On Systems of Algebraic Equations with Parametric Exponents

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ABSTRACT

We deal with systems of algebraic equations with parametric exponents. As the first step for solving such systems, we consider the most simple cases, univariate case and 0-dimensional case, and give a concrete method for computing Gröbner bases. From studies on such cases, we derive a simple formulation and basic notions which will be helpful to deal with more complicated cases.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms

General Terms

Algorithms, Theory

Keywords

Ideals with parametric exponents, Gröbner basis

1. INTRODUCTION

In mathematical problem, there arise systems of algebraic equations with parameters. For solving systems with parametric coefficients, many works were done by several authors, where complete methods are proposed. (See [10, 5, 11].) Systems with parametric exponents are also important and very interesting. However, as few works except [9, 12] were done for those systems, many questions/problems seem untouched. Here we consider certain *stability* of systems with parametric exponents and *computability* of their solutions. These problems can be translated to problems on the form of Gröbner bases of (radical) ideals generated by polynomials with parametric exponents.

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The problems dealt with here are originally raised by Tadashi Takahashi in order to give a computational proof of non-degeneracy conditions of singularities of algebraic surfaces [9]. We show one typical type of his problem.

EXAMPLE 1. *What is the singularity of $S_{k,0}$ [1, 9]*

$$f = x^2z + yz^2 + y^{4k+1} + axy^{3k+1} + bzy^{2k+1},$$

where k is a positive integer and a, b are complex numbers. Then we have to solve the following system

$$f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0.$$

The "parameter" k appears in coefficients, which makes the problem more difficult.

Here we set our problem and our goal as follows:

Goal. For an ideal $\mathcal{I}$ generated by finitely many polynomials with parametric exponents, we want to examine the following problems for $\mathcal{I}$. When one fixes a value (a positive integer) for each parameter, the reduced Gröbner basis ([2, 3]) can be determined with respect to a fixed ordering. Then,

(1) **Stability:** When the values of parameters are large enough, does the form of the Gröbner basis of $\mathcal{I}$ become "stable"? Or the form can be determined "uniformly" in the values of parameters? If one wants to express the zeros of $\mathcal{I}$ uniformly, one may concentrate on the radical of $\mathcal{I}$.

(2) **Computability:** If the form of the Gröbner basis of $\mathcal{I}$ is stable for sufficiently large values for parameters, are there algorithms for computing it? That is, do algorithms stop in finitely many steps independent of the values of parameters?

The problems are also heavily related to the following:

(3) **Effects of "sparsity" of generating sets on the computational complexity:** When the values of parameters are sufficiently large, the inputs are sparse polynomials. Especially, in the 0-dimensional case, the computational complexity is estimated using the Bézout bound, when one use a rev-lex ordering. Since the Bézout bound will be given as a polynomial function in the parameters of our problem, study on ideals with parametric exponents might give some insights in the problem.

For an arbitrary ideal generated by polynomials with parametric exponents, "stability/uniformity" is impossible in general. However, there are some classes of ideals which posses such stability/uniformity, and it is very important to find a wider class with many applications.

Here, as the first attempt to attack the problems, we deal with the most simple cases, univariate case and 0-dimensional case with one parameter, and have an affirmative answer with concrete procedures for computing Gröbner bases. Aiming for dealing with more complicated cases, we derive a "simple formalization" and important basic notions from studies on such simple cases.

As another attempt for the problem with one parameter, Weispfenning, also inspired by Takahashi's problem, has made a complete work [12] on ideals generated by monomials and binomials. His results are orthogonal to ours and both supplement each other.

2. FORMULATION

Here we give a precise settings on the problems and necessary notions in order to solve them.

Settings. We consider a polynomial ring $\mathbf{Q}[X]$ (or $\mathbf{C}[X]$), where $X = \{x_1, \ldots, x_n\}$. So, polynomials with parametric exponents are treated as "ordinary" polynomials with fixed (but unknown) integer values substituted in parameters. (So, parameters are not treated as variables.) As every exponent is non-negative, there might be certain restriction on the values of parameters. But, by shifting values, without loss of generality, we can assume that parameters can range over all positive integers.

DEFINITION 1 (EP-POWER PRODUCT AND EP-IDEAL). *We call a power product with parametric exponents an* ep-power product*, a term having an ep-power product as its subterm an* ep-term*, and a polynomial with ep-term an* ep-polynomial*. When an ideal has an ep-polynomial in its generator, we call the ideal an* ep-ideal*. In distinction to ep-polynomials and ep-ideals, we call a polynomial without ep-term an* ordinary *polynomial, and an ideal generated by ordinary polynomials an* ordinary *ideal. (The name "ep" is given by Weispfenning.)*

EXAMPLE 2. *The polynomial in Example 1*

$$f = x^2z + yz^2 + y^{4k+1} + axy^{3k+1} + bzy^{2k+1}$$

is an ep-polynomial in $\mathbf{C}[x, y, z]$ for fixed complex numbers a, b, where y^{4k+1}, axy^{3k+1}, bzy^{2k+1} are ep-terms having y^k as their subterm with parameter $k \geq 1$.

In Example 2, y^k plays an essential role. Because, replacing y^k with a new variable w, we have an ordinary polynomial

$$g = x^2z + yz^2 + w^4y + axw^3y + bzw^2y.$$

DEFINITION 2 (ESSENTIAL SET).
For an ep-polynomial f, there is a set of ep-power products $\{T_1, \ldots, T_s\}$ such that each T_i is a subterm of some term appearing in f and one can obtain an ordinary polynomial by replacing each T_i with a new variable y_i. We call the set $\{T_1, \ldots, T_s\}$ an essential set *for f. Moreover, for a generating set G of an ideal $\mathcal{I}$, if a set $\{T_1, \ldots, T_s\}$ of ep-power products is an essential set for every element in G, we call the set $\{T_1, \ldots, T_s\}$ an* essential set *for $\mathcal{I}$. There might be several essential sets for a fixed f.*

Here we consider forms of Gröbner bases in order to give certain "stability". From now on, we fix a term order $\prec$. Let $\mathcal{I}$ be an ep-ideal having an ep-polynomial in its given generator $\mathcal{F}$, and $K = (k_1, \ldots, k_t)$ a set of parameters appearing in $\mathcal{I}$. (We write $\mathcal{I} = \langle \mathcal{F} \rangle$.) For each $A = (a_1, \ldots, a_t) \in \mathbf{N}^t$, where $\mathbf{N}$ is the set of natural numbers (positive integers), let $G(A)$ be the reduced Gröbner basis of an ep-ideal $\mathcal{I}$ with A substituted in K.

