INTRINSIC SYMMETRIES AND OTHER CONSTRAINTS IN GENERAL RELATIVITY

by

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A thesis
presented to the University of Waterloo
in partial fulfillment of the
requirements for the degree of
Master of Mathematics
in
Applied Mathematics

Waterloo, Ontario, 1981

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ABSTRACT

In this thesis we give a method of simplifying the Einstein field equations with restrictions of a geometric nature. We shall be concerned mainly with the cosmological applications of this approach but the techniques may be used to advantage in other areas. First a method of systematically imposing constraints to specialize a class of space-times is described. Then the method of "intrinsic symmetries", in which restrictions are placed on submanifolds of a space-time, is introduced and it is shown how the second fundamental form and the intrinsic Ricci tensor for one or more families of surfaces may be used to give an algebraic classification of solutions. We use these methods to investigate a class of locally rotationally symmetric space-times, examining intrinsic symmetries of geometrically well defined families of surfaces and investigating a restriction that the equation of state be of a particular general form. Finally, the general procedure for examining classes of solutions is discussed from the point of view of computer programming.

ACKNOWLEDGEMENTS

I wish to thank my supervisor, C.B. Collins, for his guidance and assistance throughout the course of this work. Many of the calculations in this thesis were performed using the symbolic computation system MACSYMA. I would like to thank H. Warren for typing the manuscript.

This work was supported by a Natural Sciences and Engineering Research Council 1967 Science Scholarship. MACSYMA is a product of the Mathlab Group at the MIT Laboratory for Computer Science and is supported, in part, by the United States Energy Research and Development Administration under contract number E(11-1)-3070 and by the National Aeronautics and Space Administration under grant NSG 1323.

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NOTATION AND CONVENTIONS

Notation

$:=$	definition
$\mathbf{T}$	component-free notation for a geometric object
$\langle \omega, \mathbf{v} \rangle$	scalar product of one-form and vector: $\omega_a v^a$
$\mathbf{u} \cdot \mathbf{v}$	inner product with respect to metric: $u^a v^b g_{ab}$
$\mathbf{u} \otimes \mathbf{v}$	tensor product
$[\mathbf{u}, \mathbf{v}]$	commutator: $\nabla_{\mathbf{u}} \mathbf{v} - \nabla_{\mathbf{v}} \mathbf{u}$
$_{;k}$	covariant derivative
$_{;k} v^k$ or $\nabla_{\mathbf{v}}$	covariant derivative along a vector — $\nabla_a := \nabla_{\mathbf{e}_a}$
$_{,k}$	partial derivative
$_{,k} v^k$ or $\partial_{\mathbf{v}}$	partial derivative along a vector — $\partial_a := \partial_{\mathbf{e}_a}$
$\mathscr{L}_{\mathbf{v}} \mathbf{T}$	Lie derivative
$\mathscr{F}_m(V)$	m-parameter family of surfaces generated by the set of vector fields V
$_{a \ldots b}\mathscr{S}$	member of the family $\mathscr{F}_m(\{\mathbf{e}_a, \ldots, \mathbf{e}_b\})$
$\mathbf{R}(\mathscr{S})$	Ricci tensor intrinscially defined in $\mathscr{S}$
$\theta(\mathscr{S}, \mathbf{n})$	expansion tensor for the surface $\mathscr{S}$ with respect to normal vector field $\mathbf{n}$.
$_{a \ldots b}R$	$\mathbf{R}({}_{a \ldots b}\mathscr{S})$
$_{(c)a \ldots b}\theta$	$\theta({}_{a \ldots b}\mathscr{S}, \mathbf{e}_c)$

Conventions

indices:	Latin 0–3, Greek 1–3 (except section 3.2) $a, b, c \ldots \alpha, \beta, \gamma \ldots$ tetrad $i, j, k \ldots \lambda, \mu, \nu \ldots$ coordinate
units:	$8\pi G = c = 1$
signature:	$+2 \quad (-+++)$
Riemann tensor:	$v^a{}_{;dc} - v^a{}_{;cd} = R^a{}_{bcd} v^b$
Ricci tensor:	$R_{ab} = R^c{}_{acb}$

Equation Labels

In several calculations mnemonic equation labels have been used. Derived equations are labelled with the original equation label followed by a comma, then a code which indicates the sequence of calculations when read from left to right. The code may be interpreted as follows:

0 or 1	–	differentiation along $\mathbf{e}_0$ or $\mathbf{e}_1$
*	–	application of the $[\mathbf{e}_0, \mathbf{e}_1]$ commutator
A, B, C	–	possible consiquences, one of which must hold
a, b	–	consequences, both of which must hold (usually from splitting an equation).

For example, (I1,1a*) is obtained from (I1) by differentiating along $\mathbf{e}_1$, splitting the equation and applying the commutator.

Chapter I

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

The dominant long range force in the universe is gravitation. Therefore, any reasonable cosmological model must take it into account. The most widely accepted theory of gravitation is Einstein's general theory of relativity, which we shall employ throughout this thesis.

General relativity is a mathematical theory in which space-time is viewed as a 4-dimensional manifold, $\mathcal{M}$, with Lorenzian metric $\mathbf{g}$. Gravitation is described by the field equations,

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = \frac{8\pi G}{c^2}T_{ij} \tag{1.1.1}$$

which are ten coupled non-linear partial differential equations, second order in the metric components. The left-hand side of (1.1.1) describes the geometry of the manifold and the right-hand side is the stress-energy tensor of the source of the gravitational field. The quantities R_{ij} and $R := R^i{}_i$ are the Ricci tensor and Ricci scalar respectively, and Λ is the cosmological constant. In general the field equations are under-determined so we must also specify an equation of state. Depending on the nature of the matter modelled, there may be additional relations between the components of $\mathbf{T}$. For instance, if there is an electromagnetic field present, then these additional relations would include Maxwell's equations.

We shall choose units in which $c = 8\pi G = 1$; then a mass of 10^{26}Kg corresponds roughly to a length of 1m $\left(\frac{c^2}{8\pi G} = 5.4 \times 10^{25}\,\frac{\mathrm{Kg}}{\mathrm{m}}\right)$. Observations of distant galaxies show that the magnitude of Λ is less than $10^{-52}\mathrm{m}^{-2}$ (see Hawking & Ellis [1973a], p. 73) and many authors choose $\Lambda = 0$.

Because the field equations are quite complicated, their general solution is not known. However many specific solutions and classes of solutions have been found. Different formalisms have been developed to aid the study of these equations. These include the orthonormal tetrad formalism (see Ellis & MacCallum [1969a] or MacCallum [1973c]) and the Newman-Pensrose null tetrad formalism (Newman & Penrose [1962]).

Usually, simplifying assumptions are made to render the field equations more manageable. The traditional approaches include demanding that the space-time be invariant under a group of isometries or that the Weyl tensor have a special canonical form.

In this thesis we give an alternative method of simplifying the field equations with restrictions of a geometric nature. We shall be concerned mainly in the cosmological applications of this approach, although the techniques may be used to advantage in other areas.

In this chapter preliminary material is covered before proceeding on to the main body of the thesis. Section 1.2 describes the orthonormal tetrad formalism wich shall be used extensively. Section 1.3 introduces the class of space-times which exhibit "local rotational symmetry", singling out a particular case as the most interesting. In section 1.4, we discuss some of the features of the symbolic mathematical computation system MACSYMA.

Chapters 2 and 3 develop tools which are employed in the subsequent material. In Chapter 2, a method is given to check systematically the consistency of a given constraint for a class of space-times. This gives a technique for specializing a class to contain only those solutions which satisfy the constraint. In Chapter 3, the method of "intrinsic symmetries" is introduced. We then present a scheme to characterize families of non-null submanifolds using this approach.

Chapters 4 and 5 use these tools to investigate the most interesting class of locally rotationally symmetric space-times. In Chapter 4, we impose intrinsic symmetries on subspaces orthogonal to the fluid flow in the space-times and obtain a hierarchy of sub-classes. In Chapter 5 we give some further results for this class of space-times and investigate a general equation of state. We also examine other intrinsic symmetries because of their interesting mathematical properties.

In Chapter 6 we formalize the method of specializing classes of space-times with a view to automating the procedure.

Finally, Chapter 7 contains some concluding remarks. Throughout the thesis we refer to appendices when the inclusion of material in the text would divert the development of the main ideas.

1.2 The Orthonormal Tetrad Formalism

This section is devoted to describing the orthonormal tetrad formalism as used by MacCallum [1973c]. The advantage of this formalism is that it reduces the field equations to first order. The trade-off is that new variables are introduced for which additional relations must be satisfied. First, the geometrical aspects of the formalism will be described and then a specialization for space-times with the matter modelled by a fluid will be discussed.

1) *Geometry*

An orthonormal tetrad at a point, p, in the space-time manifold, $\mathscr{M}$, is a basis of vectors $\{\mathbf{e}_a\}$ in the tangent vector space, T_p, such that

$$\mathbf{e}_a \cdot \mathbf{e}_b = \mathrm{diag}(-1, 1, 1, 1).$$

Tetrad indices will be chosen from the first letters of the alphabets $(a, b, c, \ldots$ or $\alpha, \beta, \gamma, \ldots)$ and coordinate indices will be chosen from the other letters $(i, j, k, \ldots$ or $\lambda, \mu, \nu, \ldots)$.

The bundle of orthonormal frames over an open set $\mathscr{U} \subset \mathscr{M}$ is denoted by

$$\mathscr{O}(\mathscr{M}) := (\mathscr{E}, \mathscr{U}, \pi),$$

where the total space, $\mathscr{E}$, consists of all orthonormal tetrads at all points of $\mathscr{U}$ and the projection is

$$\pi : \{\mathbf{e}_a\}_p \mapsto p.$$

A smooth cross-section of $\mathscr{O}(\mathscr{M})$ is a smooth choice of an orthonormal tetrad at each point in $\mathscr{U}$. We shall usually refer to such a smooth orthonormal tetrad field simply as an "orthonormal tetrad" or, more briefly, as a "tetrad". Similarly, an "orthonormal triad" for an open set in a hypersurface $\mathscr{S} \subset \mathscr{M}$ is defined in terms of bases in the tangent vector space of $\mathscr{S}$. At this point we stress that we only require that the orthonormal tetrad be defined on an open set and that all results are local.

In an open set $\mathscr{U} \subset \mathscr{M}$ with coordinates $\{x^i\}$, the components of any geometric object may be given with respect to an orthonormal tetrad $\{\mathbf{e}_a\}$ and dual basis of forms $\{\mathbf{e}^a\}$ or with respect to the local coordinate bases $\left\{\frac{\partial}{\partial x^i}\right\}$ and $\{\mathrm{d}x^i\}$

$$e.g. \qquad \mathbf{T} = T^i{}_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j = T^a{}_b \, \mathbf{e}_a \otimes \mathbf{e}^b$$

Since $\mathbf{e}_a = e_a{}^i \frac{\partial}{\partial x^i}$, the relation between the tetrad and coordinate components is given by

$$T^i{}_j = T^a{}_b \, e_a{}^i e^b{}_j.$$

In particular, the metric components are

$$\begin{aligned} g_{ab} &= e_a{}^i e_b{}^j g_{ij} = \mathrm{diag}(-1, 1, 1, 1) \\ g_a{}^b &= \delta_a{}^b = e_a{}^i e^b{}_i \\ g^{ab} &= \mathrm{diag}(-1, 1, 1, 1). \end{aligned}$$

The directional derivative along $\mathbf{e}_a$ of a function ϕ is given by

$$\partial_a \phi := \mathbf{e}_a(\phi) = \phi_{,i} e_a{}^i.$$

In general, these directional derivative operators do not commute. The commutator of two basis vectors is given by

$$\begin{aligned} [\mathbf{e}_a, \mathbf{e}_b] &= \left(e_a{}^j e_b{}^i{}_{,j} - e_b{}^j e_a{}^i{}_{,j} \right) \frac{\partial}{\partial x^i} \\ &= \gamma^c{}_{ab} \mathbf{e}_c. \end{aligned} \tag{1.2.1}$$

This relation defines the quantities $\gamma^c{}_{ab}$, which are called the "objects of anholonomity" or "commutation functions". The Jacobi identity,

$$[[\mathbf{e}_a, \mathbf{e}_b], \mathbf{e}_c] + [[\mathbf{e}_c, \mathbf{e}_a], \mathbf{e}_b] + [[\mathbf{e}_b, \mathbf{e}_c], \mathbf{e}_a] = 0,$$

imposes on the $\gamma^c{}_{ab}$ the condition

$$\partial_{[d} \gamma^a{}_{cb]} - \gamma^a{}_{f[d} \gamma^f{}_{cb]} = 0, \qquad \text{labelled} \begin{pmatrix} a \\ bcd \end{pmatrix}$$

which, by contraction, implies

$$\partial_a \gamma^a{}_{bc} + \partial_c \gamma^a{}_{ab} - \partial_b \gamma^a{}_{ac} + \gamma^a{}_{af} \gamma^f{}_{bc} = 0.$$

The quantity $\gamma^\alpha{}_{\beta\gamma}$ may be decomposed into a symmetric matrix $n_{\alpha\beta}$ and a triple a_β:

$$\left.\begin{aligned} n^{\alpha\beta} &:= \tfrac{1}{2} \gamma^{(\alpha}{}_{\gamma\delta} \epsilon^{\beta)\gamma\delta}, \qquad a_\beta = \tfrac{1}{2} \gamma^\alpha{}_{\beta\alpha} \\ \Leftrightarrow \gamma^\alpha{}_{\beta\gamma} &= \epsilon_{\beta\gamma\delta} n^{\alpha\delta} - 2\delta^\alpha{}_{[\beta} a_{\gamma]}. \end{aligned}\right\} \tag{1.2.2}$$

The Ricci rotation coefficients are the tetrad components of the connection:

$$\Gamma_{abc} := \mathbf{e}_a \cdot \nabla_b \mathbf{e}_c \tag{1.2.3}$$

and $g_{ab;c} = 0$ implies

$$\Gamma_{abc} + \Gamma_{cba} = 0.$$

Using the definitions (1.2.1) and (1.2.3) we obtain the relations

$$\begin{aligned} \gamma^c{}_{ab} &= \langle \mathbf{e}^c, \nabla_a \mathbf{e}_b - \nabla_b \mathbf{e}_a \rangle = \Gamma^c{}_{ab} - \Gamma^c{}_{ba} \\ \Leftrightarrow \Gamma_{abc} &= \frac{1}{2} \left(\gamma_{abc} + \gamma_{cab} - \gamma_{bca} \right). \end{aligned}$$

2) *Tetrad with a Fluid*

For space-times where the matter content is modelled by a fluid, it is possible to specialize the tetrad such that certain $\gamma^c{}_{ab}$ are given by the kinematic quantities related to the fluid's motion. Let $\mathbf{u}$ denote the normalized 4-velocity field of the fluid. Then

$$h_{ab} = g_{ab} + u_a u_b$$

is the projection into the instantaneous rest space of the fluid. A dot will be used to denote intrinsic differentiation along the integral curves of $\mathbf{u}$. For example, the "4-acceleration" of the fluid is

$$\dot{\mathbf{u}} = \nabla_{\mathbf{u}} \mathbf{u}. \tag{1.2.4}$$

In order to describe the kinematics we now consider relative motion within the fluid following Ellis [1971]. Let O and G be two nearby world lines of fulid elements and let the vector $\mathbf{X}$ give the separation of G from O at each instant. Straightforward calculation in co-moving coordinates shows

$$\begin{aligned} \dot{\mathbf{X}} &:= \nabla_{\mathbf{u}} \mathbf{X} = \nabla_{\mathbf{X}} \mathbf{u} \qquad (1.2.5) \\ \Rightarrow \mathscr{L}_{\mathbf{u}} \mathbf{X} &= 0. \end{aligned}$$

The relative position of G with respect to O in the rest space is then

$$X_\perp{}^a := h^a{}_b X^b$$

and the relative velocity of G with respect to O in the rest space is

$$V^b := (X_\perp{}^a)^{\cdot}\, h_a{}^b = \left(h^{ac} h_b{}^d u_{c;d}\right) X_\perp{}^b,$$

using (1.2.5). Thus the relative velocity vector is a linear transformation of the relative position vector. This transformation may be decomposed into its symmetric and anti-symmetric parts:

$$h_a{}^c h_b{}^d u_{c;\,d} = \theta_{ab} + \omega_{ab}. \tag{1.2.6}$$

Here $\theta_{ab} = \theta_{(ab)}$ is the "expansion tensor" and $\omega_{ab} = \omega_{[ab]}$ is the "vorticity tensor". The expansion tensor may be further decomposed into its "isotropic" (trace) and "anisotropic" (trace-free) parts:

$$\theta_{ab} = \sigma_{ab} + \frac{1}{3}\theta h_{ab}. \tag{1.2.7}$$

The trace part $\theta := \theta^a{}_a$ is the "expansion scalar" and the trace-free part $\sigma_{ab} := \theta_{ab} - \frac{1}{3}\theta h_{ab}$ is the "shear tensor". The acceleration, expansion, shear and vorticity are known as the kinematic quantities of the congruence of integral curves of $\mathbf{u}$. Using (1.2.4), (1.2.6) and (1.2.7) the covariant derivatives of $\mathbf{u}$ are now given by

$$u_{a;\,b} = \sigma_{ab} + \frac{1}{3}\theta h_{ab} + \omega_{ab} - \dot{u}_a u_b. \tag{1.2.8}$$

In what follows, we shall use another quantity, the "vorticity vector", defined in terms of ω_{ab} by

$$\omega^a := \frac{1}{2}\eta^{abcd}u_b\omega_{cd}.$$

Now we discuss the effect of the tetrad specialization $\mathbf{e}_0 = \mathbf{u}$. First, note that with this choice the Fermi derivatives are antisymmetric,

$$-\mathbf{e}_\beta \cdot \dot{\mathbf{e}}_\alpha = \mathbf{e}_\alpha \cdot \dot{\mathbf{e}}_\beta = \mathbf{e}_\alpha \cdot \nabla_{\mathbf{u}}\mathbf{e}_\beta = \Gamma_{\alpha 0 \beta},$$

so we may define

$$\Omega^a := \frac{1}{2}\eta^{abcd}u_b\dot{\mathbf{e}}_c \cdot \mathbf{e}_d.$$

This is the angular velocity of the triad $\{\mathbf{e}_\alpha\}$ with respect to a Fermi propagated triad (*c.f.* Ellis & MacCallum [1969a]). Now, using (1.2.8), the $\gamma^c{}_{ab}$ with at least one of a, b, c equal to zero can be given in terms of $\boldsymbol{\Omega}$ and the kinematic quantities of $\mathbf{u}$:

$$\left.\begin{aligned}
\gamma_{00\alpha} &= -\dot{u}_\alpha \\
\gamma_{0\alpha\beta} &= 2\epsilon_{\alpha\beta\gamma}\omega^\gamma \\
\gamma_{\alpha 0\beta} &= -\epsilon_{\alpha\beta\gamma}\left(\Omega^\gamma + \omega^\gamma\right) - \theta_{\alpha\beta}.
\end{aligned}\right\} \tag{1.2.9}$$

Using (1.2.2) and (1.2.9) the commutators can be written out in full as

$$\left.\begin{aligned}
[\mathbf{e}_0, \mathbf{e}_1] &= \dot{u}^1\mathbf{e}_0 - \theta_{11}\mathbf{e}_1 - (\theta_{12} - \omega_3 - \Omega_3)\,\mathbf{e}_2 - (\theta_{13} + \omega_2 + \Omega_2)\,\mathbf{e}_3 \\
[\mathbf{e}_0, \mathbf{e}_2] &= \dot{u}^2\mathbf{e}_0 - (\theta_{21} + \omega_3 + \Omega_3)\,\mathbf{e}_1 - \theta_{22}\mathbf{e}_2 - (\theta_{23} - \omega_1 - \Omega_1)\,\mathbf{e}_3 \\
[\mathbf{e}_0, \mathbf{e}_3] &= \dot{u}^3\mathbf{e}_0 - (\theta_{31} - \omega_2 - \Omega_2)\,\mathbf{e}_1 - (\theta_{32} + \omega_1 + \Omega_1)\,\mathbf{e}_2 - \theta_{33}\mathbf{e}_3 \\
[\mathbf{e}_2, \mathbf{e}_3] &= -2\omega_1\mathbf{e}_0 + n_{11}\mathbf{e}_1 + (n_{12} - a_3)\,\mathbf{e}_2 + (n_{13} + a_2)\,\mathbf{e}_3 \\
[\mathbf{e}_3, \mathbf{e}_1] &= -2\omega_2\mathbf{e}_0 + (n_{21} + a_3)\,\mathbf{e}_1 + n_{22}\mathbf{e}_2 + (n_{23} - a_1)\,\mathbf{e}_3 \\
[\mathbf{e}_1, \mathbf{e}_2] &= -2\omega_3\mathbf{e}_0 + (n_{31} - a_2)\,\mathbf{e}_1 + (n_{32} + a_1)\,\mathbf{e}_2 + n_{33}\mathbf{e}_3
\end{aligned}\right\} \tag{1.2.10}$$

In the next section the remaining tetrad freedom will be used to further simplify these relations.

