More on Vectors

- **Euclidian norm or length** \(|x| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}\)
- If \(|x|=1\) we say \(x\) is *normalized* or *unit* length
- Angle \(\theta\) between vectors \(x\) and \(y\) \(\cos \theta = \frac{x^T y}{||x||||y||}\)

Thus inner product captures direction relationship between \(x\) and \(y\)
More on Vectors

- Vectors \( x \) and \( y \) are orthonormal if they are orthogonal and \(|x| = |y| = 1\).

- Euclidian distance between vectors \( x \) and \( y \):

\[
|x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}
\]
Vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly **dependent** if there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ s.t.

1. $\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_n \mathbf{x}_n = \mathbf{0}$
2. at least one $\alpha_i \neq 0$

Vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly **independent** if $\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_n \mathbf{x}_n = \mathbf{0} \Rightarrow \alpha_1 = \ldots = \alpha_n = 0$
Vector Spaces and Basis

- The set of all n-dimensional vectors is called a vector space $V$

- A set of vectors $\{u_1, u_2, \ldots, u_n\}$ are called a basis for vector space if any $v$ in $V$ can be written as $v = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$

- $u_1, u_2, \ldots, u_n$ are independent implies they form a basis, and vice versa

- $u_1, u_2, \ldots, u_n$ give an orthonormal basis if
  1. $|u_i| = 1 \quad \forall i$
  2. $u_i \perp u_j \quad \forall i \neq j$
**Matrices**

- n by m matrix $A$ and its m by n transpose $A^\top$

\[
A = \begin{bmatrix}
    x_{11} & x_{12} & \cdots & x_{1m} \\
    x_{21} & x_{22} & \cdots & x_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n1} & x_{n2} & \cdots & x_{nm}
\end{bmatrix} \\
A^\top = \begin{bmatrix}
    x_{11} & x_{12} & \cdots & x_{n1} \\
    x_{12} & x_{22} & \cdots & x_{n2} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{1m} & x_{2m} & \cdots & x_{nm}
\end{bmatrix}
\]
Matrix Product

\[ AB = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1d} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nd}
\end{bmatrix}
\begin{bmatrix}
    b_{11} & \cdots & b_{1m} \\
    b_{21} & \cdots & b_{2m} \\
    b_{31} & \cdots & b_{3m} \\
    \vdots & \ddots & \vdots \\
    b_{d1} & \cdots & b_{dm}
\end{bmatrix}
= \begin{bmatrix}
    c_{ij}
\end{bmatrix} = C
\]

- # of columns of A = # of rows of B
- even if defined, in general  \( AB \neq BA \)
Matrices

- **Rank** of a matrix is the number of linearly independent rows (or equivalently columns).
- A square matrix is **non-singular** if its rank equal to the number of rows. If its rank is less than number of rows it is **singular**.
- **Identity matrix** \( I = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \)
- Matrix \( A \) is **symmetric** if \( A = A^T \)

\[
\begin{bmatrix}
1 & 2 & 9 & 5 \\
2 & 7 & 4 & 8 \\
9 & 4 & 3 & 6 \\
5 & 8 & 6 & 4
\end{bmatrix}
\]
Matrices

- Matrix $A$ is **positive definite** if
  \[ x^T A x = \sum_{i,j} A_{i,j} x_i x_j > 0 \]

- Matrix $A$ is **positive semi-definite** if
  \[ x^T A x = \sum_{i,j} A_{i,j} x_i x_j \geq 0 \]

- Trace of a square matrix $A$ is sum on the elements on the diagonal
  \[ tr[A] = \sum_{i=1}^{n} a_{ii} \]
Matrices

- **Inverse** of a square matrix $A$ is matrix $A^{-1}$ s.t. $AA^{-1} = I$

- If $A$ is singular or not square, inverse does not exist. **Pseudo-inverse** $A^\dagger$ is defined whenever $A^\top A$ is not singular (it is square)
  - $A^\dagger = (A^\top A)^{-1}A^\top$
  - $A^\dagger A = (A^\top A)^{-1}A^\top A = I$
Matrices

- Determinant of \( n \) by \( n \) matrix \( A \) is

\[
\det(A) = \sum_{k=1}^{n} (-1)^{k+i} a_{ik} \det(A_{ik})
\]

- Where \( A_{ik} \) obtained from \( A \) by removing the \( i \)th row and \( k \)th column

- Absolute value of determinant gives the volume of parallelepiped spanned by the matrix rows

\[
\left\{ \beta_1 a^1 + \beta_2 a^2 + \ldots + \beta_n a^n \right\}
\]

\[
\beta_i \in [0,1] \ \forall i
\]
A linear transformation from vector space \( V \) to vector space \( U \) is a mapping which can be represented by a matrix \( M \):

- \( u = Mv \)

If \( U \) and \( V \) have the same dimension, \( M \) is a square matrix.

In pattern recognition, often \( U \) has smaller dimensionality than \( V \), i.e. transformation \( M \) is used to reduce the number of features.
**Eigenvectors and Eigenvalues**

Given an $n$ by $n$ matrix $A$, and a nonzero vector $x$. Suppose there is $\lambda$ which satisfies $Ax = \lambda x$

- $x$ is called an eigenvector of $A$
- $\lambda$ is called an eigenvalue of $A$

- Linear transformation $A$ maps an eigenvector $v$ in a simple way. Magnitude changes by $\lambda$, direction.

  - If $\lambda > 0$
  - If $\lambda < 0$

Note: $A \mathbf{0} = \lambda \mathbf{0}$ for any $\lambda$, not interesting
**Eigenvectors and Eigenvalues**

- If $A$ is real and symmetric, then all eigenvalues are real (not complex).
- If $A$ is non-singular, all eigenvalues are non-zero.
- If $A$ is positive definite, all eigenvalues are positive.
MATLAB
Starting matlab
- `xterm -fn 12X24`  
- `matlab`

Basic Navigation
- `quit`
- `more`
- `help general`

Scalars, variables, basic arithmetic
- Clear
- `+ - * / ^`
- `help arith`

Relational operators
- `==,&,|,~,xor`
- `help relop`

Lists, vectors, matrices
- `A=[2 3;4 5]`
- `A'`

Matrix and vector operations
- `find(A>3), colon operator`
- `* / ^ .* ./ .^`
- `eye(n),norm(A),det(A),eig(A)`
- `max,min,std`
- `help matfun`

Elementary functions
- `help elfun`

Data types
- `double`
- `Char`

Programming in Matlab
- `.m files`
- `scripts`
- `function y=square(x)`
- `help lang`

Flow control
- `if i==1 else end, if else if end`
- `for i=1:0.5:2 ... end`
- `while i == 1 ... end`
- `Return`
- `help lang`

Graphics
- `help graphics`
- `help graph3d`

File I/O
- `load, save`
- `fopen, fclose, fprintf, fscanf`