DEFINITION 3 (STABILITY OF GRÖBNER BASIS).
The ideal $\mathcal{I}$ is said to have a stable *Gröbner basis if there exists a vector $B = (b_1, \ldots, b_t) \in \mathbf{N}^t$, called a* bound*, such that one of the following occurs:*

(1) ***Generic Form:*** *For every $A = (a_1, \ldots, a_t)$ with $a_i \geq b_i$, the number of elements of $G(A)$ does not depend on the values A of parameters K, and each element has finitely many terms independent of the values A of parameters K and is "comprehensive". That is, each element can be expressed by a sum of fixed ep-terms and ordinary terms, and for fixed values A of parameters K, we have the reduced Gröbner base $G(A)$ by simply substituting those values. In this case, $G(A)$ is called of* generic form.

(2) ***Periodic Form:*** *There also exists a vector $P = (p_1, \ldots, p_t) \in \mathbf{N}^t$ such that for every $A = (a_1, \ldots, a_t)$ with $a_i \geq b_i$, $G(A)$ is determined uniquely by the values $(a_1 \bmod p_1, \ldots, a_s \bmod p_s)$. In this case, $G(A)$ is called of* periodic form*, and P is called a* period*. As its special case, there is a case where $G(A)$ does not depend on any values A of parameters. In this case, $G(A)$ is called of* completely stable form.

(3) ***Bounded Form:*** *For every $A = (a_1, \ldots, a_t)$ with $a_i \geq b_i$, $G(A)$ is trivial ($G(A) = \{1\}$). In this case, $G(A)$ is called of* bounded form*. This is also a special case of* completely stable form.

We also call periodic forms (including bounded forms) finite forms*, because the degree of every element of the Gröbner basis is bounded.*

Moreover, the ideal $\mathcal{I}$ is said to have a semi-stable *Gröbner basis if $\mathcal{I}$ is expressed as an intersection of fixed ideals having stable Gröbner bases, where ordinary ideals are considered to have stable Gröbner bases.*

EXAMPLE 3. *The following is a Gröbner basis of the ideal generated by itself with respect to lex order $x_1 \prec x_2 \prec x_3$. It is of generic form.*

$$\begin{aligned} f_1 &= x_1^{k+2} + 1 \\ f_2 &= x_2 - x_1^{k+1} - x_1 + 1 \\ f_3 &= x_3 - x_1^k - 1 \end{aligned}$$

The radical of the ideal for $S_{k,0}$ in Example 1 is $\langle x, y, z \rangle$ for almost every complex values for a and b, which gives an example of finite form (completely stable). (See Example 5.)

EXAMPLE 4. *(1) The ideal $\langle x^k - 1, x^2 + x + 1 \rangle$ becomes $\langle x^2 + x + 1 \rangle$ if $k \equiv 0 \pmod 3$ and $\langle 1 \rangle$ for otherwise. This is a periodic case.*

(2) For the ideal $\langle x^k - 5x + 2, x^2 + x - 6\rangle$, it becomes $\langle x - 2\rangle$ if $k = 3$, and $\langle 1\rangle$ otherwise. This is a bounded case.

(3) For the ideal $\langle x^{k+1} - x^k + x^2 - 1, x^2 + x - 2\rangle$, it becomes $\langle x - 1\rangle$ for every $k \geq 1$. This is a completely stable case.

REMARK 1. *With respect to the lex order $x \prec y$, the ideal $\langle x^k - 1, (x-1)y - 1\rangle$ does not have its Gröbner basis G of generic form, but G can be expressed by a certain "comprehensive form" as follows:*

$$\frac{(x^k - 1)}{(x-1)},\ y + \frac{(x^k - ky + k - 1)}{(x-1)^2}.$$

But, as mentioned in Example 7, it has certain difficulty in computation. So, it seems difficult to handle such a case as a generic form case.

2.1 Applicable Techniques

Here we mention two important techniques which seem very useful to solve a system of algebraic equations with parametric exponents and to compute its Gröbner basis. From now on, we assume that $\mathcal{I}$ is an ep-ideal generated by $\mathcal{F} = \{f_1, \ldots, f_r\}$, and $\mathcal{T} = \{T_1, \ldots, T_s\}$ is an essential set. See [4, 3] for elimination ideal and [10, 11] for comprehensive Gröbner bases.

Slack Variables and Elimination. If $\mathcal{I}$ has a Gröbner basis of finite form, it might be effective to eliminate all ep-power products appearing in generating polynomials.

From $\mathcal{F}$, replacing ep-power products $T_1, \ldots, T_s$ with new slack variables $y_1, \ldots, y_s$, we have a set $\mathcal{F}_0$ of ordinary polynomials in $\mathbf{Q}[X, Y]$. That is, from each f_i, we have a new polynomial $f_{i,0}(X, Y)$ such that $f_{i,0}(X, T) = f_i(X)$.

Let $\mathcal{I}_0$ be the ideal in $\mathbf{Q}[X, Y]$ generated by $\mathcal{F}_0$. Computing the elimination ideal $\mathcal{J} = \mathcal{I}_0 \cap \mathbf{Q}[X]$ with some fixed elimination order $X \prec\prec Y$, we find ordinary polynomials belonging to the ep-ideal $\mathcal{I}$. Let H be a Gröbner basis of $\mathcal{J}$. Then,

LEMMA 1. *H is contained in $\mathcal{I}$, that is, $\mathcal{J}$ is contained in $\mathcal{I}$.*

PROOF. For each polynomial $h(X)$ in H, we show that $h(X)$ belongs to $\mathcal{I}$. As $h(X) \in \mathcal{J} = \mathcal{I}_0 \cap \mathbf{Q}[X]$, there are polynomials $a_i(X, Y)$ such that

$$h(X) = \sum_{i=1}^{r} a_i(X, Y) f_{i,0}(X, Y).$$

Then, substituting T_i for each y_i, we have

$$h(X) = \sum_{i=1}^{r} a_i(X, T) f_i(X).$$

This implies that $h(X)$ belongs to $\mathcal{I}$. $\square$

DEFINITION 4. *We call the above elimination ideal $\mathcal{J}$ the* finite subideal *of $\mathcal{I}$. ($\mathcal{J}$ depends on the choice of $\mathcal{T}$.)*

If the computed finite subideal $\mathcal{J}$ is 0-dimensional, there is a method for computing the Gröbner basis of finite form, which will be shown in later. As to computing all zeros, we have a more efficient way: As $\mathcal{J} \subset \mathcal{I}$, the set of zeros $V(\mathcal{J})$ contains $V(\mathcal{I})$. Thus, all zeros of $\mathcal{I}$ can be obtained by checking if each zero of $\mathcal{J}$ satisfies the original generating set F. This method is very efficient when $V(\mathcal{J})$ is a fixed finite set, that is, $\mathcal{J}$ is 0-dimensional.