1.3 Local Rotational Symmetry

This section describes the locally rotationally symetric space-times with a charged perfect fluid and electromagnetic field, in which case Maxwell's equations are

$$\begin{aligned} F^{ab}{}_{;b} &= \epsilon u^a \\ F_{[ab;c]} &= 0. \end{aligned}$$

These space-times have been studied by Ellis [1967], in the case of dust, and by Stewart and Ellis [1968], for fluid with electromagnetic field. Most of the material in this section relies on these references. Except where stated otherwise, the results of this section hold regardless of whether the charge density, ϵ, or the electromagnetic field, F_{ab}, are zero or non-zero.

We use the following:

Definition. *A space-time is said to be "locally rotationally symmetric" (LRS) in the neighborhood U of a point p_0 if at each point p in U there exists a nondiscrete subgroup, g, of the Lorentz group in the tangent space, T_p, which leaves invariant the fluid flow vector, the curvature tensor and their derivatives up to third order.*

Since $\mathbf{u}$ is invariant under g, we have that g is a group of rotations in the subspace of T_p orghogonal to $\mathbf{u}$ and so is either one- or three- dimensional. For a perfect fluid with electromagnetic field, the stress-energy tensor is given by

$$T_{ab} = \mu u_a u_b + p h_{ab} + \tau_{ab}$$

where μ is the energy dentisty and p is the pressure of the fluid in the rest space, and

$$\tau_{ab} = \frac{1}{4} g_{ab} F_{cd} F^{cd} - F_{ac} F_b{}^c$$

is the stress-energy of the electromagnetic field. Since μ, p and τ_{ab} are defined uniquely by u_a and R_{ab}, they and their derivatives up to third order are also invariant under g.

If g is three-dimensional, then assuming continuity of the group dimension, the geometry is spatially isotropic for observers with velocity $\mathbf{u}$, and so we have a Friedmann-Robertson-Walker (FRW) space-time. In this case, g has a one-dimensional subgroup so these models are included in the discussion, where g is one-dimensional, as special instances.

When g is one-dimensional we shall choose a tetrad, $\{\mathbf{e}_a\}$, in which $\mathbf{e}_0 = \mathbf{u}$ and the group g consists of the four-dimensional rotations about the plane spanned by $\mathbf{e}_0$ and

$\mathbf{e}_1$ in T_p. The vector $\mathbf{e}_1$ is then a spacelike axis of symmetry lying in the fluid rest space and we have one rotational degree of freedom in the choice of $\mathbf{e}_2$ and $\mathbf{e}_3$. Then, for any covariantly defined vector $\mathbf{v}$, we must have $\mathbf{v}\cdot\mathbf{e}_2 = \mathbf{v}\cdot\mathbf{e}_3 = 0$. In particular, the derivatives in the $\mathbf{e}_2$ and $\mathbf{e}_3$ directions of any covariantly defined scalars must be zero. This also implies $\omega^a = \omega{\delta_1}^a$ and $\dot{u}^a = \dot{u}{\delta_1}^a$. For any symmetric covariantly defined rank two tensor, invariance under g implies the 22 and 33 components are equal and all off-diagonal components but 01 and 10 are zero. We may therefore write

$$\theta_{ab} = \mathrm{diag}(0, \alpha, \beta, \beta).$$

The electric and magnetic field in the rest frame of $\mathbf{u}$ are, respectively,

$$E_a := F_{ab}u^b \qquad \text{and} \qquad B_a := \frac{1}{2}{\eta_{ab}}^{cd}u^b F_{cd}.$$

Since F_{ab} is not covariantly defined, the form of E_a and B_a must be deduced through τ_{ab}, which is defined invariantly. In general E_a and B_a are invariant under g if and only if $\tau_{01} = 0$ in the tetrad[1]. We shall make the additional assumption that E_a, B_a and their derivatives are invariant under g, in which case

$$\tau_{ab} = \mathrm{diag}(\tau, -\tau, \tau, \tau) \tag{1.3.1}$$

From Stewart & Ellis [1968], the LRS space-times exhibit the following properties:

LRS Property 1. *We may use the rotational freedom to choose a frame in which*

$$\left.\begin{array}{rcl} [\mathbf{e}_0, \mathbf{e}_1] &=& \dot{u}\mathbf{e}_0 - \alpha\mathbf{e}_1 \\ [\mathbf{e}_0, \mathbf{e}_2] &=& -\beta\mathbf{e}_2 \\ [\mathbf{e}_0, \mathbf{e}_1] &=& -\beta\mathbf{e}_3 \\ [\mathbf{e}_2, \mathbf{e}_3] &=& -2\omega\mathbf{e}_0 - k\mathbf{e}_1 + s\mathbf{e}_3 \\ [\mathbf{e}_3, \mathbf{e}_1] &=& -a\mathbf{e}_3 \\ [\mathbf{e}_1, \mathbf{e}_2] &=& a\mathbf{e}_2. \end{array}\right\} \tag{1.3.2}$$

The quantities k and a may be defined in terms of the congruence of integral curves of $\mathbf{e}_1$ and so have zero derivatives in the $\mathbf{e}_2$ and $\mathbf{e}_3$ direction (except in the FRW case, in which $\mathbf{e}_1$ is not invariantly defined). Without loss of generality, we may choose the

[1] This is contrary to the claim of Stewart & Ellis [1968] that invariance of τ_{ab} under g implies (1.3.1). See Appendix B for the proof.

tetrad such that $\partial_3 s = 0$ and for later use, we define the quantity $r := \partial_2 s - s^2$, which has zero derivatives in the $\mathbf{e}_2$ and $\mathbf{e}_3$ direction. If we make the additional reasonable assumption that $\mu + p \neq 0$, then for a perfect fluid with an electromagnetic field satisfying (1.3.1), we have

LRS Property 2. *The product ωk is identically zero*

$$\omega k \equiv 0.$$

Stewart and Ellis divide these space-times into disjoint and exhaustive classes as follows

$$\begin{array}{lcl} \text{Class I} & : & \omega \neq 0, \quad k = 0 \\ \text{Class II} & : & \omega = 0, \quad k = 0 \\ \text{Class III} & : & \omega = 0, \quad k \neq 0. \end{array}$$

LRS Property 3. *For an LRS space-time with perfect fluid and delectromagnetic field, the coordinate freedom can be used to set the metric in the form*

$$\begin{aligned} \mathrm{d}s^2 \;=\; & -\tfrac{(\mathrm{d}x^0)^2}{F^2} + X^2(\mathrm{d}x^1)^2 + Y^2\left[(\mathrm{d}x^2)^2 + t^2(\mathrm{d}x^3)^2\right] \\ & + \tfrac{y}{F^2}\left[2\mathrm{d}x^0 - y\mathrm{d}x^3\right]\mathrm{d}x^3 - X^2 h\left[2\mathrm{d}x^1 - h\mathrm{d}x^3\right]\mathrm{d}x^3, \end{aligned} \tag{1.3.3}$$

where $F = F(x^0, x^1)$, $X = X(x^0, x^1)$ and $Y = Y(x^0, x^1)$. The functions $t(x^2)$, $h(x^2)$ and $y(x^2)$ are defined by

$$\begin{aligned} t_{,22} + Kt &= 0 \\ h_{,2} + 2Ct &= 0 \\ y_{,2} + 2ct &= 0, \end{aligned}$$

where K, C and c are constants. In each of the classes the metric may be specialized further as follows.

$$\begin{array}{lcl} \textit{Class I} & : & F = F(x^1), X \equiv 1, Y = Y(x^1), h \equiv 0. \\ \textit{Class II} & : & h \equiv 0, y \equiv 0. \\ \textit{Class III} & : & F \equiv 1, X = X(x^0), Y = Y(x^0), y \equiv 0. \end{array}$$

The relation between the metric and the tetrad is

$$\left.\begin{aligned}
\mathbf{e}_0 &= F\frac{\partial}{\partial x^0} \\
\mathbf{e}_1 &= \frac{1}{X}\frac{\partial}{\partial x^1} \\
\mathbf{e}_2 &= \frac{1}{Y}\frac{\partial}{\partial x^2} \\
\mathbf{e}_3 &= \frac{y}{tY}\frac{\partial}{\partial x^0} + \frac{h}{tY}\frac{\partial}{\partial x^1} + \frac{1}{tY}\frac{\partial}{\partial x^3}.
\end{aligned}\right\} \tag{1.3.4}$$

Each of the Classes I, II and III is divided into subclasses. Stewart and Ellis [1968] give the dimension of the isometry group and the orbits for each of the sublcasses. These results are summarized in figure 1.3.1. It is seen that in each case the space-times admit a multiply transitive G_r of local isometries. The result is stronger than the definition we have chosen for LRS but it could have been used as an alternative starting point.

Classes I and III are better understood than Class II because of the nature of their isometry groups. In Class I we have a timelike Killing vector so the space-times are necessarily stationary. In Class III a space-time admits an isometry group acting transitively on spacelike hypersurfaces and much attention has been given to space-times of this form. The only non-stationary solutions which do not have a group acting transitively on spacelike hypersurfaces are found in Class II, specifically subclasses IIa and IIc. In subclasses IIa and IIb the models have geodesic flow since $\dot{u} = 0$. In the LRS type IIc space-times, the acceleration may assume any non-zero value so IIc is the most general subclass of Class II. All of the space-times in Class II have the property that the surfaces $\{x^0 = \text{const}\}$ are conformally flat, since their Cotton-York tensor is identically zero.

We shall be most interested in the space-times of Class II because of the surface-forming properties of the basis vectors and the generality of the isometry group. This class includes the Einstein static solution, the Einstein-de Sitter solution, the Bondi spherically symmetric solutions, and the Kantowski-Sachs and FRW space-times and some of their generalizations. Also, if we allow $\mu + p = 0$, this class contains the Reissner-Nordstrom family of solutions. In Appendix A we give the commutators, Jacobi identities, Maxwell's equations, Einstein field equations and Bianchi identities for the Class II space-times.

Class	Class I $\omega \neq 0, k = 0$				Class II $\omega = 0, k = 0$			Class III $\omega = 0, k \neq 0$	
Subclass	Ia $a = 0 = \dot{u}$	Ib $a = 0 \neq \dot{u}$	Ic $a \neq 0 = \dot{u}$	Id $a \neq 0 \neq \dot{u}$	IIa $a = 0$	IIb $a \neq 0 = \dot{u}$	IIc $a \neq 0 \neq \dot{u}$	IIIa $\beta = 0$	IIIb $\beta \neq 0$
Group	G_5	G_4			G_3	G_4	G_3	G_5	G_4
Orbit†	space-time	$\{x^1 = C^1\}$			$\{x^0 = C^0, x^1 = C^1\}$	$\{x^1 = C^1\}$	$\{x^0 = C^0, x^1 = C^1\}$	space-time	$\{x^0 = C^0\}$

C^0 and C^1 represent arbitrary constants.

† There are higher-dimensional groups in certain specialized cases, which posess these listed groups as subgroups.

Figure 1.3.1: Isometry groups of LRS space-times

1.4 Symbolic Computation with MACSYMA

In many disciplines of science and engineering, straight-forward computations often lead to the generation of lengthy expressions. The manipulation of these expressions can be tedious and error-prone. This has led to the development of computer systems for symbolic mathematical computation. Due to the nature of many calculations in general relativity, a large number of these systems for symbolic computation have been used by relativists over the past two decades. (For a review of the use of symbolic computation in general relativity, see Fitch [1979c] and and references cited.) Some of the basic concepts used in symbolic mathematical computation are explained in Knuth ([1969c], Chapter 4) and an overview of current activities in this field is given by the proceedings of EUROSAM 1979 (edited by Ng [1979d]).

At the present time, one of the most comprehensive symbolic computation systems is MACSYMA. MACSYMA is a large, LISP-based system that was developed at MIT and first became available for use in 1971. MACSYMA has been coninuously extended since that time by the group at MIT but is currently available at only a few installations. For the computations in this thesis, we have used the system on a DEC PDP-10 at MIT, known as the MACSYMA Consortium machine (MIT-MC). The purpose of this section is to describe how MACSYMA has been used in this thesis. For a full description of MACSYMA see the reference manual (Mathlab [1977]).

MACSYMA is an interactive system; the user is prompted for input, which is evaluated and then displayed. The user is then again prompted for input and the session continues in a dialogue fasion. The input usually consists of mathematical expressions, written with a syntax similar to many other programming languages. These expressions are made up of constants, variables and functions, combined using the usual mathematical operators. It is not necessary to ascribe values to these quantities prior to the evaluation of expression containing them. MACSYMA provides many functions which may be used to manipulate such algebraic expressions. These functions may be used to differentiate, integrate, factor polynomials, solve equations and perform many other mathematical operations. There are also functions which may be used to select the form in which an expression is to be presented. This is a strong point of the MACSYMA system — the user may choose the representation of his expressions and is not forced to use a canonical form. The system makes very few assumptions regarding the simplification of expressions and usually leaves them in the same form unless a specific transformation has been requested.

The calculations in this thesis which have been done with the aid of MACSYMA were performed interactively. When a given calculation was completed the commands necessary to reproduce the results were placed in a file. There is a MACSYMA function, `BATCH`, which may be used to direct MACSYMA to take its input from a file. In this way calculations can be reproduced as needed.

MACSYMA also posesses the usual features of a programming language. It is possible to write functions which exhibit block structure and the usual control of program flow with loops and conditional execution. Using these features, it is possible for a user to write functions which perform complicated tasks. In Chapter 6 we present a set of MACSYMA functions which use these capabilities.

Chapter II

A METHOD FOR SPECIALIZING CLASSES OF SPACE-TIMES

Suppose that we are examining a class of solutions to the Einstein field equations. Using the orthonormal tetrad formalism, the field equations and the Jacobi identities form a consistent system of first order partial differential equations. The contracted Bianchi identities are then consistent with this system, by virtue of the field equations.

To form a specialization of our class, we impose one or more constraints. These constraints must be checked for consistency with the existing system and with one another. This chapter introduces a method for doing this systematically.

2.1 Checking the Consistency of a Given Constraint

For simplicity, we shall initially assume that we have a non-empty class of solutions with perfect fluid source and a constraint that may be expressed as a single equation. Later we shal remove these restrictions. We shall say that a solution in the class "admits" the constraint if the constraint is satisfied on an oben subset of the space-time manifold, $\mathscr{M}$. The constraint specializes the original class to the subclass consisting of all solutions which admit the constraint.

Let p be the pressure and μ the energy density in the fluid rest space. The Jacobi identities, field equations and Bianchi identities form a consistent system of equations. In general, there are 16 Jacobi identities, 10 field equations and 4 contracted Bianchi identities. For perfect fluid solutions, these equations may be expressed in terms of the (2) quantities p and μ, the (24) quantities Ω^α, ω^α, $\theta_{\alpha\beta}$, $\dot{u}^\alpha$, $n^{\alpha\beta}$ and a^α, introduced in Section 1.2, and all of the (26×4) directional derivatives except (13): $\partial_0 p$, $\partial_\alpha \mu$, $\partial_0 \Omega^\alpha$, $\partial_\alpha \theta_{\alpha\alpha}$ (no sum), and $\partial_0 \dot{u}^\alpha$. In the most general case we have a total of 30 equations in 26 variables and 91 of their first derivatives $(= 26 \times 4 - 13)$, and we may solve directly for up to 30 of the directional direvatives in terms of the variables and other derivatives, which shall also be considered unknowns. Note, however, that for an unspecified function, say A, the directional derivatives, $\partial_a A$, are not independent but are subject to the integrability conditions (1.2.1):

$$\partial_a \partial_b A - \partial_b \partial_a A = \gamma^c{}_{ab} \partial_c A.$$

Let the constraint be given by the function

$$C(\Omega^\alpha, \omega^\alpha, \theta_{\alpha\beta}, \dot{u}^\alpha, n^{\alpha\beta}, a^\alpha, \mu, p) = 0$$

and be satisfied at all points in the open set $\mathscr{U} \subset \mathscr{M}$. For each point in $\mathscr{U}$, $C = 0$ holds in a neighbourhood of the point so the propagation equations,

$$\partial_a C = 0,$$

must also be satisfied on $\mathscr{U}$ (that is, $C = 0$ is preserved along the intergral curves of the basis vector fields on $\mathscr{U}$). These new constraints, in turn, give rise (in a way described in detail below) to new propagation equations. At this process is repeated, the new equations may lead to a contradiction. In this case, there are no solutions in the original class which admit the constraint in the open set. Otherwise, a stage may be reached where the new equations are identically satisfied by virtue of old propagation equations, the original system and the commutators (1.2.10). Then a solution in the class will admit the constraint only if all the additional equations obtained by propagating the constraint also hold. (These may be viewed as additional “constraints”.)