Moreover, it might be much efficient to use prime decomposition of $\mathcal{J}$. (See [3, 8] for detailed algorithms.) For each prime divisor $\mathcal{P}$ of $\mathcal{J}$, we compute $\mathcal{I} + \mathcal{P}$. Then, gathering the computational results of $\mathcal{I} + \mathcal{P}$ for all prime divisors $\mathcal{P}$, we have the final result.

EXAMPLE 5. *For the ep-polynomial $S_{k,0}$ in Example 1*

$$f = x^2 z + yz^2 + y^{4k+1} + axy^{3k+1} + bzy^{2k+1},$$

$\{y^k\}$ is the unique essential set. So, replacing y^k with w, we have an ordinary polynomial in 4 variables

$$f_0 = x^2 z + yz^2 + w^4 y + axw^3 y + bzw^2 y.$$

In Takahashi's Problem $S_{k,0}$, we have the following 3 additional polynomials obtained by partial differentiation:

$$\begin{aligned} f_1 &= 2xz + aw^3 y \\ f_2 &= z^2 + (4k+1)w^4 + (3k+1)axw^3 + (2k+1)bzw^2 \\ f_3 &= x^2 + 2yz + bw^2 y. \end{aligned}$$

Then, considering a, b, k as other variables, that is, considering only generic case for parametric coefficients a, b, k, we can compute an elimination ideal $\mathcal{J}$ of $\langle f_0, f_1, f_2, f_3\rangle$ eliminating w in the polynomial ring $\mathbf{Q}(a, b, k)[x, y, z, w]$.

With lex order $w \succ z \succ y \succ x$, we computed $\mathcal{J}$ and also computed all its prime divisors. Then $\mathcal{J}$ has two prime divisors

$$\langle x, y\rangle, \langle x, z\rangle.$$

We divide the problem into two cases, the case $x = y = 0$ and the case $x = z = 0$. Then, we have

$$\begin{aligned} x = y = 0 &\ \rightarrow\ z = 0 \\ x = z = 0 &\ \rightarrow\ y = 0, \end{aligned}$$

which shows that $\langle x, y, z\rangle$ is the radical of the ep-ideal $\mathcal{I}$ in generic case for a, b.

REMARK 2. *In Example 5, the parameter k also appears in coefficients. So, the above computation corresponds to "generic case", that is, a, b, k does not satisfy certain algebraic constraints, actually, $a \neq 0$ and $b \notin \{0, 2, -2\}$ (see [9]). For solving such parametric systems precisely, see Chapter 6 Section 3 in [4] or comprehensive Gröbner basis computation [10, 5, 11].*

If one wants to classify all possible forms of the Gröbner basis, one needs the technique derived from COMPREHENSIVE GRÖBNER BASIS[10, 5, 11]. See also the most recent work [12] for ep-ideals generated by monomials and binomials.

Comprehensive Gröbner basis. We execute Buchberger algorithm [2, 4, 3] stepwise, where we decide which term should be the leading term. So, there might appear some branches depending on the values of parameters.

EXAMPLE 6. *If two ep-terms y^{3k+2} and y^{2k+20} appear, their order will depend on the value of k as follows:*

$$\begin{aligned} k > 6 &\ \rightarrow\ y^{3k+2} \succ y^{2k+8} \\ k < 6 &\ \rightarrow\ y^{3k+2} \prec y^{2k+8} \\ k = 6 &\ \rightarrow\ \textit{we must merge } y^{3k+2} \textit{ and } y^{2k+8}. \end{aligned}$$

The most crucial problem is the termination of Buchberger algorithm including monomial reductions in finitely many steps independent of the values of parameters. There is a case where the computational complexity of Buchberger algorithm depends on the value of parameters. The following example requires $O(k)$ steps.

EXAMPLE 7.

$$\begin{aligned} f(x,y) &= x^k - 1 \\ g(x,y) &= xy - y - 1 \end{aligned}$$

With respect to the lex order $y \succ x$, the reduced Gröbner basis will be

$$\{x^{k-1} + x^{k-2} + \cdots + 1, x^{k-2} + 2x^{k-3} + \cdots + (k-1) + ky\}.$$

This implies that the Buchberger algorithm requires at least k monomial reductions.

From now on, we will consider the simplest case where an essential set consists of one variable with one parameter exponent.

3. UNIVARIATE CASE

Here we consider an ep-ideal $\mathcal{I}$ in $\mathbf{Q}[x]$. And suppose that $\{x^k\}$ is the unique essential set for $\mathcal{I}$. In general, it is not true that the ideal $\mathcal{I}$ has a (semi-)stable Gröbner basis. But, in this case, there is a certain stability.

REMARK 3. *In many systems appearing in mathematics, k is supposed to be sufficiently large, or terms with different expressions are supposed different to each other for any values of parameters. From these assumptions, there are certain restrictions on values of parameters. For example, for the expression $f(x) = x^{2k} + x^{k+5} + x^{12}$ the condition $k > 7$ might be given to assert that $2k > k + 5 > 12$.*

Settings. For ep-polynomials $f(x), g(x)$ over $\mathbf{Q}$ with essential set $\{x^k\}$, we compute $\gcd(f(x), g(x))$, which is a Gröbner basis of the ideal $\mathcal{I} = \langle f(x), g(x)\rangle$. (We assume that k does not appear in coefficients.) Moreover, for simplicity, $f(x)$ and $g(x)$ have non zero constant terms. (We remove the factor x from $f(x)$ and $g(x)$ in advance.)

Then we have the following result.

THEOREM 1. *There are positive integers P, B computable from $f(x), g(x)$ such that for each value $a \geq B$ of the parameter k, $\gcd(f(x), g(x))$ is the product of a "generic form factor" and a "finite form factor" determined uniquely by the value $a \bmod P$. That is, the ep-ideal $\mathcal{I}$ has a semi-stable Gröbner basis.*

In the following, we will give a concrete procedure for computing $\gcd(f(x), g(x))$, which gives a proof of Theorem 1.

First, replacing x^k with a new variable y, we compute bivariate polynomials f_0, g_0 from f, g. So, $f(x) = f_0(x, x^k)$ and $g(x) = g_0(x, x^k)$. Then, as bivariate polynomials, we compute $\gcd(f_0(x,y), g_0(x,y))$ which we denote by $h_0(x,y)$. Then, $h(x) = h_0(x, x^k)$ is a common factor of $f(x), g(x)$. We call $h(x)$ *the generic form factor.* (There is a case where $h(x)$ is an ordinary polynomial.)

Next we consider $f'(x) = f(x)/h(x)$ and $g'(x) = g(x)/h(x)$ and try to compute $\gcd(f'(x), g'(x))$. Replacing x^k with a new variable y in f', g', we have bivariate polynomials f_1, g_1 from f', g', that is, $f'(x) = f_1(x, x^k)$ and $g'(x) = g_1(x, x^k)$.

As $f_0 = f_1 h_0$ and $g_0 = g_1 h_0$, f_1 and g_1 have no common factor as bivariate polynomials. So, the resultant $\mathrm{res}_y(f_1, g_1)$ does not vanish, and it is an ordinary non-zero polynomial in x belonging to $\langle f_1(x,y), g_1(x,y)\rangle$.

Consider the finite subideal $\langle f_1(x,y), g_1(x,y)\rangle \cap \mathbf{Q}[x]$ which is not $\{0\}$, and let $m(x)$ be its generator. Then $m(x)$ belongs $\langle f'(x), g'(x)\rangle$ by Lemma 1.

If $m(x)$ is a constant (non zero), then $\langle f'(x), g'(x)\rangle = 1$ and so there is no common factor of $f'(x), g'(x)$.

If $m(x)$ is not a constant, we factorize $m(x)$ into its irreducible factors $m_i(x)$ over $\mathbf{Q}$:

$$m(x) = \prod_{i=1}^{r} m_i(x)^{e_i}.$$

Since $m(x)$ belongs to the ideal $\langle f'(x), g'(x)\rangle$ and $m_i(x)^{e_i}$'s are pairwise prime, we have

$$\gcd(f', g') = \gcd(f', g', m) = \prod_{i=1}^{r} \gcd(f', g', m_i(x)^{e_i}).$$

Thus, the gcd computation is reduced to the computation of $\gcd(f', g', m_i(x)^{e_i})$. (We already exclude x from factors.)

Now we divide the factors $m_i(x)$ into two cases:

DEFINITION 5. *If $m_i(x)$ is a factor of $x^p - 1$ with a positive integer p, we call $m_i(x)$ a* cyclotomic factor. *And we call the smallest positive integer p such that $m_i(x)$ divides $x^p - 1$ the* period *of $m_i(x)$. (In this case $m_i(x)$ is a cyclotomic polynomial.) Otherwise, we call $m_i(x) (\neq x)$ a* non cyclotomic factor.

We note that the period P_i of $m_i(x)$ can be bounded by a certain function in $\deg(m_i)$, as $\deg(m_i) = \phi(P_i)$. (As an easy example, $P_i < 2\deg(m_i)^2$.) Thus, we can decide wether $m_i(x)$ is a cyclotomic factor and can compute its period if so by dividing $x^n - 1$ by m_i for each n less than a computed bound on P_i.

Cyclotomic Case. Suppose that $m_i(x)$ is a cyclotomic factor of the period P_i. In this case, we have the following.

PROPOSITION 1. $\gcd(f'(x), g'(x), m_i(x))$ *is determined uniquely by the value of $k \bmod P_i$.*

PROOF. For each value k, we denote $k \bmod P_i$ simply by a, where $a \in \{0, 1, \ldots, P_i - 1\}$. As $m_i(x)$ is irreducible, $\gcd(f'(x), g'(x), m_i(x))$ is non-trivial if and only if both of $f'(x), g'(x)$ are divided by $m_i(x)$. As $m_i(x)$ divides $x^{P_i} - 1$, x^k and x^a are congruent modulo $m_i(x)$, that is, both belong to the same residue class in the residue class ring $\mathbf{Q}[x]/\langle m_i(x)\rangle$. Substituting a for k, we have an ordinary polynomial $f'_a(x)$ congruent to $f'(x)$ modulo $m_i(x)$. Then, as $m_i(x)$ divides $f'(x) - f'_a(x)$, $m_i(x)$ divides $f'_a(x)$ if and only if $m_i(x)$ divides $f'(x)$. Similarly, we also have an ordinary polynomial $g'_a(x)$ congruent to $g'(x)$. This arguments shows that $\gcd(f'(x), g'(x), m_i(x))$ is determined by $\gcd(f'_a(x), g'_a(x), m_i(x))$ and hence, it is determined uniquely by the value $a \equiv k \pmod{P_i}$. $\square$

Thus, we can determine whether $\gcd(f(x), g(x))$ has $m_i(x)$ as its factor simply by dividing $f'_a(x)$ and $g'_a(x)$ by $m_i(x)$ for each $a \in \{0, 1, \ldots, P_i - 1\}$. ($f'_a(x)$ and $g'_a(x)$ are obtained from $f'(x)$ and $g'(x)$ by substituting a for k.) If $e_i = 1$, we are done.

For the case $e_i > 1$, we need "derivatives" to know the power e such that $\gcd(f'(x), g'(x)) = m_i(x)^e$. Suppose that we already know $m_i(x)$ divides $\gcd(f(x), g(x))$. Then, $m_i(x)^2$ divides $\gcd(f(x), g(x))$ if and only if $m_i(x)$ also divides both of $\frac{df'(x)}{dx}$ and $\frac{dg'(x)}{dx}$. We note that there appear parametric coefficients (linear in k) in $\frac{df'(x)}{dx}$ and $\frac{dg'(x)}{dx}$.