We now give a general description of this procedure. At each state of the repeated propagation, we must check for the compatibility of a given equation with the original system and all previous propagation equations. First, replace in the given constraint equation all known derivatives with their values given by the original system. If the resulting equation is an identity, then it is obviously compatible. If the equation is a contradiction, then it is incompatible. In many cases, though, the equation will give neither an identity nor a contradiction directly. These cases must be investigated further.

If the equation has not been determined to be an identity or contradiction, then we proceed differently depending on whether the equation contains unknown derivatives or not. (This situation may depend on the vanishing of a coefficient; in this case the investigation should be divided into two cases, one where the coefficient is identically zero — another constraint in itself — and another where the coefficient is non-zero.) If the equation is free of derivatives, then it may be solved for one of the variables to decrease the number of unknowns in the system. If, however, there remain derivatives in the given equation, then solve for one of the previously unspecified directional derivatives. This gives a new propagation equation for one of the unknowns, which may itself be a derivative.

Not only must the equation hold, but its directional derivatives must be considered. If the equation is free of derivatives, then the propagation equations along all four basis vectors must be checked. If the equation specifies a directional derivative, then we must take into account the appropriate integrability conditions for every other known derivative for the variable. This will give up to three additional equations to be checked. These equations may involve higher order derivatives of other variables. Note, though, that an equation involving a number of derivatives of any order may be split into single derivative, first order equations by introducing “arbitrary” functions. It is not necessary to do so, but the equations may be more manageable.

In both the algebraic and directional derivative cases, the equations will be compatible only if all the new additional equations are compatible but it will be incompatible if any of the new equations is incompatible. This procedure has been described in a recursive way, with the base of the recursion being an identity or contradiction. There is as yet no guarantee that in general this procedure is finite, though for a large class of problems this seems to be the case. This is further discussed in Chapter 5. For the remainder of this chapter, we shall assume that the procedure terminates.

We shall now remove the restrictions on the matter content and form of constraint imposed at the beginning of the section.

If a perfect fluid model is inappropriate, then this procedure may be used with little modification. When a more complicated stress-energy tensor is used, more variables and derivatives appear in the field equations. Also, additional equations physically relating these new quantities must be considered. For example, if terms for heat flux are included in the stress-energy tensor, the classical heat conduction law must also be obeyed; if an electromagnetic stress-energy tensor is used, Maxwell’s equations must be satisfied. These additional equations, written in tetrad form, must be included in the original system. The constraint may then be checked for compatibility in the manner already described.

Types of constraints other than single equations are often needed in specializing classes of solutions. If the constraint desired is expressed as an inequality relation (e.g. $\mu + p \neq 0$, $\mu \geq 0$), then upon differentiation, no new constraint arises. In this case the propagation procedure is entirely unnecessary. If the constraint is specified by a number of equations, then these equations must be checked for consistency with one another as well as with the original system. This may be done systematically by taking the equations sequentially and checking them for consistency with the system using the procedure for a single constraint equation. If no contradiction has been reached, then each equation checked and the additional equations obtained by propagation should be included with the original system before proceeding to the next constraint equation. If the last constraint equation checks successfully, then the constraint given by the system of equations is compatible with the class and may be used to form a specialization. On occasion, it may also be necessary to check resulting constraint equations against inequalities. For instance, a stage may be reached where a constraint implies $\mu + p = 0$, when it has already been assumed that $\mu + p > 0$.

The next two sections give a few simple examples of this method for specializing classes of solutions.

2.2 Two Spatially Homogeneous Examples

This section gives two examples of how to specialize a class of solutions using a constraint. In the first example the constraint may be applied consistently and in the second it may not. The classes considered consist of spatially homogeneous perfect fluid solutions.

Definition. [1]

A space-time is defined to be "spatially homogenous" if it admits a G_3 *of local isometries with three-dimensional space-like orbits.*

Spatially homogeneous space-times are classified according to the nature of their isometry group using the Bianchi-Behr classification of the three dimensional Lie algebras (see, for example, Ellis & MacCallum [1969a] or Ryan & Shepley [1975]). These space-times obey the properties given below.

Property 1. *(MacCallum [1973c], pp. 107, 108): The normals to the orbit hypersurface are the tangent vector field of a geodesic, hypersurface-orthogonal congruence.*

Property 2. *(Ellis & MacCallum [1969a], pp. 113, 114): If we choose our tetrad such that* $\mathbf{e}_0$ *is orthogonal to the orbit hypersurfaces, then we have the freedom to choose a frame such that*

$$\left.\begin{array}{rcl} n_{\alpha\beta} & = & \mathrm{diag}(n_1, n_2, n_3) \\ a^{\beta} & = & (a, 0, 0), \qquad a \geq 0 \\ a n_1 & = & 0 \\ \partial_\alpha \gamma^a{}_{bc} & = & 0 \end{array}\right\}$$

We shall call such a tetrad a "canonical tetrad" for the spatially homogeneous space-time. Note that the triad $\{\mathbf{e}_\alpha\}$ spans the tangent vector space of the group orbits at all points.

In the Ellis-MacCallum classification of spatially homogeneous space-times (Ellis & MacCallum [1969a], p. 114), a space-time is of Class A if $a = 0$, in a canonical tetrad. Otherwise $a > 0$ and the space-time is of Class B.

[1] A weaker definition requires a G_r, $r \geq 3$, with three-dimensional space-like orbits. Our definition omits the case in which the G_r does not contain a G_3 subgroup acting on space-like hypersurfaces. This situation may arise only when $r = 4$ and has been described by Kantowski and Sachs [1966].

Property 3. *(Ellis & MacCallum [1969a], p. 118): In a space-time of Class A with perfect fluid source, a canonical tetrad may be chosen with* $\mathbf{u} = \mathbf{e}_0$ *such that the triad* $\{\mathbf{e}_\alpha\}$ *is a Fermi-propagated eigenframe of* $\sigma_{\alpha\beta}$.

Thus for a Class A space-time, a special canonical tetrad may be chosen in which

$$\left.\begin{array}{rcl} \mathbf{u} &=& \mathbf{e}_0 \\ \dot{\mathbf{u}} &=& \omega = \mathbf{\Omega} = \mathbf{0} \\ \theta_{ab} &=& \mathrm{diag}(0, \theta_1, \theta_2, \theta_3) \\ \partial_\alpha \theta_{ab} &=& 0 \\ a_\alpha &=& 0 \\ n_{\alpha\beta} &=& \mathrm{diag}(n_1, n_2, n_3) \\ \partial_\alpha n_{\beta\gamma} &=& 0. \end{array}\right\}$$

In this tetrad, the non-trivial Jacobi identities are

$$\left.\begin{array}{rcrcl} \binom{1}{023} &\Leftrightarrow& \partial_0 n_1 &=& n_1(2\theta_1 - \theta) \\ \binom{2}{031} &\Leftrightarrow& \partial_0 n_2 &=& n_2(2\theta_2 - \theta) \\ \binom{3}{012} &\Leftrightarrow& \partial_0 n_3 &=& n_3(2\theta_3 - \theta) \end{array}\right\} \qquad (2.2.1)$$

The non-trivial Einstein field equations are

$$\left.\begin{array}{rcrcl} (00) &\Leftrightarrow& \partial_0 \theta &=& -\left({\theta_1}^2 + {\theta_2}^2 + {\theta_3}^2\right) - \frac{1}{2}(\mu + 3p) + \Lambda \\ (11) &\Leftrightarrow& \partial_0 \theta_1 &=& -\theta\theta_1 - \frac{1}{2}{n_1}^2 + \frac{1}{2}(n_2 - n_3)^2 + \frac{1}{2}(\mu - p) + \Lambda \\ (22) &\Leftrightarrow& \partial_0 \theta_2 &=& -\theta\theta_2 - \frac{1}{2}{n_2}^2 + \frac{1}{2}(n_3 - n_1)^2 + \frac{1}{2}(\mu - p) + \Lambda \\ (33) &\Leftrightarrow& \partial_0 \theta_3 &=& -\theta\theta_3 - \frac{1}{2}{n_3}^2 + \frac{1}{2}(n_1 - n_2)^2 + \frac{1}{2}(\mu - p) + \Lambda \end{array}\right\} \qquad (2.2.2)$$

and the only non-trivial contracted Bianchi identity is

$$(0) \quad \Leftrightarrow \quad \partial_0 \mu = -(\mu + p)\theta.$$

We shall now attempt to specialize two Bianchi-Behr types in Class A by imposing the constraint

$$n^\alpha{}_\alpha = 0 \qquad \text{on an open set.}$$

Example 2.2.1. Consider the class of Bianchi-Behr type $\mathrm{VI}_{h=0}$ solutions, with perfect fluid source. These solutions are in Class A and are distinguished in a canonical tetrad, possibly by renumbering, by

$$n_1 = 0, \quad n_2 > 0, \quad n_3 < 0.$$

(Note that initially the requirement that $n_1 = 0$ may be regarded as a constraint which is satisfied by virtue of (2.2.1).)

The constraitn which we want to impose is given by

$$n^{\alpha}{}_{\alpha} = n_2 + n_3 = 0. \tag{2.2.3}$$

For this to hold in an open set we must also have

$$\partial_a(n_2 + n_3) = 0 \quad \Leftrightarrow \quad \partial_0(n_2 + n_3) = 0.$$

Adopting a special canonical tetrad, this gives

$$\begin{aligned} -\theta(n_2 + n_3) + 2(n_2\theta_2 + n_3\theta_3) &= 0 \\ \Leftrightarrow \quad \theta_2 - \theta_3 &= 0 \end{aligned} \tag{2.2.4}$$

where use has been made of (2.2.1) and then (2.2.3). This relation must also be satisfied on an open set so we must have, using (2.2.2),

$$\begin{aligned} \partial_a(\theta_2 - \theta_3) = 0 \quad &\Leftrightarrow \quad \partial_0(\theta_2 - \theta_3) = 0 \\ \Leftrightarrow \quad &-\theta(\theta_2 - \theta_3) - ({n_2}^2 - {n_3}^2) = 0, \end{aligned}$$

which is an identity by virtue of (2.2.3) and (2.2.4). This leaves us to conclude that there is a specialization of the class of Bianchi-Behr Type $\mathrm{VI}_{h=0}$ solutions with perfect fluid source in which $n^{\alpha}{}_{\alpha} = 0$. In this specialization, we necessarily have $\theta_2 = \theta_3$ and the Jacobi identities, field equations and Bianchi identity reduce to

$$\begin{aligned} \partial_0 n_2 &= -n_2\theta_1 \\ \partial_0\theta &= -({\theta_1}^2 + 2{\theta_2}^2) - \frac{1}{2}(\mu + 3p) + \Lambda \\ \partial_0\theta_1 &= -\theta\theta_1 + {n_2}^2 + \frac{1}{2}(\mu - p) + \Lambda \\ \partial_0\theta_2 &= -\theta\theta_2 + \frac{1}{2}(\mu - p) + \Lambda \\ \partial_0\mu &= -(\mu + p)\theta. \end{aligned}$$

The procedure for arriving at this conclusion may be depicted schematically as in Figure 2.2.1. □

Example 2.2.2. Consider the Bianchi-Behr type VIII solutions with perfect fluid source. These solutions are contained in the Ellis-MacCallum Class A and in a canonical tetrad, renumbering if necessary, have

$$n_1 > 0, \quad n_2 > 0, \quad n_3 < 0.$$

We now impose the constraint

$$n^{\alpha}{}_{\alpha} = n_1 + n_2 + n_3 = 0. \tag{2.2.5}$$

Adopting a special canonical tetrad and demanding that (2.2.5) hold on an open set, we obtain

$$\begin{aligned} \partial_0(n_1 + n_2 + n_3) = 0 \quad &\Leftrightarrow \quad -\theta(n_1 + n_2 + n_3) + 2(n_1\theta_1 + n_2\theta_2 + n_3\theta_3) = 0 \\ &\Leftrightarrow \quad n_1\theta_1 + n_2\theta_2 + n_3\theta_3 = 0, \qquad \text{using (2.2.5)} \\ &\Leftrightarrow \quad \theta_3 = \frac{n_1\theta_1 + n_2\theta_2}{n_1 + n_2}, \end{aligned} \tag{2.2.6}$$

again by (2.2.5). This relationship must be time propagated as well, giving

$$\partial_0(n_1\theta_1 + n_2\theta_2 + n_3\theta_3) = 0$$

$$\begin{aligned} \Leftrightarrow \quad 2({n_1\theta_1}^2 + {n_2\theta_2}^2 + {n_3\theta_3}^2) + \frac{1}{2}(-{n_1}^3 - {n_2}^3 - {n_3}^3 \\ + n_1{n_2}^2 + n_1{n_3}^2 + n_2{n_1}^2 + n_2{n_3}^2 + n_3{n_1}^2 + n_3{n_2}^2 - 6n_1n_2n_3) = 0, \end{aligned}$$

where use has been made of (2.2.1) and (2.2.2). Using (2.2.5) and (2.2.6) to eliminate n_3 and θ_3 this gives

$$\frac{n_1 n_2}{n_1 + n_2}\left[(\theta_1 - \theta_2)^2 + 3(n_1 + n_2)^2\right] = 0. \tag{2.2.7}$$

Each of the factors on the left-hand side of (2.2.7) is strictly positive, so we have a contradiction. This gives the conclustion that the specialization $n^{\alpha}{}_{\alpha} = 0$ yields an empty class; there are no Biahcni-Behr type VIII perfect fluid models in which $n^{\alpha}{}_{\alpha} = 0$ on an open set. The above steps are depicted schematically in figure 2.2.2. □

2.3 A Spatially Inhomogeneous Example

This section gives another examle of how to specialize a class of solutions according to a constraint. We shall consider the LRS type II solutions with dust as the source (see section 1.3). Our original system will be given by the equations (LRSII) of Appendix A, with p, $\dot{u}$, τ and ϵ replaced by zero. For all the unknowns but s, the only non-trivial commutator is

$$[\mathbf{e}_0, \mathbf{e}_1] = -\alpha \mathbf{e}_1. \tag{2.3.1}$$

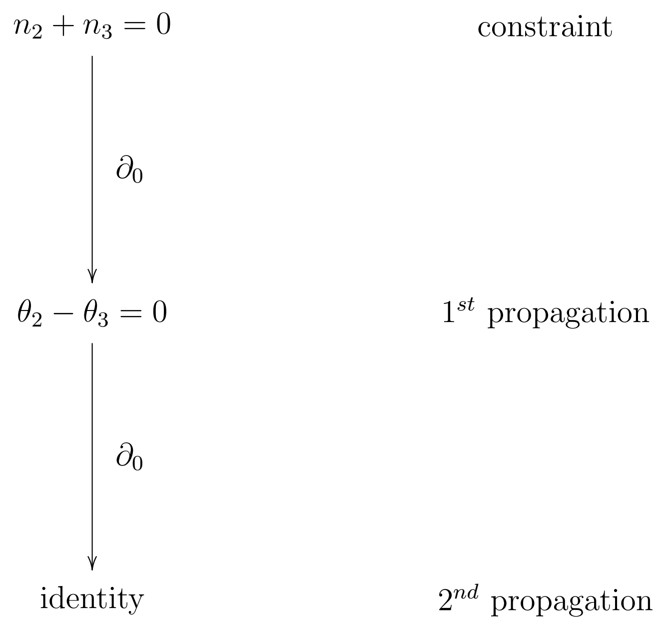

Figure 2.2.1: Bianchi-Behr type $\mathrm{VI}_{h=0}$ perfect fluid solutions subject to $n^{\alpha}{}_{\alpha} = 0$

The non-trivial field equations give

$$\left.\begin{array}{lcl} (01) & \Leftrightarrow & \partial_1\beta = a(\beta - \alpha) \\ \left.\begin{array}{l} (00) \\ (11) \\ (22) \end{array}\right\} & \Leftrightarrow & \left\{\begin{array}{lcl} \partial_0\alpha & = & -\frac{\mu}{2} + \beta^2 - \alpha^2 - a^2 + r \\ \partial_0\beta & = & \frac{1}{2}(\Lambda - 3\beta^2 + a^2 - r) \\ \partial_1 a & = & \frac{1}{2}(\Lambda + \mu - \beta^2 - 2\alpha\beta + 3a^2 - r). \end{array}\right. \end{array}\right\} \tag{2.3.2}$$

The non-trivial Jacobi identities are

$$\left.\begin{array}{lcl} \binom{2}{012} \text{ and } (01) & \Leftrightarrow & \partial_0 a = -\beta a \\ \binom{3}{023} & \Leftrightarrow & \partial_0 r = -2\beta r \\ \binom{3}{123} & \Leftrightarrow & \partial_1 r = 2ar \end{array}\right\} \tag{2.3.3}$$

and the only non-trivial contracted Bianchi identity is

$$(0) \quad \Leftrightarrow \quad \partial_0\mu = -(\alpha + 2\beta)\mu. \tag{2.3.4}$$

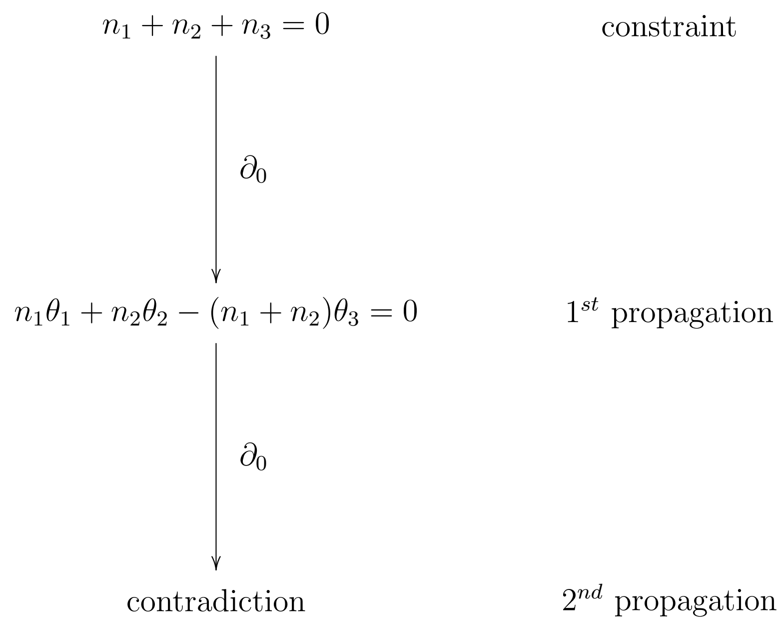

Figure 2.2.2: Bianchi-Behr type VIII perfect fluid solutions subject to ${n^\alpha}_\alpha = 0$

We shal also assume the energy condition

$$\mu \geq 0.$$

These relations shall be used in the following example.