For each $a \in \{0, 1, \ldots, P_i - 1\}$, we replace the parameter k in exponents with a. (For $a = 0$, some exponent may be negative. In this case, we replace k with P_i instead of 0.) But, for parametric coefficient linear in k, we introduce another parameter s and replace the parameter k in coefficients with $sP_i + a$. We denote new ordinary polynomials obtained from $\frac{df'(x)}{dx}$ and $\frac{dg'(x)}{dx}$ by $f''_a(x)$ and $g''_a(x)$, respectively.

Then we compute resultants $R_{a,f} = \mathrm{res}_x(m_i(x), f''_a(x))$ and $R_{a,g} = \mathrm{res}_x(m_i(x), f''_a(x))$, where s is considered as a variable and polynomials are considered in $\mathbf{Q}[x, s]$. Then $R_{a,f}$ and $R_{a,g}$ are univariate polynomials in s.

Lemma 2. *For each $a(= k \bmod P_i)$ and $sP_i + a$, the followings hold:*

(1) If both of $R_{a,f}$ and $R_{a,g}$ are zero polynomials, then $m_i(x)^2$ divides $\gcd(f'(x), g'(x))$ for any $k = sP_i + a$.

(2) If at least one of $R_{a,f}$ or $R_{a,g}$ is non zero constant, then $m_i(x)^2$ does not divide $\gcd(f'(x), g'(x))$ for any $k = sP_i + a$.

(3) If at least one of $R_{a,f}$ and $R_{a,g}$ is a non constant polynomial and the other is not a non zero constant, then $m_i(x)^2$ divides $\gcd(f'(x), g'(x))$ only for special values $k = sP_i + a$, where s are positive integral common roots of $R_{a,f}(s)$ and $R_{a,g}(s)$. (Where we consider all integer as roots of the zero polynomial.) Conversely, in this case, let M be the maximal value of $k = sP_i + a$, where s ranges all positive integral common roots. (If there is no such common root, we set $M = 0$.) Then, for any $k = sP_i + a > M$, $m_i^2(x)$ does not divide $\gcd(f'(x), g'(x))$.

Proof. For each $a(= k \bmod P_i)$ and $sP_i + a$, $m_i(x)^2$ divides $\gcd(f'(x), g'(x))$ if and only if $m_i(x)$ divides both of $f''_a(x)$ and $g''_a(x)$. By using this fact, we have only to consider wether $m_i(x)$ divides both of $f''_a(x)$ and $g''_a(x)$. Then, by using resultant theory, we have (1),(2) and (3). Here, as $m_i(x)$ has no parametric coefficient, we do not need to check if the leading coefficients of $f''_a(x)$ and $g''_a(x)$ vanish or not. (See [4] Chapter 3 Section 6.) $\square$

By Lemma 2, we can decide if $m_i(x)^2$ divides $\gcd(f'(x), g'(x))$ for any $k = sP_i + a$. And, if not, we also have a bound, say $M_a^{(2)}$ such that for any $k = sP_i + a > M_a^{(2)}$, $m_i^2(x)$ does not divide $\gcd(f'(x), g''(x))$. In this case, we need to compute $\gcd(f'(x), g'(x), m_i(x)^{e_i})$ only for the special values $k = sP_i + a \le M_a^{(2)}$,

While $e \le e_i$ and $m_i(x)^{e-1}$ divides $\gcd(f'(x), g'(x))$, repeating the same procedure for higher derivatives $\frac{d^e f'_a}{dx^e}$ and $\frac{d^e g'_a}{dx^e}$, we can decide wether $m_i(x)^e$ divides $f'_a(x)$ for every k with $k \equiv a \pmod{P_i}$. Moreover, if not, we have a bound $M_a^{(e)}$ such that $m_i(x)^e$ does not divide $f'(x)$ for any $k > M_a^{(e)}$ with $k \equiv a \pmod{P_i}$.

Thus, gathering these informations on the divisibility, we have Proposition 2 and Procedure [Cyclotomic Case].

Remark 4. *For the derivatives $\frac{d^e f'_a}{dx^e}$ and $\frac{d^e g'_a}{dx^e}$, every exponents must be non-negative. Therefore, we need the condition $k \ge e_i$ and use $a + dP_i$ for some positive integer d for substitution instead of a. From this modification, for smaller value $k < e_i$, we have to compute $\gcd(f'(x), g'(x), m_i(x)^{e_i})$ individually.*

Proposition 2. *There exists a positive integer M_i such that if $k > M_i$, then $\gcd(f'(x), g'(x), m_i(x)^{e_i})$ is determined uniquely by the value $k \bmod P_i$. Moreover, M_i can be computed by $f(x), g(x)$ and $m_i(x)$.*

Procedure [Cyclotomic Case]
For each value $a \in \{0, 1, \ldots, P_i - 1\}$, execute the following:

1. Compute ordinary polynomials $f'_a(x), g'_a(x)$ by substituting a for k.
2. Compute $\gcd(f'_a(x), g'_a(x), m_i(x))$.
3. If $\gcd(f'_a(x), g'_a(x), m_i(x)) = 1$, return 1.
4. If $\gcd(f'_a(x), g'_a(x), m_i(x)) = m_i(x)$ then set $E = e_i$, $F = f'(x)$, $G = g'(x)$ and $A = m_i(x)$.
5. If $E = 1$, then return A. Otherwise set $E = E - 1$.
6. while($E > 0$)
    - 6.1. Compute $\frac{dF}{dx}$ and $\frac{dF}{dx}$. Set $F = \frac{dF}{dx}$ and $G = \frac{dF}{dx}$.
    - 6.2. Compute F_a and G_a by substituting a for k in ep-terms and by replacing k with $sP_i + a$ in coefficients. (See Remark 4 for a modification.)
    - 6.3. Compute $\mathrm{res}_x(F_a, m_i)$ and $\mathrm{res}_x(G_a, m_i)$.
    - 6.4. If both resultants vanishes, then set $A = A \times m_i(x)$ and $E = E - 1$, and return to the top of 6.
    - 6.5. Otherwise, compute the set R of all common positive integer roots of $\mathrm{res}_x(F_a, m_i)$ and $\mathrm{res}_x(G_a, m_i)$.
    - 6.6. If $R = \emptyset$, return A.
    - 6.7. Let $B = [A]$. For each root s in R,
        - compute F_k and G_k from $f'(x)$ and $g'(x)$ by replacing k with $sP_i + a$, and
        - compute $\gcd(F_k(x), G_k(x), m_i(x)^{e_i})$ and append $(sP_i + a, \gcd(F_k(x), G_k(x), m_i(x)^{e_i}))$ to B.
    - 6.8. return B.
7. return A.