Example 2.3.1. We shall make the specialization that, on an open set, the flow be expansion-free but not necessarily shear-free:

$$\alpha + 2\beta = 0. \tag{2.3.5}$$

Since the directional derivatives ∂_2 and ∂_3 give zero for all of the variables which appear in our equations (2.3.2-2.3.4), we need only check the $\mathbf{e}_0$ and $\mathbf{e}_1$ propagation equations.

The first differentiation of the constraint (2.3.5), using (2.3.1) and (2.3.5), give

$$\partial_0(\alpha + 2\beta) = 0 \quad \Leftrightarrow \quad \mu - 2\Lambda + 12\beta^2 = 0 \tag{2.3.5,0}$$
$$\partial_1(\alpha + 2\beta) = 0 \quad \Leftrightarrow \quad \partial_1\alpha = -6a\beta \tag{2.3.5,1}$$

Both of these equations must now hold on an open set. The propagation equations for (2.3.5,0) are

$$\partial_0(\mu - 2\Lambda + 12\beta^2) = 0$$
$$\Leftrightarrow \quad \beta(\mu + 2(a^2 - r) + 6\beta^2) = 0 \tag{2.3.5,00}$$

and

$$\begin{aligned} & \partial_1(\mu - 2\Lambda + 12\beta^2) = 0 \\ \Leftrightarrow \qquad & \partial_1\mu = -72a\beta^2, \end{aligned} \tag{2.3.5,01}$$

by (2.3.2), (2.3.4), (2.3.5) and (2.3.5,0). The propagation equations of (2.3.5,01),

$$\begin{aligned} \partial_0\partial_1\mu &= \partial_0(-72a\beta^2) \\ \partial_1\partial_1\mu &= \partial_1(-72a\beta^2), \end{aligned}$$

are consistent with the original system (2.3.1-2.3.4) if and only if μ satisfies the integrability condition (2.3.1):

$$\begin{aligned} & \partial_0\partial_1\mu - \partial_1\partial_0\mu = -\alpha\partial_1\mu \\ \Leftrightarrow \qquad & \partial_0(-72a\beta 2) = -\alpha(-72a\beta^2) \\ \Leftrightarrow \qquad & a\beta(\mu + 2(a^2 - r)) = 0 \end{aligned} \tag{2.3.5,01*}$$

Thus, (2.3.5,0) holds on an open set only if, in addition, (2.3.5,00) and (2.3.5,01) also hold. The propagation equations of (2.3.5,1) hold identically since α satisfies the integrability condition:

$$\begin{aligned} & \partial_0\partial_1\alpha - \partial_1\partial_0\alpha = -\alpha\partial_1\alpha \\ \Leftrightarrow \qquad & \partial_0(-6a\beta) - \partial_1\left(-\frac{\mu}{2} - 3\beta^2 - a^2 + r\right) = -\alpha(-6a\beta) \\ \Leftrightarrow \qquad & \text{identity,} \end{aligned}$$

by (2.3.2), (2.3.3), (2.3.5) and (2.3.5,1).

It remains to check for consistency the equations

$$\beta(\mu + 2(a^2 - r) + 6\beta^2) = 0 \tag{2.3.5,00}$$

$$a\beta(\mu + 2(a^2 - r)) = 0 \tag{2.3.5,01*}$$

with the original system. At this point we would usually divide the examination into a number of cases in which combinations of factors were zero. Note, though, that β is a factor common to both equations. In the cases in which $\beta \neq 0$ on an open set, then solving (2.3.5,00) for μ and substituting in (2.3.5,01*) gives

$$\begin{aligned} & \mu = 2(r - a^2) - 6\beta^2 \\ \Rightarrow \qquad & a = 0 \end{aligned}$$

The propagation equations for $a = 0$ are:

$$\begin{array}{rcll} \partial_0 a = 0 & \Leftrightarrow & \text{identity} & \\ \partial_1 a = 0 & \Leftrightarrow & 3\mu - 2r + 18\beta^2 = 0, & \text{using (2.3.5,0)} \\ & \Leftrightarrow & r = 0, & \text{using (2.3.5,00)} \end{array}$$

But this gives $\mu = -6\beta^2$, which contradicts the energy condition $\mu \geq 0$. Therefore the only cases which do not lead to a contradiction are those in which

$$\beta = 0 \tag{2.3.6}$$

on an open set. For this to hold we must have

$$\begin{array}{rcl} \partial_0 \beta = 0 & \Leftrightarrow & \mu = 2(r - a^2) \\ \partial_1 \beta = 0 & \Leftrightarrow & \text{identity.} \end{array} \tag{2.3.6,0}$$

Equation (2.3.6,0) may now be imposed on an open set consistently with the original system, since its propagation along $\mathbf{e}_0$ and $\mathbf{e}_1$ yield identities by virtue of (2.3.2), (2.3.3), (2.3.4) and (2.3.5,0).

We have shown that the constraint $\alpha + 2\beta = 0$ may be applied to our class of solutions only if we also demand (2.3.5,0), (2.3.6) and (2.3.6,0), whence

$$\begin{array}{ll} \alpha = \beta = 0 & \\ \mu = 2\Lambda & (\Rightarrow \Lambda \geq 0) \\ r = \Lambda + a^2 & (\Rightarrow r \geq a^2 \geq 0). \end{array}$$

Subject to these restrictions, our original system of equations is equivalent to

$$\left. \begin{array}{rcl} \partial_0 a & = & 0 \\ \partial_1 a & = & \Lambda + a^2. \end{array} \right\} \tag{2.3.7}$$

From its definition, a may be given in terms of the tetrad components, ${e^i}_a$, so (2.3.7) may readily be solved to obtain the general solution in the subclass. Thus, the demand that the dust flow be expansion-free is much more restrictive than it originally appears.

The procedure we have followed in this example is given schematically in figure 2.3.1. The notation introduced in this figure shall be used in diagrams in subsequent chapters.

□

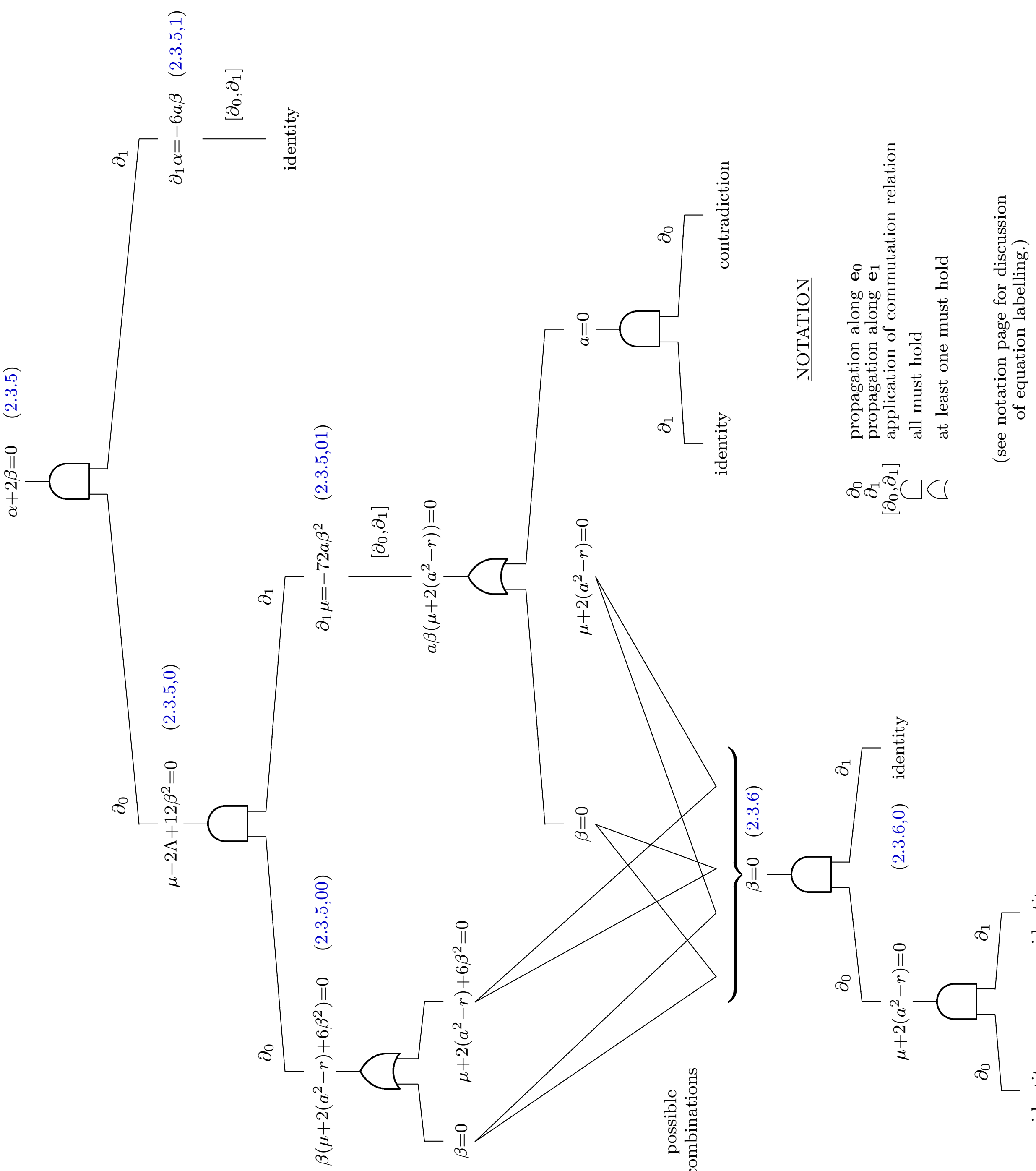

Figure 2.3.1: LRS Class II with dust subjects to $\theta = 0$

Chapter III

INTRINSIC SYMMETRIES IN GENERAL RELATIVITY

In this chapter, we introduce the concept of "intrinsic symmetries" used in this thesis. Section 3.1 describes the basic idea of intrinsic symmetries and Section 3.2 goes into greater detail, showing how the intrinsic Ricci tensor and the second fundamental form may be used to characterize families of submanifolds.

3.1 The Role of Intrinsic Symmetries

Spatially homogeneous space-times, defined in Section 2.2, have been used extensively as universe models (cf. MacCallum [1973c] or, more recently, MacCallum [1980b] and references cited therein). The conventional belief in large-scale inhomogeneity of the universe and the relative mathematical simplicity of these models lend justification to this approach. However, spatially homogeneous cosmological models are not suitable in all contexts. For example, spatially homogeneous models are probably not sufficient to solve the problems associated with galaxy formation and particle horizons in the early universe. It then becomes necessary to study spatially inhomogeneous cosmological models, even though they are mathematically much more complicated.

By "spatially inhomogeneous" we mean that there does not exist a group of local isometries $G_r, r \geq 3$, acting transitively on spacelike hypersurfaces. To introduce spatial inhomogeneity, one may relax the requirements for spatial homogeneity by decreasing the dimension of the local isometry group or of the group orbits. The simplest case is to have a G_3 acting on two dimensional spacelike orbits. For a perfect fluid (or perfect fluid with electromagnetic field), this coincides with the LRS type II space-times (see Section 1.3), for which a general solution is not known.

To attempt a study with spatial inhomogeneity, one may alternatively employ spatial collineations which are not isometries (Katzin, Levine & Davis, [1969b]). The simplest such collineation is a simple "homothetic motion" or "similarity mapping". This is a difeomorphism for which the generating vector field, **h**, satisfies

$$\mathscr{L}_{\mathbf{h}} g_{ab} = 2b g_{ab}$$

for some constant, b. It has been shown (McIntosh [1976]) that in any perfect fluid space-time unless we have a "stiff matter" equation of state, if such a vector field, **h**, is orthogonal to the fluid flow vector then **h** is necessarily a Killing vector. * Thus, if a general perfect fluid space-time admits a local group $H_r(r \geq 3)$ of homothetic motions, which is not an r-dimensional isometry group, then the group orbits are necessarily not orthogonal to the fluid flow. If the group orbits are 3-dimensional spacelike hypersurfaces, the solutions are then said to be "tilting" (cf. King & Ellis [1973b]) and, although there is some justification for studying tilting models, they are complicated to deal with even in the case where the group is an isometry group. (Eardley [1974]) has initiated some studies of such (tilted) models in the case where there is a $H_r(r \geq 3)$ homothety group.

These difficulties motivate another approach to space-time symmetries. The method of "intrinsic symmetries" consists of imposing conditions not on the full space-time manifold but on submanifolds of the space-time (Collins & Szafron [1979a], Szafron & Collins [1979e], Collins & Szafron [1979b]). In this way one hopes to arrive at a tractable set of field equations without requiring spatial homogeneity. This method is illustrated in the following example.

Example 3.1.1. Consider the class of Bianchi-Behr type I (spatially homogeneous) perfect fluid solutions. These space-times are contained in the Ellis-McCallum Class A (see Section 2.2) and are distinguished in a special canonical tetrad by

$$n_1 = n_2 = n_3 = a = 0.$$

The metric is given by

$$ds^2 = -dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2$$

and the fluid flow is orthogonal to the flat hypersurfaces which are the orbits of the local isometry group.

We may generalize this class of space-times by adapting the intrinsic symmetries approach. We shall look at models in which the fluid flow is orthogonal to flat hypersurfaces but which are not necessarily the orbits of an isometry group. That is, we shall impose the condition that the hypersurfaces, when considered as 3-spaces, admit 6 independent Killing vectors, one of which is necessarily a Killing vector of the full space-time.

That we need not have spatial homogeneity is illustrated by a simple plane- symmetric metric:

$$ds^2 = -dt^2 + t^{-2/3}\left[1 + tC(x)\right]dx^2 + t^{4/3}\left(dy^2 + dz^2\right)$$

*This result does not necessarily carry over to a perfect fluid with electromagnetic field (see Appendix B).

This metric describes dust with density

$$\mu = \frac{4}{3}\frac{C(x)}{t\,[t + C(x)]}$$

flowing orthogonally to the flat hypersurfaces {t = constant}. In general, this space-time admits the three Killing vectors:

$$\xi^{(1)} = \frac{\partial}{\partial y}, \quad \xi^{(2)} = \frac{\partial}{\partial z}, \quad \xi^{(3)} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

which generate a G_3 acting on the *two dimensional* orbits {t = constant, x = constant}. We have spatial homogeneity if and only if C(x) is a constant, in which case $\xi^4 = \frac{\partial}{\partial x}$ is also a Killing vector and we have a G_4 whose orbits are the hypersurfaces {t = constant}.

A more complicated example would be obtained by postulating the existence of spacelike hypersurfaces, each invariant under an $O(3)$ group of local isometries, which again need not be isometries of the full space-time (Krasinski [1980a]). The space-times are "intrinsically spherically symmetric" although not necessarily spherically symmetric in the conventional sense.

3.2 Families of Surfaces: Intrinsic and Extrinsic Curvatures

When adopting the intrinsic symmetries approach, it is most natural to place conditions on the submanifolds of a geometrically well-defined family. Hopefully, the gain here would be in the use of 1- and 2-parameter families of (respectively) 3- and 2-dimensional submanifolds, since 3-parameter families of 1-dimensional surfaces (i.e. congruences of curves) are already thoroughly exploited.

The geometry of a submanifold of dimension 2 or 3 is described by the intrinsic Ricci tensor, which, because of the dimension of the submanifold, entirely determines the Riemann tensor of the submanifold. This Ricci tensor may be used to construct invariants which characterize it. Then a constraint on the intrinsic curvature could be given in terms of these invariants.

There is some latitude in choosing independent invariants to describe the intrinsic curvature of a submanifold. The following exaple illustrates this.

Example 3.2.1. Consider $R^*_{\alpha\beta}$, the intrinsic Ricci tensor of a spacelike hypersurface. (The notation follows Ellis [1971], p.132 and McCallum [1973c], p.105.) In general this tensor will have six different components. However, using three degrees of freedom for orientation, a local basis may be chosen to be an orthonormal triad which is an eigenframe of $R^*_{\alpha\beta}$. In this frame, $R^*_{\alpha\beta}$ takes the form diag $(R^*_{11}, R^*_{22}, R^*_{33})$ and we see that we must construct, in general, three algebraically independent scalars to describe the intrinsic curvature. There is some freedom in the choice of three such scalars so, since no choice is preferred on physical grounds, we shall construct the scalars that are the simplest to work with. The easiest method to imposing a condition on a scalar is to set it to zero, so we would like to define our quantities in such a way that their vanishing has significance. A few possibilities are:

(i) *Contractions of $R^*_{\alpha\beta}$ with itself*

$$R^{*}{}_{\alpha}{}^{\alpha}, \quad R^{*}{}_{\alpha}{}^{\beta} R^{*}{}_{\beta}{}^{\alpha}, \quad R^{*}{}_{\alpha}{}^{\beta} R^{*}{}_{\beta}{}^{\gamma} R^{*}{}_{\gamma}{}^{\alpha}.$$

(ii) *The three eigenvalues of $R^*_{\alpha\beta}$*

These are the solutions, λ_1, λ_2 and λ_3, to the characteristic equation:

$$\det\left(R^{*\alpha}{}_{\beta} - \lambda g^{\alpha}{}_{\beta}\right) = 0$$

$$\Leftrightarrow \lambda^3 - R^{*\alpha}{}_{\alpha}\lambda^2 + \frac{1}{2}\left[(R^{*\alpha}{}_{\alpha})^2 - R^{*}{}_{\alpha}{}^{\beta}R^{*}{}_{\beta}{}^{\alpha}\right]\lambda - \det\left(R^{*}{}_{\alpha}{}^{\beta}\right) = 0 \quad .$$

(iii) *The coefficients of λ in the characteristic equation for $R^*_{\alpha\beta}$*

We would then have the following interpretation:

$$R^{*\ \alpha}_{\alpha} = \lambda_1 + \lambda_2 + \lambda_3$$

$$\frac{1}{2}\left[(R^{*\ \alpha}_{\alpha})^2 - R^{*\ \beta}_{\alpha} R^{*\ \alpha}_{\beta}\right] = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$\det\left(R^{*\ \beta}_{\alpha}\right) = \lambda_1\lambda_2\lambda_3$$

(iv) *"Degree of isotropy" $R^*_{\alpha\beta}$*

Define the scalars R^*, $S \geq 0$ and $T \geq 0$ as follows:

$$R^* = R^{*\ \alpha}_{\alpha}$$

$$S^2 = \frac{1}{2} S_{\alpha\beta} S^{\alpha\beta} \quad , \quad S_{\alpha\beta} = R^*_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} R^*$$

$$T^6 = 4S^6 - 3\left(S_{\alpha\beta} S^{\beta\gamma} S^{\alpha}{}_{\gamma}\right)^2$$

Then

$$\begin{aligned} R^* = 0 &\Leftrightarrow \text{ the isotropic part of } R^*_{\alpha\beta} \text{ is zero} \\ S = 0 &\Leftrightarrow R^*_{\alpha\beta} \text{ is completely isotropic (all eigenvalues are equal)} \\ T = 0 &\Leftrightarrow R^*_{\alpha\beta} \text{ is partially isotropic (some pair of eigenvalues are equal).} \end{aligned}$$

Of these choices, (ii) and (iv) are the most intuitive and the chief difference between them is that the information gained from the vanishing of the eigenvalues is highly directional, whereas that from the vanishing of R^*, S and T is more general.