Example 8. *Consider the following polynomials:*

$$\begin{aligned} f(x) &= x^{3k} - 2x^{k+6} + 1, \\ g(x) &= (x^k - 1)^2 + (x^2 + x + 1)^2 \end{aligned}$$

The elimination ideal is generated by $m(x) = (x^2 + x + 1)^2 m'(x)$, where $m'(x)$ is a non cyclotomic factor. For $k \equiv 0$

(mod 3), *$f(x), g(x)$ are divided by $x^2 + x + 1$. Then their derivatives are as follows.*

$$\begin{aligned} \frac{f(x)}{dx} &= 3kx^{3k-1} - 2(k+6)x^{k+5}, \\ \frac{g(x)}{dx} &= 2kx^{k-1}(x^k - 1) + 2(x^2 + x + 1)(2x + 1) \end{aligned}$$

Letting $k = 3s$, where $s \geq 1$, and replacing k in the exponents *with* 3*, we have*

$$\begin{aligned} \frac{f(x)}{dx} &\rightarrow (3s - 12)x^8 \\ \frac{g(x)}{dx} &\rightarrow 2sx^2(x^3 - 1) + 2(x^2 + x + 1)(2x + 1). \end{aligned}$$

By resultant computation, we can show that $f(x), g(x)$ are divided by $(x^2 + x + 1)^2$ only for $k = 12$, where $s = 4$.

Non Cyclotomic Case. For each non cyclotomic factor $m_i(x)(\neq x)$, we have

PROPOSITION 3. *There is a positive integer B_i such that* $\gcd(f'(x), g'(x), m_i(x)^{e_i})$ *is trivial for every $k > B_i$. Moreover, B_i can be computed by $f(x), g(x)$ and $m_i(x)$.*

PROOF. Suppose that $m_i(x)(\neq x)$ is a non cyclotomic factor. For bivariate polynomials $f_1(x, y), g_1(x, y)$ obtained by replacing x^k with y, we set

$$\begin{aligned} F(y) &= \mathrm{res}_x(f_1(x, y), m_i(x)) \\ G(y) &= \mathrm{res}_x(g_1(x, y), m_i(x)). \end{aligned}$$

Then, at least $F(y) \neq 0$ or $G(y) \neq 0$ holds. Because, if $F(y) = G(y) = 0$, then $m_i(x)$ must divide both of $f_1(x, y)$ and $g_1(x, y)$. But, as assumption, $f_1(x, y)$ and $g_1(x, y)$ have no common factor, this is a contradiction. So, without loss of generality, we can assume that $F(y) \neq 0$. Moreover, as $m(x)$ is the generator of $\langle f_1(x, y), g_1(x, y)\rangle \cap \mathbf{Q}[x]$, we can show that $F(y)$ is not a non zero constant. (Otherwise, m/m_i belongs to $\langle f_1(x, y), g_1(x, y)\rangle \cap \mathbf{Q}[x]$.)

Suppose that $m_i(x)$ divides $f'(x)$ for some value of k. Then, by property of the resultant, we can show that for any root α of $m_i(x)$, α^k must be a root of $F(y)$. Now we fix a root α of $m_i(x)$.

On the other hand, as $F(y)$ is an ordinary univariate polynomial in y over $\mathbf{Q}$, we can set U and L as the maximal absolute value of roots of $F(y)$ and the minimum absolute value of non zero roots of $F(y)$. Then, if $|\alpha| > 1$, it follows $|\alpha^k| = |\alpha|^k \leq U$ and we obtain $k \leq log_{|\alpha|}(U)$. If $|\alpha| < 1$, it follows $|\alpha^k| = |\alpha|^k \geq L$ and we obtain $k \leq log_{1/|\alpha|}(1/L)$. Thus, letting B_i be $log_{|\alpha|}(U)$ or $log_{1/|\alpha|}(1/L)$, $m_i(x)$ does not divide $f'(x)$ for any $k > B_i$. In this case, $\gcd(f'(x), g'(x), m_i(x))$ becomes trivial. Moreover, the above B_i can be computed exactly by numerical computation of approximate value of roots of $m_i(x)$ with rigorous error analysis. See [6, 7] for exact methods and rigorous error analysis. $\square$

Now we give a concrete procedure.

PROCEDURE [NON CYCLOTOMIC CASE]

1. Compute a root α of $m_i(x)$ with rigorous error analysis and compute a correct bound A on $|\alpha|$ so that
   - $|\alpha| > A > 1$ if $|\alpha| > 1$, and
   - $|\alpha| < A < 1$ if $|\alpha| < 1$.
2. Compute $F(y), G(y)$ by
$$\begin{aligned} F(y) &= \mathrm{res}_x(f_1(x, y), m_i(x)) \\ G(y) &= \mathrm{res}_x(g_1(x, y), m_i(x)). \end{aligned}$$
3. If $F(y) \neq 0$, then compute a bound D on the absolute value of roots of $F(y)$ so that
   - $D > |\beta|$ for any root β of $F(y)$ if $|\alpha| > 1$, and
   - $0 < D < |\beta|$ for any non-zero root β of $F(y)$ if $|\alpha| < 1$.

   If $F(y) = 0$, then compute a bound D on the absolute value of roots of $G(y)$ so that
   - $D > |\beta|$ for any root β of $G(y)$ if $|\alpha| > 1$, and
   - $0 < D < |\beta|$ for any non-zero root β of $G(y)$ if $|\alpha| < 1$.
4. Compute the smallest positive integer B_i such that
   - if $|\alpha| > 1$, $A^{B_i} > D$, and
   - if $|\alpha| < 1$, $A^{B_i} < D$.

   Then, $\gcd(f'(x), g'(x), m_i(x))$ is trivial if $k > B_i$.
5. Substituting $1, ..., N_i$ for k, compute
$$\gcd(f'(x), g'(x), m_i(x))$$
   and return them. (B_i can be updated as the largest integer $N \leq B_i$ such that $\gcd(f'(x), g'(x), m_i(x))$ is non-trivial. If it does not exist, we can set $B_i = 0$.)