At this point we shall introduce a new notation. If a set V of 4-m linearly independent vector fields is (locally) surfaceforming, then the family of surfaces shall be denoted by $\mathscr{F}_m(V)$. We then have an m-parameter family of submanifolds partitioning the region of interest. Geometric objects defined on a submanifold may be written showing their manifold dependence. For example, the components of the Ricci tensor for $\mathscr{S} \in \mathscr{F}_m(V)$ would be written as $R_{\alpha\beta}(\mathscr{S})$, where a and B are suitably restricted.

We now go on to give the relationship between the submanifolds and the containing space. The embedding of non-null hypersurfaces in a (sub)manifold is described by the congruence of integral curves of the unit normal vector field. This is done using the second fundamental form or "extrinsic curvature" of the hypersurface, which may be defined in the following way. Let $\mathscr{S}'$ be a non-null hypersurface with unit normal vector field $\mathbf{n}$ in $\mathscr{S}$. If g_{ab} is the metric of $\mathscr{S}$, then the projection tensor from $\mathscr{S}$ into $\mathscr{S}'$ is

$$h_{ab} = g_{ab} - n_c n^c n_a n_b$$

and the extrinsic curvature tensor χ_{ab} is defined (cf. Hawking and Ellis [1973a]) by

$$\chi_{ab} = h^c{}_a h^d{}_b n_{c||d}.$$

Here $\|$ indicates the covariant derivative with respect to the connection on $\mathscr{S}$. In this discussion, Latin indices shall take on the values appropriate to $\mathscr{S}$ while Greek indices take on the values appropriate to $\mathscr{S}'$.

We may now relate the curvature tensor, $R^a{}_{bcd}$ of $\mathscr{S}$ to the intrinsic and extrinsic curvatures, $R'^{\alpha}{}_{\beta\gamma\delta}$ and χ_{ab}, of $\mathscr{S}'$ using the formulas of Gauss:

$$R'^a{}_{bcd} = R^e{}_{fgh} h^a{}_e h^f{}_b h^g{}_c h^h{}_d + n^e n_e \chi^a_{[c}\chi_{d]b}$$

and Codacci:

$$R_{cd}\; n^c h^d{}_b = \chi^a_{[b||a]}.$$

To illustrate this idea, $R^*{}_{\alpha\beta}$ in example 3.1.2 would be given by

$$R^*{}_{ab} = R^e{}_{fgh}\; h^c{}_e h^f{}_a h^g{}_c h^h{}_b - \theta\theta_{ab} + \theta^c{}_b \theta_{ac}.$$

In this case, $\mathscr{S}$ is the space-time manifold $\mathscr{M}$ and $\|$ corresponds to ;. Note that if $\mathscr{S}''$ is a hypersurface in $\mathscr{S}'$ then the curvature of $\mathscr{S}''$ may be related to $\mathscr{S}'$ and hence to $\mathscr{S}$.

We shall be using the intrinsic Ricci tensor and the second fundamental form to describe the geometry of 2- and 3-dimensional submanifolds. Following example 3.2.1, we shall use the following scalars to characterize $R_{\alpha\beta}(\mathscr{S}')$ and $\chi_{ab}(\mathscr{S}',\mathbf{n})$ for a submanifold $\mathscr{S}' \in \mathscr{F}_{4-r}(V)$:

$$R(\mathscr{S}') := R^{\alpha}{}_{\alpha}(\mathscr{S}'), \qquad \theta(\mathscr{S}',\mathbf{n}) := \chi^{a}{}_{a}(\mathscr{S}',\mathbf{n})$$
$$S(\mathscr{S}') := \tfrac{1}{2}S_{\alpha\beta}(\mathscr{S}')S^{\alpha\beta}(\mathscr{S}'), \quad \sigma(\mathscr{S}',\mathbf{n}) := \tfrac{1}{2}\sigma_{ab}(\mathscr{S}',\mathbf{n})\sigma^{ab}(\mathscr{S}',\mathbf{n})$$

and, if r = 3,

$$T^6(\mathscr{S}') := 4S^6(\mathscr{S}') - 3\left[S_{\alpha\beta}(\mathscr{S}')S^{\beta\gamma}(\mathscr{S}')S^{a}{}_{\gamma}(\mathscr{S}')\right]^2$$
$$\tau^6(\mathscr{S}',\mathbf{n}) := 4\sigma^6(\mathscr{S}') - 3\left[\sigma_{ab}(\mathscr{S}',\mathbf{n})\sigma^{bc}(\mathscr{S}',\mathbf{n})\sigma^{a}{}_{c}(\mathscr{S}',\mathbf{n})\right]^2,$$

where $S_{\alpha\beta}(\mathscr{S}')$ and $\sigma_{ab}(\mathscr{S}',\mathbf{n})$ are given by

$$S_{\alpha\beta}(\mathscr{S}') = R_{\alpha\beta}(\mathscr{S}') - \frac{g_{\alpha\beta}(\mathscr{S}')}{r}R(\mathscr{S}')$$
$$\sigma_{ab}(\mathscr{S}',\mathbf{n}) = \chi_{(ab)}(\mathscr{S}',\mathbf{n}) - \frac{h_{ab}(\mathscr{S}',\mathbf{n})}{r}\theta(\mathscr{S}').$$

The scalars R and θ are the isotropic parts of $R_{\alpha\beta}$ and $\chi_{\alpha\beta}$ respectively. The quantity $S \geq 0$ or $\sigma \geq 0$ vanishes if and only if all the eigenvalues of $R_{\alpha\beta}$ or $\chi_{\alpha\beta}$ respectively, are equal and $T \geq 0$ or $\tau \geq 0$ vanishes if and only if any two eigenvalues of $R_{\alpha\beta}$ and $\chi_{\alpha\beta}$, respectively, are equal. Note that when $\mathbf{n} = \mathbf{u}$ and r = 3 then θ and σ are the usual kinematical quantities. Also note that the invariant τ, introduced here, should not be confused with the quantity τ, introduced in section 1.3, which describes the energy density of the electromagnetic field.

It is possible to give formulas for the intrinsic and extrensic curvature tensors in terms of $\mathbf{n}$ and the set of vector fields V. This is shown in the following example.

Example 3.2.2. Suppose $\mathscr{S}' \in \mathscr{F}_2(\mathbf{V}_1,\mathbf{V}_2)$, $\mathscr{S} \in \mathscr{F}_1(\mathbf{n},\mathbf{V}_1,\mathbf{V}_2)$ and $\mathbf{n}\cdot\mathbf{n} = -1$. Then, without loss of generality, choose an orthonormal tetrad $\mathbf{e}_A$, A = 0,1,2,3 , in which $\mathbf{e}_0 = \mathbf{n}$ and $\mathbf{e}_2$ and $\mathbf{e}_3$ span the tangent space of $\mathscr{S}'$. Then

$$R_{\alpha\beta}(\mathscr{S}') = \partial_\gamma\Gamma^{\gamma}{}_{\beta\alpha} - \partial_\beta\Gamma^{\gamma}{}_{\gamma\alpha} + \Gamma^{\gamma}{}_{\gamma\delta}\Gamma^{\delta}{}_{\beta\alpha} - \Gamma^{\gamma}{}_{\delta\alpha}\Gamma^{\delta}{}_{\gamma\beta}$$

and

$$\chi_{ab}(\mathscr{S}', \mathbf{n}) = n_{c\|d} h_a{}^c h_b{}^d =$$
$$= (\partial_d n_c - \Gamma^e{}_{dc} n_e)(\tilde{h_a}^c + n_a n^c)(\tilde{h_b}^d + n_b n^d),$$

where $\tilde{h}_{AB} := g_{AB} - e_{1A} e_{1B}$ and the indices take the following values: $\alpha, \beta \ldots = 2, 3, \quad a, b \ldots = 0, 2, 3$ and A,B $\ldots = 0, 1, 2, 3$. Here the quantities $\Gamma^A{}_{BC}$ are the Ricci rotation coefficients of the full space-time with metric g_{AB}, and $\tilde{h}_{AB}$ is the projection tensor from the full space-time onto $\mathscr{S}$.

When $R_{\alpha\beta}(\mathscr{S}')$ and $\chi_{ab}(\mathscr{S}', \mathbf{n})$ are written as in the above example, all the quantities R, S, T, θ, σ and τ are well defined, although their interpretation is less succinct, regardless of whether or not the set of vector fields, V, is surface-forming. If the vector fields are surface-forming then the Greek indices may be replaced by small Latin indices to emphasize the tensor character of the intrinsic and extrinsic curvatures. Another point to note is that if we have more than one family of surfaces, the intrinsic and extrinsic curvature invariants associated with each family are not all independent but are related via the Gauss-Codacci equations.

In a general space-time partitioned by a family of surfaces, we would expect the curvature invariants to be non-zero in a given neighborhood on any surface. However, there will be certain distinguished space-times in which one or more of these quantities is zero on an open set. We can thus obtain an algebraic classification of solutions depending on the vanishing of these invariants on all surfaces of the family. Further, in a space-time with several geometrically preferred families of surfaces, in any of the families combinations of the quantities R, S, T, θ, σ and τ may vanish. This classification scheme is a simple one based on the "degree of isotropy" of the intrinsic and extrinsic curvatures of subspaces. Another scheme exists in which both the inhomogeneity and conformal nature of the spacelike hypersurfaces are exploited (Wainwright [1979f]).

Our aim is to impose one or more of the intrinsic symmetries in order to simplify the field equations. (One hope is eventually to find new exact solutions but this has not yet been achieved.) We shall impose our conditions on one of the simple classes of inhomogeneous space-times, the LRS models. Since our technique uses surfaceforming combinations of vector fields, we shall concentrate the investigation on LRS type II space-times. In fact, we shall be concerned mainly with type IIc models, in which $\dot{\mathbf{u}} \neq 0,\ a \neq 0,$ and $\omega = k = 0$. In that case the spacelike hypersurfaces are conformally flat so the classification scheme of Wainwright divides the models into only 3 categories ($B_3: \ \sigma_{ab} \neq 0, \ \ h^a{}_b \partial_a \theta \neq 0; \quad B4: \ \sigma_{ab} \neq 0, \ \ h^a{}_b \partial_a \theta = 0;$ and $D3: \ \sigma_{ab} = 0, \ \ h^a{}_b \partial_a \theta = 0$). We shall be employing the intrinsic symmetries approach in preference to this, since it leads to a finer subdivision of types. However, a combination of the two approaches could be valuable.

Chapter IV

SPECIALIZATION OF THE LRS CLASS II USING INTRINSIC SYMMETRIES

We come now to the application of intrinsic symmetries in this thesis. We shall use certain symmetries to specialize the LRS type II class of solutions. The basic set of equations for this class is given by the system (LRSII) in Appendix A.

In this section we shall examine subspaces orthogonal to the time-like congruence, employing combinations of the following intrinsic symmetries:

$$\left.\begin{array}{llll} {}_{123}R := R({}_{123}\mathscr{S}) = 0 & \Leftrightarrow & r = -2\partial_1 a + 3a^2 & \text{(I1)} \\ {}_{123}S := S({}_{123}\mathscr{S}) = 0 & \Leftrightarrow & r = \partial_1 a & \text{(I2)} \\ {}_{123}T := T({}_{123}\mathscr{S}) = 0 & & \text{(holds identically)} & \text{(I3)} \\ {}_{12}R := R({}_{12}\mathscr{S}) = 0 & \Leftrightarrow & 0 = \partial_1 a - a^2 & \text{(I4)} \\ {}_{23}R := R({}_{23}\mathscr{S}) = 0 & \Leftrightarrow & r = 0 & \text{(I5)} \\ {}_{(0)23}\theta := \theta({}_{23}\mathscr{S}, \mathbf{e}_1) = 0 & \Leftrightarrow & \beta = 0 & \text{(I6)} \end{array}\right\} \qquad (4.0.1)$$

Here the subspaces are defined by

$$\begin{array}{l} {}_{123}\mathscr{S} \in \mathscr{F}_1(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}) \\ {}_{12}\mathscr{S} \in \mathscr{F}_2(\{\mathbf{e}_1, \mathbf{e}_2\}) \in \{\mathscr{F}_2(\{\mathbf{e}_1, \mathbf{v}\}) | \mathbf{v} \cdot \mathbf{e}_0 = \mathbf{v} \cdot \mathbf{e}_1 = 0\} \end{array}$$

and

$${}_{23}\mathscr{S} \in \mathscr{F}_2(\{\mathbf{e}_2, \mathbf{e}_3\})$$

for a tetrad, $\mathbf{e}_a$, which satisfies equation (1.3.2). These and other intrinsic symmetries for the LRS type II space-times are listed in Appendix C. Note that each of the families $\mathscr{F}_1(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$, $\mathscr{F}_2(\{\mathbf{e}1, \mathbf{v}\}) | \mathbf{v} \cdot \mathbf{e}_0 = \mathbf{v} \cdot \mathbf{e}_1 = 0$ and $\mathscr{F}_2(\{\mathbf{e}_2, \mathbf{e}_3\})$ is geometrically well defined and although $\mathscr{F}_2(\{\mathbf{e}_1, \mathbf{e}_2\})$ is not defined uniquely it is representative of the class $\mathscr{F}_2(\{\mathbf{e}_1, \mathbf{v}\}) | \mathbf{v} \cdot \mathbf{e}_0 = \mathbf{v} \cdot \mathbf{e}_1 = 0$, since $\mathbf{e}_2$ and $\mathbf{e}_3$ are fixed up to a rotation only.

We shall investigate the consequences of these symmetries individually and in combinations. To do this we shall impose the appropriate constraint equations from (4.0.1), using the method introduced in Chapter 2. Note that, for all quantities in the basic equations and the constraints, the operators ∂_2 and ∂_3 yield zero so for any constraint only the ∂_0 and ∂_1 propagation equations need be checked.

The symmetries, (I5) and (I6), related to the group orbits will be examined first. Then (I1), (I2) and (I4) will be examined individually and with combinations of (I5) and (I6). Finally, since any two of (I1), (I2) and (I4) imp1y the third, we shall investigate the case where all three hold and again consider comibinations with (I5) and (I6).

The majority of these calculations have been performed using the symbolic computing system MACSYMA (see section 1.4). The calculations for (I5) and (I6) will be outlined in some detail in computer free terms, as will be those for (I1). For (I2) the specialization is done with the aid of the computer and will again be presented in some detail. The remaining cases will only be summarized, the computational specifics being given in Appendix E.

4.1 Combinations of ${}_{23}R = 0$ and ${}_{(0)23}\theta = 0$

The first intrinsic symmetries that we shall investigate are those associated with the orbits of the isometry group. We examine specializations in which the group orbits are intrinsically flat (I5), the group orbits have zero extrinsic curvature in the hypersurfaces orthogonal to the fluid flow (I6), or both I5&6.

(I5) ${}_{23}\boldsymbol{R} = \mathbf{0}$

In this case we have

$$ {}_{23}R = 0 \quad \Leftrightarrow \quad r = 0. \tag{I5} $$

For this constraint to be preserved along $\mathbf{e}_0$ we must have, using (J4),

$$ \partial_0 r = 0 \quad \Leftrightarrow \quad -2\beta r = 0, $$

which is an identity. For the constraint to be preserved along $\mathbf{e}_1$, we must have, using, (J5),

$$ \partial_1 r = 0 \quad \Leftrightarrow \quad 2ar = 0, $$

which is also an identity. Hence the intrinsic symmetry may be imposed without requiring that any additional conditions be satisfied. This is indicated in figure 4.1.1.

Note that since (J4) and (J5) always hold, we may have (I5) in conjunction with any of the other intrinsic symmetries, unless r = 0 is specifically prohibited.

(I6) $_{(0)23}\boldsymbol{\theta} = \mathbf{0}$

We consider the condition

$$_{(0)23}\theta = 0 \quad \Leftrightarrow \quad \beta = 0 \tag{I6}$$

and check for the necessary conditions for it to hold on an open set. The $\mathbf{e}_0$ propagation of β is given by the field equation (F2):

$$\partial_0 \beta = 0 \quad \Leftrightarrow \quad \Lambda - p + a^2 - r - 2a\dot{u} + \tau = 0 \tag{I6,0}$$

using (I6), and the $\mathbf{e}_1$ propagation is given by (F4):

$$\partial_1 \beta = 0 \quad \Leftrightarrow \quad a_\alpha = 0,$$

again using (I6).

The last equation is satisifed if and only if a $= 0$ or $\alpha = 0$ on an open set. Note, though, that a $= 0$ gives a contradiction since $\partial_1 a = 0$ implies by (F3), (I6) and (I6,0) that μ + p $= 0$, which contradicts the perfect fluid energy condition. Therefore we must have

$$\alpha = 0 \tag{I6,1}$$

There are now two more equations, (I6,0) and (I6,1), which must be preserved along $\mathbf{e}_0$ and $\mathbf{e}_1$. The four resulting propagation equations are:

$$\begin{aligned} & \partial_0(\Lambda - p + a^2 - r - 2a\dot{u} + \tau) = 0 \\ \Leftrightarrow \quad & \partial_0 p = -2a\partial_0 \dot{u} \end{aligned} \tag{I6,00}$$

by (J2'), (J4), (M5) and (I6),

$$\begin{aligned} & \partial_1(\Lambda - p + a^2 - r - 2a\dot{u} + \tau) = 0 \\ \Leftrightarrow \quad & \partial_1 \dot{u} = -\dot{u}^2 + 2\tau - r + \frac{p+\mu}{2} + \frac{\epsilon E}{a} + a^2 \end{aligned} \tag{I6,01}$$

by (BI2), (F3), (J5), (M6) and (I6) and (I6,0),

$$\begin{aligned} & \partial_0 a = 0 \\ \Leftrightarrow \quad & \epsilon E = 0 \end{aligned} \tag{I6,10}$$

by (F3), (I6) and (I6,01), and

$$\partial_1 \alpha = 0 \tag{I6,11}$$

We shall examine each of these equations in turn.