REMARK 5. *For the bound on $|\alpha|^k$, we use $F(y)$. But, $F(y)$ tends to be very large, as the degree of $F(y)$ increases to the product of the y-degree of $f_1(x, y)$ and the x-degree of $m_i(x)$. Instead of $F(y)$ we can use another polynomial obtained from $f_1(x, y)$ by substituting for x an approximate value $\bar{\alpha}$ of the root α of $m_i(x)$ with rigorous error analysis. By Rouché's theorem, roots of a polynomial are continuous function in coefficients. From this theorem and precise approximation, we can estimate the absolute value of roots of $f_1(\alpha, y)$.*

EXAMPLE 9. *Consider the following polynomials:*

$$\begin{aligned} f(x) &= x^{2k} + x^{2+k} + 2x^k + 2, \\ g(x) &= x^2 + 2 \end{aligned}$$

Then, $m(x) = x^2 + 2$ is a generator of the elimination ideal and it is irreducible. The absolute value of roots of $m(x)$ is $\sqrt{2}$, and the absolute value of roots of $F(y) = (y^2 + 2)^2$ is also $\sqrt{2}$. Therefore, we have $A = \sqrt{2}$ and $U = \sqrt{2}$, by which we obtain $B = 1$. Thus, for any $k \geq 2$, $\gcd(f(x), g(x)) = 1$. For $k = 1$, as $f(x) = x^3 + x^2 + 2x + 2$, $g(x) = x^2 + 2$, we have $\gcd(f(x), g(x)) = x^2 + 2$.

Combining two cases, cyclotomic case and non-cyclotomic case, we gather bounds M_i, B_j and periods P_i. Then, letting

$$\begin{aligned} P &= \mathrm{LCM}(P_i \mid m_i \text{ is a cyclotomic factor } \} \\ B &= \max\{M_i, B_j \mid m_i \text{ is a cyclotomic factor and} \\ &\qquad m_j \text{ is a non-cyclotomic factor } \}, \end{aligned}$$

we obtain Theorem 1.

PROCEDURE [GENERAL]
(Assume that $f(0) \neq 0$ and $g(0) \neq 0$.)

1. Replacing x^k with a new variable y, compute bivariate polynomials f_0, g_0 from f, g.

2. In the polynomial ring $\mathbf{Q}[x, y]$, compute the elimination ideal $\mathcal{J} = \langle f_0(x, y), g_0(x, y)\rangle \cap \mathbf{Q}[x]$.

3. *(General Form Factor)* If $\mathcal{J} = \{0\}$, we compute
$$h_0 = \gcd(f_0(x, y), g_0(x, y))$$
and $h(x) = h_0(x, x^k)$, which is a common divisor of $f(x), g(x)$. To check if other common divisors exist or not, we return the top and apply f/h and g/h.

4. *(Finite Form Factor)* If $\mathcal{J} \neq \{0\}$, we compute its generator $m(x)$ by eliminating the variable y. (Then $m(x)$ is an ordinary polynomial.)

5. Factorize $m(x) = \prod_{i=1}^{r} m_i(x)^{e_i}$. Then divide the irreducible factors $m_i(x)$ into factors of cyclotomic polynomials and others. (We exclude x from factors.)

6. For each factor $m_i(x)$, execute the following:

   6.1. For each cyclotomic factor $m_i(x)$, compute $\gcd(f(x), g(x), m_i(x)^{e_i})$ by PROCEDURE CYCLOTOMIC CASE.

   6.2. For each non-cyclotomic factor $m_i(x)$, compute $\gcd(f(x), g(x), m_i(x)^{e_i})$ by PROCEDURE NON CYCLOTOMIC CASE.

7. Unify all obtained informations and return the final result.

4. 0-DIMENSIONAL CASE

Here we consider another simple and easy case, where arguments used for univariate case can be applied directly.

Assumption. Suppose that an ep-ideal $\mathcal{I}$ generated by $\mathcal{F} = \{f_1, \dots, f_r\}$ in $\mathbf{Q}[x_1, \dots, x_n]$ satisfies the following:

1. There is a unique essential set $\{x_1^k\}$ with single parameter k.

2. The finite subideal $\mathcal{J}$ of $\mathcal{I}$ obtained by SLACK VARIABLE AND ELIMINATION is 0-dimensional.

In this case, we have a procedure for computing the Gröbner bases of "components" of $\mathcal{I}$ similar to procedures in the previous section.

Here we give an outline of a concrete procedure for the radical $\sqrt{\mathcal{I}}$. For simplicity, we assume that each $f_i(X, y)$ is primitive as a univariate polynomial in y over $\mathbf{Q}[X]$.

From $\mathcal{F}$, we have a set $\mathcal{F}_0 = \{f_{1,0}(X, y), \dots, f_{r,0}(X, y)\}$ such that $f_{i,0}(X, x_1^k) = f_i(X)$ for $1 \le i \le r$, and the ideal $\mathcal{I}_0$ generated by $\mathcal{F}_0$ in $\mathbf{Q}[X, y]$. Then we can compute the finite subideal $\mathcal{J} = \mathcal{I}_0 \cap \mathbf{Q}[X]$ with some fixed elimination order $X \prec\prec y$. As $\mathcal{J}$ is 0-dimensional, $\mathcal{I}$ is 0-dimensional (or trivial) for any k.

First we compute the minimal polynomial $m(x_1)$ of x_1 with respect to $\mathcal{J}$. ($m(x_1)$ is an ordinary polynomial in x_1 over $\mathbf{Q}$.) And then, we factorize $m(x_1)$ as
$$m(x_1) = \prod_{i=1}^{s} m_i(x_1)^{e_i}.$$

As $\mathcal{I}$ and $\mathcal{J}$ are 0-dimensional for each fixed value k, we have
$$\begin{aligned}
\mathcal{J} &= \cap_{i=1}^{s}(\mathcal{J} + \langle m_i^{e_i}(x_1)\rangle) \\
\sqrt{\mathcal{J}} &= \cap_{i=1}^{s}(\sqrt{\mathcal{J}} + \langle m_i(x_1)\rangle) \\
\mathcal{I} &= \cap_{i=1}^{s}(\mathcal{I} + \langle m_i^{e_i}(x_1)\rangle) \\
\sqrt{\mathcal{I}} &= \cap_{i=1}^{s}(\sqrt{\mathcal{I}} + \langle m_i(x_1)\rangle),
\end{aligned}$$
as ordinary polynomial ideals. (See [3, 8].) Samely as univariate case, we divide the factors of $m(x_1)$ into cyclotomic factors and non-cyclotomic factors. Here, we exclude x from factors. If x is a factor of $m(x)$, we compute the Gröbner basis of the ideal $\langle f_1(0, x_2, \dots, x_n), \dots, f_r(0, x_2, \dots, x_n)\rangle$ in $\mathbf{Q}[x_2, \dots, x_n]$ as an ordinary ideal.