Equation (I6,00) may be split into two propagation equations by introducing a new, arbitrary variable, U_0, i.e.

$$\partial_0 p = -2aU_0 \tag{I6,00a}$$

$$\partial_0 \dot{u} = U_0. \tag{I6,00b}$$

Since $\partial_1 p$ is already known through (BI2), p must satisfy the integrability condition:

$$\begin{aligned} & \partial_0\partial_1 p - \partial_1\partial_0 p = \dot{u}\partial_0 p \quad \text{(using (I6,1))} && \text{(I6,00a*)} \\ \Leftrightarrow \quad & \partial_0(-(\mu+p)\dot{u}) - \partial_1(-2aU_0) = -2a\dot{u}U_0 \quad \text{(using (BI2) and (I6,00a))} \\ \Leftrightarrow \quad & \partial_1 U_0 = -U_0(a+3\dot{u}) \quad \text{(using (BI1), (I6,00a), (I6,00b), and (F3)).} \end{aligned}$$

The last equation will be automatically consistent with the system since $\partial_0 U_0$ has yet to be specified.

Now considering equation (I6,00b) in conjunction with (I6,01), we have equations for both derivatives of $\dot{u}$. The integrability condition for this variable is identically satisfied using (I6,10), (M5),(J4), (I6,00a), (BI1), (J2) and (I6,00a*) in addition to (I6,00b) and (I6,01), so the values of the directional derivatives of $\dot{u}$ are compatible with our system.

Next consider equation (I6,10). There are now two possible cases as well as the combination:

$$E = 0 \tag{I6,10A}$$

$$\epsilon = 0 \tag{I6,10B}$$

If E = 0 then, by (M1), $\partial_1 E = 0$ implies $\epsilon = O$. Therefore

$$\epsilon E = 0 \quad \Leftrightarrow \quad \epsilon = 0.$$

Further propagation yields an identity, by (M7), and $\partial_1 \epsilon = 0$, which is consistent with (M7) by the commutation relation. Finally the derivatives of α trivially satisfy the commutatior relations so (I6,11) is an identity.

We have now checked all of the necessary conditions for (I5) to hold on an open set. Since these equations are compatible, we have a consistent specialization. In summary, we have

$$\left.\begin{aligned}
&\beta = 0 && \text{(I6)}\\
&\alpha = 0 && \text{(I6, 1)}\\
&\epsilon = 0 && \text{(I6, 10B)}\\
&\Lambda - p + a^2 - r - 2a\dot{u} + \tau = 0 && \text{(I6, 0)}\\
&\partial_0 p = -2a\partial_0\dot{u} && \text{(I6, 00)}\\
&\partial_1\dot{u} = -\dot{u}^2 + 2\tau - r + \tfrac{p+\mu}{2} + a^2 && \text{(I6, 01)}\\
&\partial_1\partial_0\dot{u} = -\partial_0\dot{u}(a + 3\dot{u}) && \text{(I6, 00a*)}
\end{aligned}\right\} \quad (4.1.1)$$

as well as the equations (LRSII) of Appendix A.

The computations we have performed are depicted in figure 4.1.2.

(I5&6) $_{23}R = 0$ *and* $_{(0)23}\theta = 0$

We now consider the specialization in which both (I5) and (I6) hold. This is, in effect, the intersection of the two classes just considered. To see if the specialization is consistent, we first impose the constraint (I6), obtaining the additional conditions (4.1.1). We then check to see if (I5) may be imposed on the new system ((LRSII) and (4.1.1)). We find that (I5) is admitted unconditionally, since $r = 0$ is not specifically prohibited. This is shown in figure 4.1.3.

Alternatively, the constraint (I5) may be imposed first, after which (I6) would be applied. We would then work through a sequence of steps similar to those for (I6) alone, except that the quantity r would be zero. Either way, we see that we have a consistent specialization in which we have (LRSII), (4.1.1) and the additional equation $r = 0$.

4.2 $_{123}R = 0$ and Combinations of $_{23}R = 0$ and $_{(0)23}\theta = 0$

We shall now investigate the intrinsic symmetry (I1) — the vanishing of the Ricci scalar of the hypersurfaces orthogonal to the fluid flow:

$$_{123}R = 0 \quad \Leftrightarrow \quad 2\partial_1 a - 3a^2 + r = 0 \tag{I1}$$

We shall also explore this condition along with either or both of (I5) and (I6), labelling the combinations as follows:

$$_{123}R = 0 \quad \text{and} \quad _{23}R = 0 \tag{I1\&5}$$

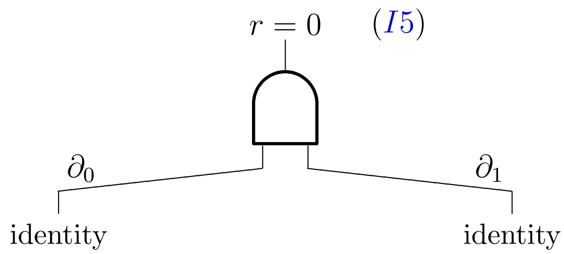

Figure 4.1.1: *I5:* ${}_{23}R$*=0*

$$ {}_{123}R = 0 \quad \text{and} \quad {}_{(0)23}\theta = 0 \tag{I1\&6} $$

$$ {}_{123}R = 0 \quad \text{and} \quad {}_{23}R = 0 \quad \text{and} \quad {}_{(0)23}\theta = 0. \tag{I1\&5\&6} $$

The calculations involved in imposing these conditions will be outlined here in a fairly detailed way.

We now show that the investigation of ${}_{123}R = 0$ splits naturally into two disjoint cases: one in which ${}_{(0)23}\theta \neq 0$ and one in which ${}_{(0)23}\theta \equiv 0$. Using (F3), the equation (I1) may be rewritten as an algebraic constraint:

$$ \Lambda + \mu + \tau - \beta(2\alpha + \beta) = 0. \tag{I1} $$

Applying ∂_0 to this equation gives the condition for preservation of the constraint along $\mathbf{e}_0$. We then have

$$ \begin{aligned} & \partial_0\mu + \partial_0\tau - 2(\alpha+\beta)\partial_0\beta - 2\beta\partial_0\alpha = 0 \\ \Leftrightarrow \quad & 2\beta(\partial_1\dot{u} + \dot{u}^2) + (a^2 - r)(\alpha - \beta) - 2a\dot{u}(\alpha+\beta) = 0, \end{aligned} \tag{I1,0} $$

by (BI1), (M5), (F2) and (F1). Similarly, preservation along $\mathbf{e}_1$ gives

$$ \begin{aligned} & \partial_1\mu + \partial_1\tau - 2(\alpha+\beta)\partial_1\beta - 2\beta\partial_1\alpha = 0 \\ \Leftrightarrow \quad & \partial_1\mu + 4a\tau + \epsilon E - 2a(\beta^2 - \alpha^2) - 2\beta\partial_1\alpha = 0, \end{aligned} \tag{I1,1} $$

by (M6) and (F4). As before, we need not check the $\mathbf{e}_2$ and $\mathbf{e}_3$ propagation equations, since these operators give zero identically when applied to the quantities in (I1). Examination of equations (I1,0) and (I1,1) indicates that we must indeed consider two

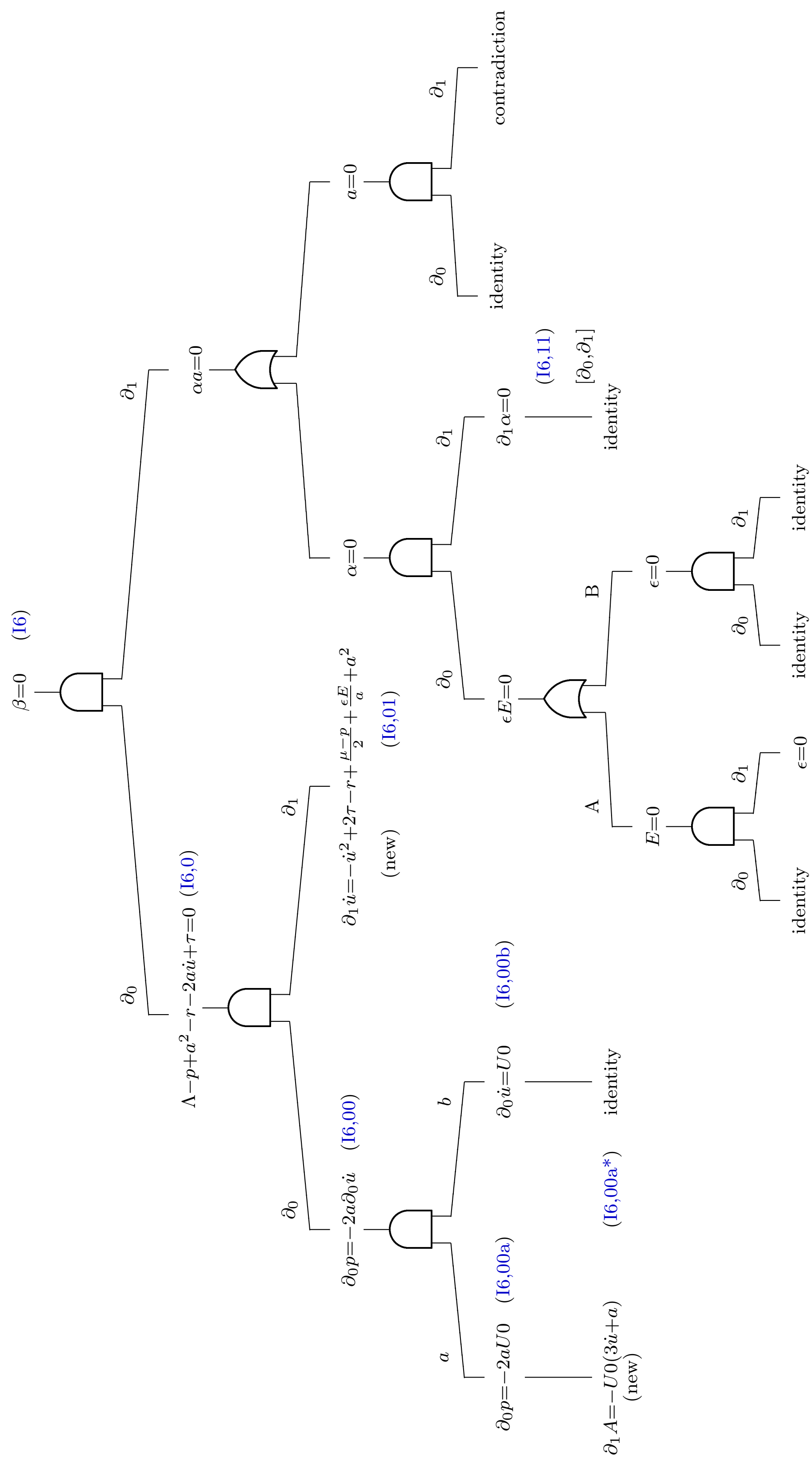

Figure 4.1.2: *I6:* $_{(0)23}\theta=0$

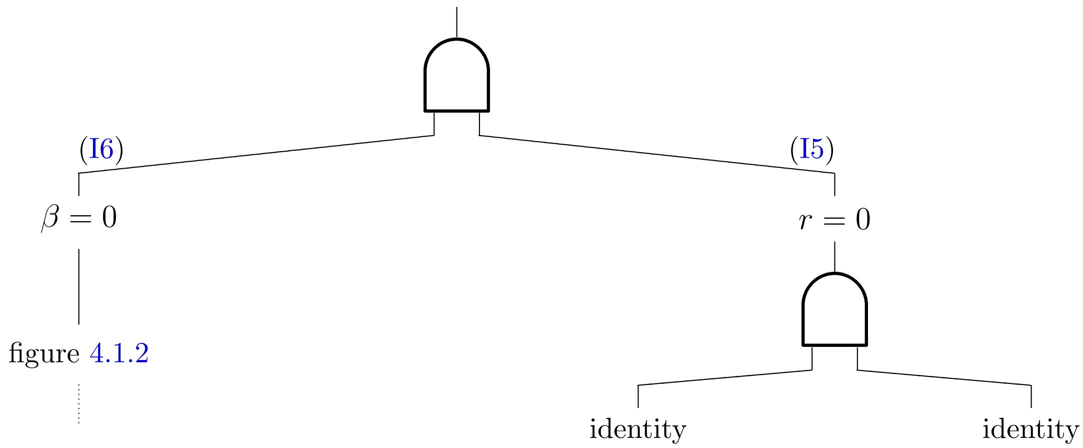

Figure 4.1.3: *I5&6:* $_{23}R$*=0 after imposing* $_{(0)23}\theta$*=0*

disjoint cases: one in which $\beta = {}_{(0)23}\theta/2$ vanishes identically on the open set and another in which β never assumes the value zero. We shall first consider the more general case, in which $\beta \neq 0$ and the calculations are somewhat more involved, and then proceed to the special case $\beta \equiv 0$.

(I1) $_{123}R = 0$ ($\beta \neq 0$)

In this subcase, division by β is possible since the quantity is never zero. Therefore we may write the first propagation equations as

$$\partial_1 \dot{u} = -\dot{u}^2 + \frac{1}{2}\beta^{-1}(a^2 - r)(\beta - \alpha) + \beta^{-1} a\dot{u}(\alpha + \beta) \tag{I1,0}$$

and

$$\partial_1 \alpha = \frac{1}{2}\beta^{-1}[4\alpha\tau + \epsilon E - 2a(\beta^2 - \alpha^2) + \partial_1 \mu]. \tag{I1,1}$$

Equation (I1,0) gives $\partial_1 \dot{u}$. Since this is the first specification of a derivative of $\dot{u}$ we need not check any integrability condition. Equation (I1,1) relates two previously unknown derivatives. We shall split this equation into two, each with only one previously unknown derivative, by introducing a new function, $M1$, as follows:

$$\partial_1 \mu = M1 \tag{I1,1a}$$

$$\partial_1 \alpha = \frac{1}{2}\beta^{-1}[M1 + 4a\tau + \epsilon E - 2a(\beta^2 - \alpha^2)]. \tag{I1,1b}$$

(The notation $M1$ is used for consistency with cases done on the computer and is not to be confused with the Maxwell's equation M1.) Both $\partial_0\mu$ and $\partial_0\alpha$ are known, so we must now check the commutators.

First we shall check the commutation relation for μ. Using $\partial_1\mu$ from (I1,1a) and $\partial_0\mu$ from (BI2), the integrability condition

$$\partial_0\partial_1\mu - \partial_1\partial_0\mu = \dot{u}\partial_0\mu - \alpha\partial_1\mu$$

becomes

$$\partial_0 M1 + \partial_1[(\mu+p)(\alpha+2\beta)] = -\dot{u}(\mu+p)(\alpha+2\beta) - \alpha M1.$$

The ∂_1 term may now be expanded employing (I1,1a), (BI2), (I1,1b) and (F4). The resulting equation is solved for $\partial_0 M1$ to obtain

$$\partial_0 M1 = -(\mu+p)\frac{\beta^{-1}}{2}[4a\tau + \epsilon E + 2a(\beta-\alpha)^2 + M1] + (\alpha+2\beta)\epsilon E - 2B(\beta+\alpha). \quad \text{(I1,1a*)}$$

This is the first derivative of $M1$ to be specified. Provided that the resulting constraints are compatible with the commutation relation for α, $M1$ may be chosen arbitrarily and no further checking is necessary.

Now we do a similar check of the commutation relation for α. With $\partial_1\alpha$ given by (I1,1b) and $\partial_0\alpha$ by (F1) (substituting for $\partial_1\dot{u}$ from (I1,0), the commutator may be expanded. After a straightforward but quite lengthy calculation using (I1) and most of the equations specifying derivatives (including (I1,1a*)), the commutation relation is found to be equivalent to

$$(\alpha+\beta) \in E = 0. \quad \text{(I1,1b*)}$$

There are now three possible cases, as well as their combinations:

$$\alpha+\beta = 0 \quad \text{(I1,1b*A)}$$

$$E = 0 \quad \text{(I1,1b*B)}$$

$$\epsilon = 0 \quad \text{(I1,1b*C)}$$

We shall examine these cases in order, and show that the first two are special cases of (I1,1b*C).

Consider the equation (I1,1b*A). Differentiating along $\mathbf{e}_0$, we use (F1) and (I1,0) to substitute for $\partial_0\alpha$ and (F2) for $\partial_0\beta$. Then we apply (I1) and (I1,1b*A) to replace Λ and β respectively, obtaining

$$\mu + p + 2\tau + a\dot{u} + 2\alpha^2 + \frac{1}{2}(r - a^2) = 0 \ . \tag{I1,1b*A0}$$

Similarly, we differentiate (I1,1b*A) along $\mathbf{e}_1$, using (I1,1b) to substitute for $\partial_1\alpha$ and (F4) for $\partial_1\beta$. Then we employ (I1,1b*A) to replace β with $-\alpha$, giving

$$M1 + 4a\tau + \epsilon E + 4a\alpha^2 = 0 \ . \tag{I1,1b*A1}$$

Normally we would now proceed to examine all four of the second propagation equations of (I1,1b*A). However, upon comparing two of these second propagation equations, it becomes apparent that further inquiry into the subcase A is not necessary. This is seen as follows. First differentiate (I1,1b*A0) along $\mathbf{e}_1$ and substitute for the derivatives. Next, replace in the resulting equation Λ, β, μ and $M1$ with their values according to (I1), (I1,1b*A), (I1,1b*A0) and (I1,1b*A1) respectively. Simplification of the result yields

$$4(\dot{u} + 2a)(\alpha^2 + \tau) - a(a^2 - r - 2a\dot{u}) = 0 \ . \tag{I1,1b*A01}$$

In the same vein, differentiating (I1,1b*A1) along $\mathbf{e}_0$, replacing the derivatives and substituting for Λ, β, μ and $M1$, yields

$$\alpha[4(\dot{u} + 2a)(\alpha^2 + \tau) - a(a^2 - r - 2a\dot{u}) + 2\epsilon E] = 0 \ . \tag{I1,1b*A10}$$

We know $\alpha = -\beta \neq 0$ so the second factor must vanish. A comparison with (I1,1b*A01) then shows we must have

$$\epsilon E = 0 \ .$$

Hence this subcase is merely a special instance of some combination of the subcases B and C.

We now turn our attention to equation (I1,1b*B). Differentiation along $\mathbf{e}_0$ and using M3 yields an identity but differentiation along $\mathbf{e}_1$ yields

$$\epsilon = 0 \ . \tag{I1,1b*B1}$$

using M1. Again we have a special instance of subcase C.

We now look at the last of the three subcases. As it has turned out, subcase C includes both of subcases A and B as special cases. Therefore ϵ must be zero in any solution admitting (I1) with $\beta \neq 0$. No further restrictions are necessary since differentiation of (I1,1b*C) yields an identity.

Since no equations remain to be checked for compatibility, we have a consistent specialization of the LRS type II space-times. From the preceding argument, any solution in the specialization is subject to

$$\left.\begin{aligned}
&\beta \neq 0 \\
&\Lambda + \mu + \tau - \beta(2\alpha + \beta) = 0 && \text{(I1)} \\
&\epsilon = 0 && \text{(I1, 1b} * \text{C)} \\
&\partial_1 \dot{u} = -\dot{u}^2 + \tfrac{1}{2}\beta^{-1}(a^2 - r)(\beta - \alpha) + \beta^{-1} a \dot{u}(\alpha + \beta) && \text{(I1, 0)} \\
&\partial_1 \alpha = \tfrac{1}{2}\beta^{-1}[\partial_1 \mu + 4a\tau - 2a(\beta^2 - \alpha^2)] && \text{(I1, 1)} \\
&\partial_0 \partial_1 \mu = \tfrac{-(\mu+p)}{2}\beta^{-1}[\partial_1 \mu + 4a\tau + 2a(\beta - \alpha)^2] && \text{(I1, 1a}*\text{)} \\
&\qquad\qquad - 2(\alpha + \beta)\partial_1 \mu \ .
\end{aligned}\right\} \quad (4.2.1)$$

as well as the system (LRSII) of Appendix A. The procedure followed for this specialization is outlined schematically in 4.1.4.

(I1&5) $_{123}R = 0$ *and* $_{23}R = 0$ $(\beta \neq 0)$

This is the specialization in which (I1) and (I5) both hold. Here we consider the more general case of $_{123}R = 0$ (i.e. $\beta \neq 0$), leaving the special case to be considered separately. We first impose the constraint given by (I1) on the system (LRSII). We then see that (I5) may be imposed without requiring any additional conditions, since $r = 0$ is not specifically prohibited. This is shown in 4.2.1.

Alternatively, the constraint given by (I5) may be imposed first, after which (I1) would be applied. Either way, we see that the specialization is consistent and (LRSII), (4.2.1) and (I5) hold.

(I1&6) $_{123}R = 0$ *and* $_{(0)23}\theta = 0$

This is the special case of (I1) in which $\beta = 0$. It was found in Section 4.1 that if $\beta = 0$, then the equations (4.1.1) must also hold. In this case then, the intrinsic symmetry (I1) reduces to

$$\Lambda + \mu + \tau = 0 \ . \qquad \text{(I1\&6)}$$

Using (BI1), (M5), (I6) and (I6,1), we find that the ∂_0 propagation of (I1&6) yields an identity. Using (M6), we see that the ∂_1 propagation gives

$$\partial_1 \mu = -4a\tau \ . \qquad \text{(I1\&6,1)}$$

Finally, employing (J2'), (M5) and (BI1), subject to $\alpha = \beta = 0$, shows that equation (I1&6,1) is consistent with (BI1), the prescription of $\partial_0 \mu$.

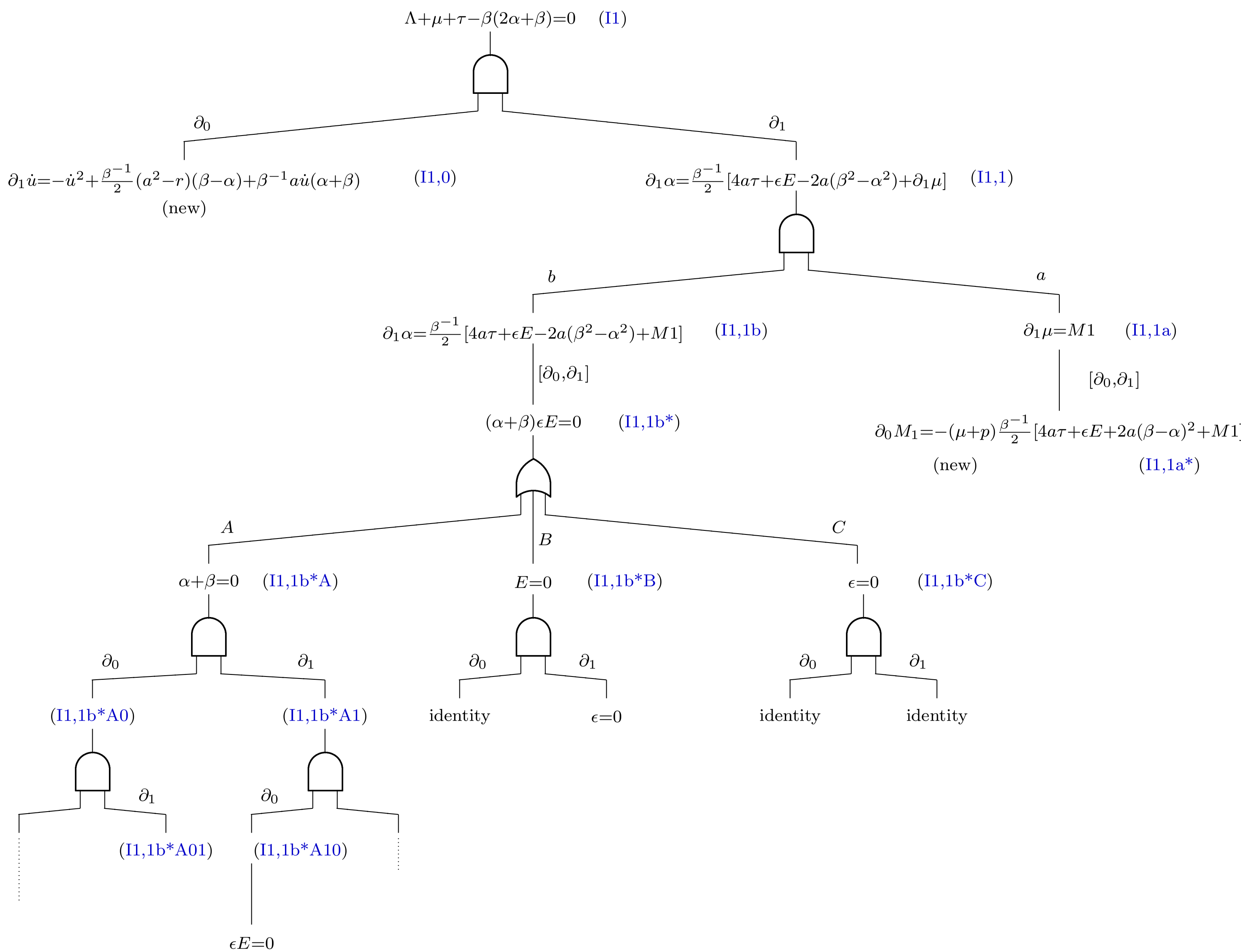

Figure 4.1.4: *I1:* ${}_{123}R=0$ *with* $\beta \neq 0$

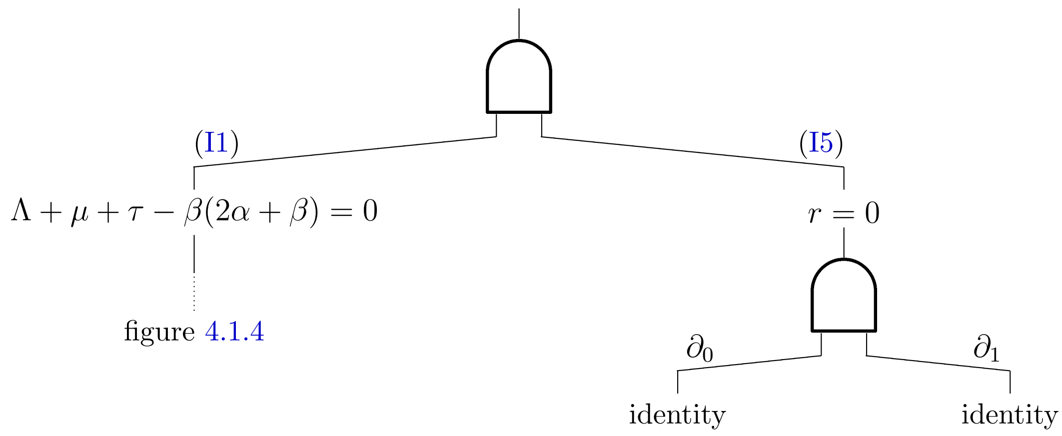

Figure 4.2.1: *I1&5: $_{23}R$=0 after imposing $_{123}R$=0 with $\beta \neq 0$*

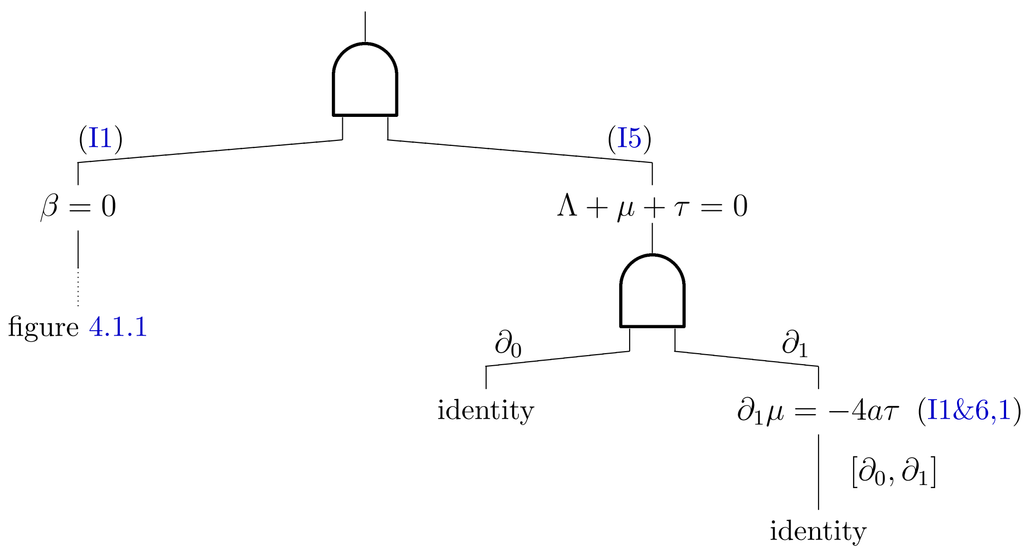

Figure 4.2.2: *I1&6: $_{123}R$=0 after imposing $_{(0)23}\theta$=0*

These steps are given in figure 4.2.2 and the equations which are to be satisfied in this specialization are (I1&6) and (I1&6,1) as well as (4.1.1) n and (LRSII).

(I1&5&6) $_{123}R = 0, {}_{12}R = 0$ *and* $_{(0)23}\theta = 0$

We shall now consider the case in which all three of (I1), (I5) and (I6) hold. The necessary conditions for (I1) and (I6) to hold together were found in the previous case. We may now impose the additional constraint (I5) and no additional conditions are

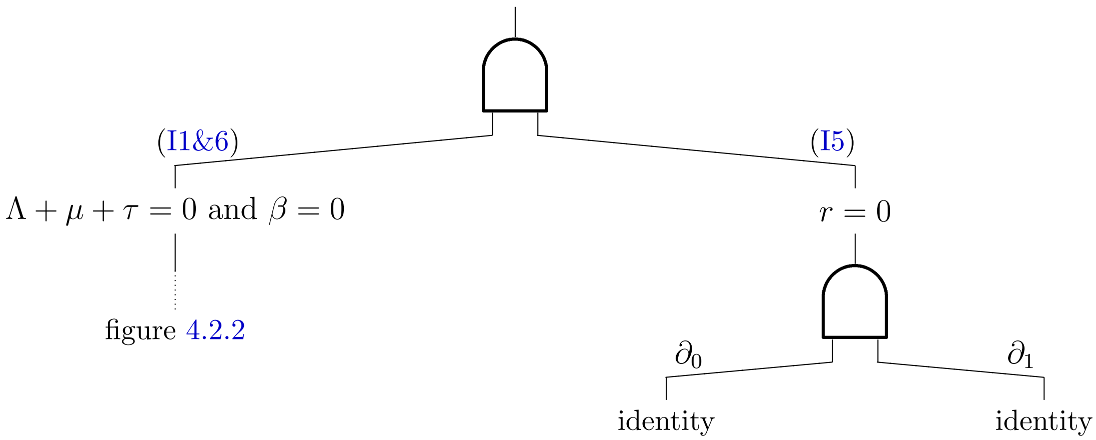

Figure 4.2.3: *I1&5&6:* ${}_{23}R$*=0 after imposing* ${}_{123}R$*=0 and* ${}_{(0)23}\theta$*=0*

necessary since $r = 0$ is again not specifically prohibited. This is indicated in figure 4.2.3.

The equations satisfied in this specialization are (I5), (I1&6), (I1&6,1) and those of (4.1.1) and (LRSII).

4.3 ${}_{123}S = 0$ and Combinations of ${}_{23}R = 0$ and ${}_{(0)23}\theta = 0$

The next intrinsic symmetry that we shall investigate is (I2) - the Ricci tensor of the fluid rest space is to be isotropic:

$$ {}_{123}S = 0 \quad \Leftrightarrow \quad \partial_1 a - r = 0 \ . \tag{I2} $$

We shall also explore this condition along with either or both of (I5) and (I6), labelling the combinations as follows:

$$ {}_{123}S = 0 \quad \text{and} \quad {}_{23}R = 0 \tag{I2\&5} $$
$$ {}_{123}S = 0 \quad \text{and} \quad {}_{(0)23}\theta = 0 \tag{I2\&6} $$
$$ {}_{123}S = 0 \quad \text{and} \quad {}_{23}R = 0 \quad \text{and} \quad {}_{(0)23}\theta = 0 \ . \tag{I2\&5\&6} $$

The course followed in imposing these conditions is similar in strategy to that followed for (I1), since the investigation of ${}_{123}S = 0$ splits naturally into two disjoint cases: ${}_{(0)23}\theta \equiv 0$ and ${}_{(0)23}\theta \neq 0$.

Here, the specialization has been performed with the aid of the symbolic computing system MACSYMA. To do this, the basic equations of the LRS type II space-times were placed in the file `MC:SWATT;LRSII SETUP` on the MACSYMA Consortium computer. This file has been reproduced in Appendix D.

This file contains a few commands which tailor the manner in which MACSYMA operates, before anything else is done.

Next, the atomic variables `F`, `X`, `s`, `a`, `r`, `udt`, `alf`, `bet`, `E`, `H`, `tau`, `eps`, `mu`, `p` and `LAM` are defined. They represent, respectively, the unknown quantities F, X, s, a, r, $\dot{u}$, α, β, E, H, τ, ϵ, μ, p and Λ. All of these variables are made to depend on two more, the coordinates `t` and `x`, except for `LAM`, since it corresponds to Λ, a constant.

The two functions, `e0` and `e1`, are defined to act as the differentiation operators. For example

$$e1(A) \ \leftrightarrow \ \frac{1}{X}\frac{\partial A}{\partial x}$$

Another function, `CR`, has been included for convenience to implement the commutation relations. This function first computes the action of

$$[\partial_0, \partial_1] - \dot{u}\partial_0 + \alpha\partial_1$$

on a quantity and then equates the result to zero.

Finally, all the equations of the (LRSII) system of Appendix A have been included as `GRADEF`s. For instance, the Jacobi identity (J1) is given as

$$GRADEF(s, x, a * s * X) \leftrightarrow \ \frac{\partial s}{\partial x} := asX \ \Leftrightarrow \ \partial_1 s = as \ .$$

The computations to impose (I2) on the class of solutions are now given in the form of a MACSYMA session. The first thing to be done is to load the file `MC:SWATT;LRSII SETUP`.

```
(C1)  BATCHLOAD("MC:SWATT\;LRSII SETUP") $

MLOAD FASL DSK MACSYM being loaded
loading done
```

We may now obtain the algebraic constraint equation equivalent to (I2). We shall solve this equation for `LAM`, since we expect to substitute for this quantity later on. (In trial computations it was found that substituting for other quantities yielded slightly more complicated expressions.)

```
(C2) I2 : SOLVE(r: = e1(a), LAM)[1];
                             2                       2
(D2)       LAM = - tau + bet  + 2 alf bet + 3 r - 3 a  - mu
```

This entire equation may now be referred to as (I2).

Now we find the first propagation equations. The MACSYMA function `EV` is used to replace the occurrences of `LAM` in the differentiation results. Note that MACSYMA syntax necessitates the inclusion of the character "\" in the variables names I2,0 and I2,1 since "," is non-alphanumeric (see Mathlab [1977]).

```
(C3) I2\,0 : RATSIMP(EV(e0('I2), I2);
                        2                                 2
(D3) 0 = ((2 bet udt + (4 a bet - 2 a alf) udt + (2 a - 2 r) bet
                        2
         + (2 r - 2 a ) alf) X + 2 bet udt )/X
                                          x
(C4) I2\,1 : RATSIMP(EV(e1('I2), I2));
                                   2         2
            (4 a tau + E eps - 2 a bet + 2 a alf ) X - 2 alf bet + mu
                                                               x      x
(D4)  0 = - -----------------------------------------------------------
                                         X
```

Just as with the constraint (I1), we find we must consider two disjoint cases: $\beta \equiv 0$ on the open set or $\beta \neq 0$ on the open set. If $\beta \equiv 0$, then I2,0 is an algebraic constraint and I2,1 specifies $\partial_1 \mu$. If $\beta \neq 0$, then I2,0 specifies $\partial_1 \dot{u}$ and I2,1 relates two unknown derivatives.

We now consider the more general case $\beta \neq 0$, leaving the simpler case $\beta \equiv 0$ until later.

(I2) $_{123}S = 0$ ($\beta \neq 0$)

First we reformulate I2,0:

```
(C5) I2\,0 : SOLVE(I2\,0, DIFF(udt, x))[1];

(D5)  udt =
         x
                   2                            2                 2
        (bet udt + (2 a bet - a alf) udt + (a - r) bet + (r - a ) alf) x
      - ------------------------------------------------------------------
                                     bet
```

This gives a value for $\partial_1 \dot{u}$ which we may substitute into the specification of $\partial_0 \alpha$.

```
(C6) GRADEF(udt, x, RHS(I2\,0))$

(C7) GRADEF (alf, t, EV(DIFF(alf, t), DIFF))$
```

The propagation equation (I2,1), along $\mathbf{e}_1$ relates the derivatives $\partial_1\mu$ and $\partial_1\alpha$. We shall split this equation by introducing a new variable, `M1`, defined by

$$\partial_1\mu = M1 \; .$$

The propagation is then treated as follows.

```
(C8)    DEPENDS(M1, COORDS) $

(C9)    I2\,1a : DIFF(mu, x) = M1*X$

(C10)   GRADEF(mu, x, RHS(I2\,1a))$

(C11)   I2\,1b : SOLVE(EV(el('I2), I2), DIFF(alf, x))[1];
                                             2           2
(D11)   alf  = (4 a tau + E eps + M1 - 2 a bet  + 2 a alf )X
           x   ------------------------------------------
                                   2 bet

(C12)   GRADEF(alf, x, RHS(I2\,1b))$
```

Equations (I2,1a) and (I2,1b) specify $\partial_1\mu$ and $\partial_1\alpha$ respectively. Since we already know $\partial_0\mu$ and $\partial_0\alpha$, we must check for consistency through the commutation relationships for the two unknowns, μ and α. We shall examine the commutator for μ first since it is the simpler of the two.

```
(C13) I2\,1a\* :"SOLVE(EV(CR(mu), I2), DIFF(M1, t))[1];
                                             2
(D13) M1   = - ((4 p + 4 mu) a tau +(- 4 bet  - 2 alf bet + p + mu) E eps
        t

          2                                        2
+ (4 bet  + 4alf bet + p + mu) M1 + (2 p + 2 mu) a bet

                                                2
+ (- 4 P - 4 mu) a alf bet + (2 p + 2 mu) a alf )/(2 bet F)

(C14) GRADEF(M1, t, RHS(I2\,1a\*))$
```

Equation (I2,1a*) is the first specification for a derivative of M1 so this branch of the investigation is complete and we may proceed to examine the commutator for α.

```
(C15) I2\,1b\* : RATSIMP(EV(CR(alf), I2));
        (bet + alf) E eps
(D15) 0 = -----------------
               bet
```

At least one of the factors in the numerator must vanish. We shall now examine the vanishing of each factor in turn and show that $\epsilon = 0$ is a necessary and sufficient condition for (I2,1b*) to hold.