Cyclotomic Case. If m_i is a cyclotomic factor of period P_i, then $x_1^k \equiv x_1^a \pmod{\sqrt{\mathcal{I}} + \langle m_i(x_1)\rangle}$ if $k \equiv a \pmod{P_i}$,

Then, the Gröbner basis of $\sqrt{\mathcal{I}} + \langle m_i(x_1)\rangle$ is determined uniquely by the value $k \bmod P_i$. Replacing k with each value a in $\{0, 1, 2, \dots, P_i - 1\}$, $\mathcal{I}$ becomes an ordinary ideal and we can compute the Gröbner basis of $\sqrt{\mathcal{I}} + \langle m_i(x_1)\rangle$.

Non-Cyclotomic Case. If $m_i(x_1)(\neq x_1)$ is a non cyclotomic factor, we can use similar method as in univariate case by using *minimal polynomial computation* instead of *resultant computation*.

To do so, we need to compute a root α of $m_i(x_1)$ with rigorous error analysis and compute a correct bound A on $|\alpha|$ such that

- $|\alpha| > A > 1$ if $|\alpha| > 1$, and
- $|\alpha| < A < 1$ if $|\alpha| < 1$.

Moreover, we also need to compute the prime decomposition of $\mathcal{J}$ as an ordinary ideal:
$$\sqrt{\mathcal{J}} + \langle m_i(x)\rangle = \cap_{j=1}^{t_i}\mathcal{J}_{i,j}.$$
Then, we have
$$\sqrt{\mathcal{I}} + \langle m_i(x)\rangle = \cap_{j=1}^{t_i}(\sqrt{\mathcal{I}} + \mathcal{J}_{i,j}).$$
For each prime divisor $\mathcal{J}_{i,j}$, we execute the following:

1. For each $f_{j,0} \in \mathcal{F}_0$, execute the following:

   1.1. Consider the ideal $\mathcal{H}_\ell$ in $\mathbf{Q}[X, y]$ generated by $\mathcal{J}_{i,j}$ and $f_{\ell,0}$. As $\mathcal{J}_{i,j}$ is a maximal ideal in $\mathbf{Q}[X]$, $f_{\ell,0}$ can be considered as a univariate polynomial over an extension field $\mathbf{Q}[X]/\mathcal{J}_{i,j}$.
   If $f_{\ell,0}$ is the zero polynomial over $\mathbf{Q}[X]/\mathcal{J}_{i,j}$, return to the top.
   Otherwise, $\mathcal{H}_\ell$ become a 0-dimensional ideal in $\mathbf{Q}[X, y]$. Then, compute the minimal polynomial $F_\ell(y)$ of y with respect to $\mathcal{H}_\ell$.

   1.2. Compute a bound B_ℓ on the absolute value of roots of $F_\ell(y)$ so that
   - $D > |\beta|$ for any root β of $F_\ell(y)$ if $|\alpha| > 1$, and
   - $0 < D < |\beta|$ for any non-zero root β of $F_\ell(y)$ if $|\alpha| < 1$.

2. If all $f_{\ell,0}$ are zero polynomials over $\mathbf{Q}[X]/\mathcal{J}_{i,j}$, return the Gröbner basis of $\mathcal{J}_{i,j}$ as a completely stable one.

3. Otherwise, compute the smallest positive integer $B_{i,j}$ such that
   • if $|\alpha| > 1$, $A^{B_{i,j}} > min\{D_1, \ldots, D_r\}$
   • if $|\alpha| < 1$, $A^{B_{i,j}} < max\{D_1, \ldots, D_r\}$,
   where we omit undefined D_j's. Then, the ideal $\sqrt{\mathcal{I}} + \mathcal{J}_{i,j}$ is trivial if $k > B_{i,j}$.

4. Substituting $1, ..., B_{i,j}$ for k, compute the Gröbner basis of $\sqrt{\mathcal{I}} + \mathcal{J}_{i,j}$ and return them.

Thus, we have the following.

THEOREM 2. *Under* ASSUMPTION, $\sqrt{\mathcal{I}}$ *has a semi-stable Gröbner basis.*

5. CONCLUDING REMARKS

In this paper we give some basic notions on stability of Gröbner bases of ideals with parametric exponents, and provide concrete procedures for computing the Gröbner bases in the simplest cases, univariate case and the 0-dimensional case with unique essential set and unique parameter. However, for the proposed procedures, neither analysis on the efficiency nor actual implementation is not examined, Thus, in the next step, we will give a more precise procedure and examine its efficiency/ability by complexity analysis and experiments on real computer. As the problem seems very hard in general settings, it is very important to go further *stepwise*. In the below, we list our next steps for further development.

1. Find efficient/effective criteria for stability and computability of Gröbner bases.

2. The notion "stability" is derived from studies on univariate case and 0-dimensional case. Refine the notion more rigorously by considering wider classes of polynomial ideals having certain stability or the computability of Gröbner basis.

3. For special cases like as ideals in fewer variables (bivariate, trivariate), find efficient/effective criteria for stability and computability of Gröbner basis. Also it is very interesting to examine the effectivity of SLACK VARIABLES AND ELIMINATION for special cases where the number of generating polynomials exceeds the sum $\#\mathcal{T} + \#\mathcal{X}$, where $\mathcal{T}$ is an essential set and $\mathcal{X}$ is the set of variables.

4. Apply developed methods to actual problems arising from mathematics and engineering. As those problems tend to have parametric coefficients like as Takahashi's problem, we have to deal with systems with parametric coefficients and parametric exponents at the same time. To solve such complicated problems, extending/improving the technique of comprehensive Gröbner basis seems indispensable.

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