We shall first examine the case $\beta + \alpha = 0$.

```
(C16) I2\,1b\*A : bet =-alf $
```

The first propagation equations of (I2,1b*A) are computed as follows.

```
(C17) I2\,1b\*A0 : SOLVE(EV(eO('I2\,1b\*A), I2, I2\,1b\*A), mu)[1];

                                   2           2
(D17) mu = - 4 a udt - 2 tau - 2 alf + 4 r - 4 a - p
(C18) I2\,1b\*A1 : SOLVE(EV(el('I2\,1b\*A), I2, I2\,1b\*A), M1)[1];

                                                   2
(D18) m1 = - 4 a tau - E eps + 2 a alf bet - 2 a alf
```

Comparing two of the second propagation equations of (I2,1b*a) shows we must have $\epsilon E = 0$:

```
(C19) I2\,1b\*A01:
   RATSIMP(EV(e1(RHS('I2\,1b\*A0) - LHS('I2\,1b\*A0) = 0),
                      I2, I2\,1b\*A, I2\,1b\*AO, I2\,1b\*A1));
                     2     2                         2            3
(D19) (-2 tau - 2 alf + 8 a ) udt - 4a tau - 4 a alf - 8a r + 8 a = 0

(C20) I2\,1b\*A10:
   RATSIMP(EV(e0(RHS('I2\,1b\*A1) - LHS('I2\,1b\*A1) = 0),
                      I2, I2\,1b\*A, I2\,1b\*AO, I2\,1b\*A1));

                          3      2
(D20) (- 4 alf tau - 4 alf + 16 a alf) udt - 8a alf tau - 2 alf E eps
                 3       3
       - 8 a alf + (16 a - 16 a r) alf = 0
(C21) RATSIMP(I2\,1b\*A01 - I2\,1b\*A10/(2*alf));
(D21) E eps = 0
```

Hence the vanishing of $\beta + \alpha$ is merely a special case of the vanishing of ϵE.

We now examine the vanishing of the second factor, E, in (I2,1b*).

```
(C22) I2\,1b\*B : E = 0$
```

The first propagation equations of (I2,1b*B) are

```
(C23) EV(e0('I2\,1b\*B), I2\,1b\*B);
(D23) 0 = 0 ,
```

an identity, and

```
(C24) EV(e1('I2\,1b\*B), I2\,1b\*B);
(D24) eps = 0
```

So the vanishing of E is merely a special case of the vanishing of ϵ. Since $\alpha + \beta = 0$ necessitates $\epsilon E = 0$, we must have $\epsilon = 0$ in that case as well.

Finally, we examine the vanishing of ϵ.

```
(C25) I2\,1b\*C : eps = 0$
```

We only know one of the derivatives of ϵ, namely $\partial_0\epsilon$, so it remains to verify that the $\mathbf{e}_0$ propagation equation holds.

```
(C26) EV(e0('I2\,1b\*C), I2\,1b\*C);
(D26) 0 = 0
```

Thus this propagation yields an identity.

We have found all the necessary conditions for this specialization to be consistent. These conditions are, in summary:

$$\left.\begin{aligned}
&\beta \neq 0 \\
&\Lambda + \mu + \tau - \beta(2\alpha+\beta) + 3(a^2 - r) = 0 && \text{(I2)} \\
&\epsilon = 0 && \text{(I2, 1b} * \text{C)} \\
&\partial_1 \dot{u} = -\dot{u}^2 + \beta^{-1}(a^2 - r)(\alpha - \beta) + a\dot{u}\beta^{-1}(\alpha - 2\beta) && \text{(I2, 0)} \\
&\partial_1 \alpha = \tfrac{1}{2}\beta^{-1}[\partial_1\mu + 4a\tau - 2a(\beta^2 - \alpha^2)] && \text{(I2, 1)} \\
&\partial_0\partial_1\mu = -(\mu + p)\tfrac{\beta^{-1}}{2}\partial_1\mu + 4a\tau + 2a(\beta^2 - \alpha^2) \\
&\qquad\qquad - 2\partial_1\mu(\beta + \alpha) && \text{(I2, 1a}*\text{)}
\end{aligned}\right\} \quad (4.3.1)$$

as well as the system (LRSII). The process we have followed is given diagramatically in figure 4.3.1.

(I2&5) $_{123}S = 0$ *and* $_{23}R = 0$ $(\beta \neq 0)$

The case of $_{123}S$=0 with $\beta \neq 0$ does not require $r \neq 0$. We may therefore impose (I5) on the system given by (LRSII) and (4.3.1) without requiring any additional conditions. This is shown in figure 4.3.2. The specialization is consistent and (LRSII), (4.3.1) and (I5) hold.

(I2&6) $_{123}S = 0$ *and* $_{(0)23}\theta = 0$

We now consider the case when $_{123}S$=0 with $\beta = 0$. The constraint $_{(0)23}\theta$=0 $\Leftrightarrow$ $\beta = 0$, examined earlier, has been checked on the computer. The commands to do this have been placed in the file `MC:SWATT;CASE I6`, given in Appendix E. Loading this file after `MC:SWATT;LRSII SETUP` performs the necessary computations, specifying the derivatives with GRADEF and placing the additional algebraic constraints in a list named I6LIST.

```
(C1) BATCHLOAD("MC:SWATT\LRSII SETUP")$

MLOAD FASL DSK MACSYM being loaded
Loading done.

(C2) BATCH LOAD ("MC:SWATT\;CASE I6")$
```

We proceed to apply the constraint ${}_{123}S = 0$. To begin, we find the algebraic constraint equation.

```
(C3) I2&6 : EV(r  e1(a), I6LIST, EXPAND);
                 2  p   mu
(D3) r = a udt + a + - + --
                    2   2
```

We now compute the first propagation equations of (I2&6).

```
(C4) EV(e0(I2&6), I6LIST, EXPAND);
(D4) 0 = 0

(C5) I2&6\,1 : SOLVE(EV(e1('I2&6), I2&6, I6LIST), DIFF(mu, x))[1]
(D5) mu = - 4 a tau X
       x
(C6) GRADEF(mu, x, RHS(I2&6\,1))$
```

The ∂_0 propagation yields an identity, while the propagation along $\mathbf{e}_0$ specifies a new derivative, $\partial_1\mu$.

Since $\partial_0\mu$ is already known from (BI1), we must now check the integrability condition for μ.

```
(C7) EV(CR(mu), I2&6, I6LIST, EXPAND);
(D7) 0 = 0
```

We have obtained an identity so the specification of $\partial_1\mu$ is consistent. No equations remain to be checked so this completes the case.

These steps are depicted schematically in figure 4.3.3. The equations which must be satisfied are (I2&6) and (I2&6,1), as well as those of (4.1.1) and (LRSII).

(I2&5&6) ${}_{123}S = 0$ *and* ${}_{23}R = 0$ *and* ${}_{(0)23}\theta = 0$

The conditions for (I2) and (I6) to hold together were found in the case (I2&6). Upon the original system with these additional conditions, we may now impose the additional constraint (I5), since $r = 0$ is not specifically prohibitted. This is shown in figure 4.3.4.

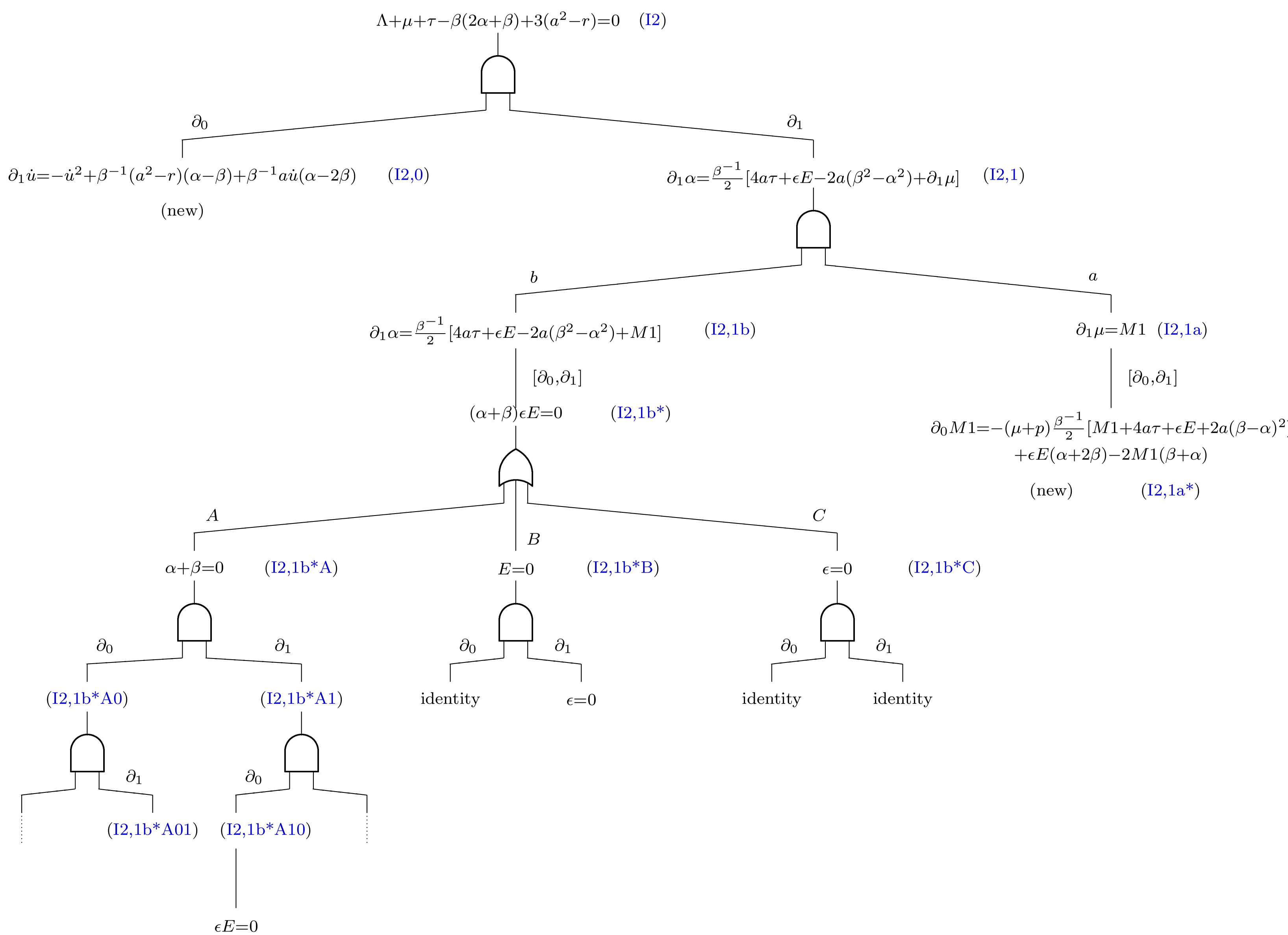

Figure 4.3.1: *I2:* $_{123}S=0$ *with* $\beta \neq 0$

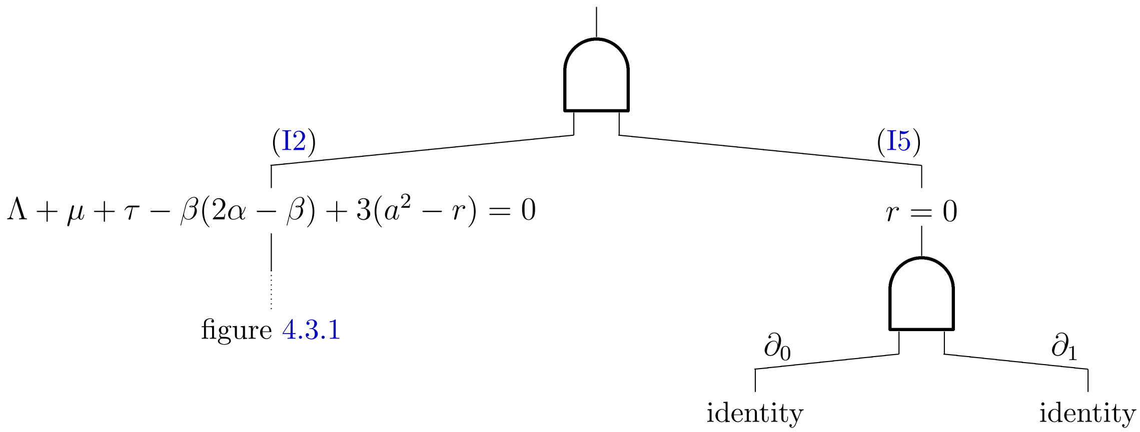

Figure 4.3.2: *I2&5:* ${}_{23}R=0$ *after imposing* ${}_{123}S=0$ *with* $\beta \neq 0$

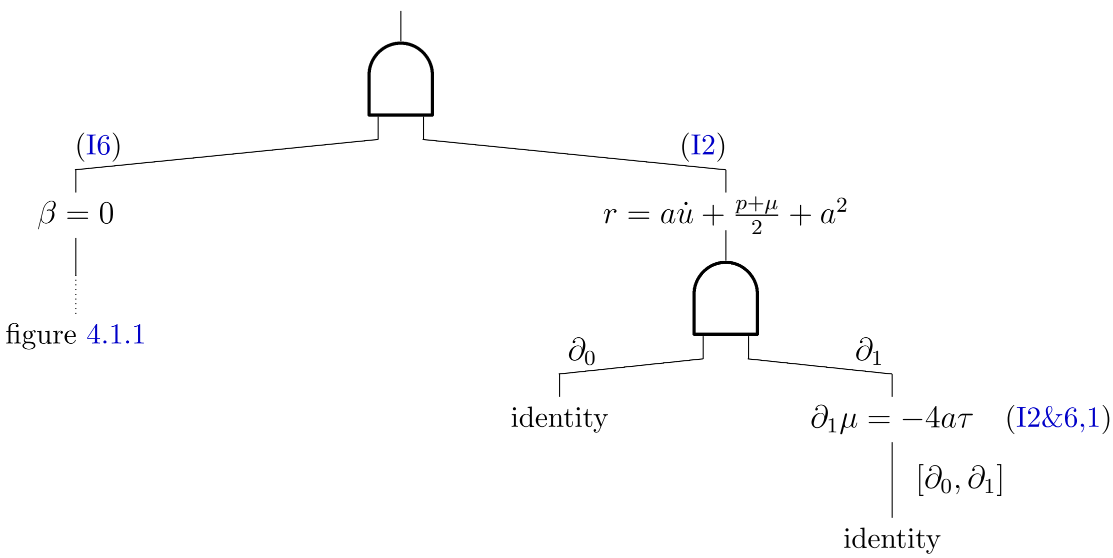

Figure 4.3.3: *I2&6:* ${}_{123}S=0$ *after imposing* ${}_{(0)23}\theta=0$

The equations satisfied in this specialization are (I5), (I2&6), (I2&6,1) and those of (4.1.1) and (LRSII).

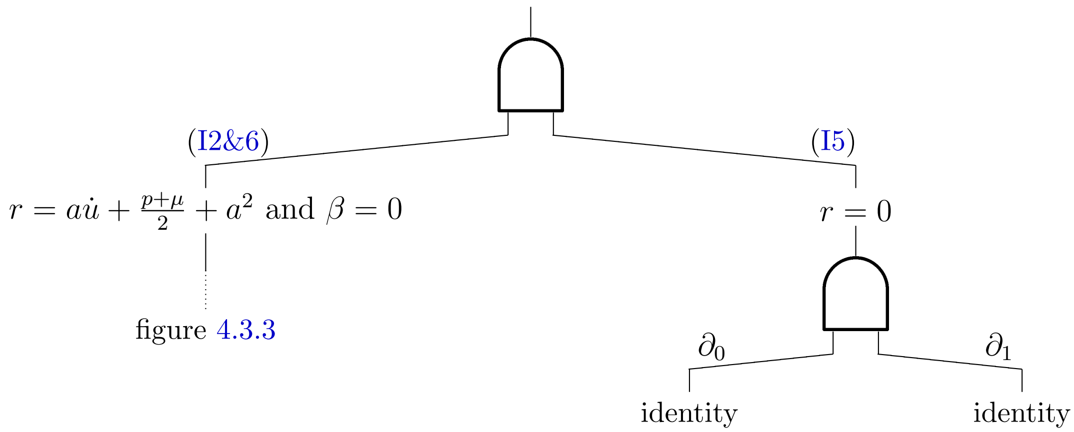

Figure 4.3.4: *I2&5&6: ${}_{23}R$=0 after imposing ${}_{123}S$=0 and ${}_{(0)23}\theta$=0*

4.4 ${}_{12}R = 0$ and Combinations of ${}_{23}R = 0$ and ${}_{(0)23}\theta = 0$

The intrinsic symmetry

$$ {}_{12}R = 0 \;\Leftrightarrow\; \partial_1 a = a^2 \;, \tag{I4} $$

like (I1) and (I2) divides naturally into two disjoint subcases depending on the vanishing or otherwise of ${}_{(0)23}\theta = 0$. The conditions necessitated by combinations of (I4), (I5) and (I6) are summarized below.

(I4) ${}_{12}\boldsymbol{R} = \mathbf{0}$ ($\boldsymbol{\beta} \neq \mathbf{0}$)

In this specialization we have

$$\left.\begin{aligned}
&\beta \neq 0 \\
&\Lambda + \tau + \mu + (a^2 - r) - \beta(2\alpha + \beta) = 0 && \text{(I4)} \\
&\epsilon = 0 && \text{(I4, 1b} * \text{C)} \\
&\partial_1 \dot{u} = \dot{u}\beta^{-1}(\alpha a - \beta \dot{u}) && \text{(I4, 0)} \\
&\partial_1 \alpha = \tfrac{\beta^{-1}}{2}[4a\tau + 2a(a^2 - r) + 2a(\alpha^2 - \beta^2) + \partial_1\mu] && \text{(I4, 1)} \\
&\partial_0\partial_1\mu = -\tfrac{(\mu+p)}{2}\beta^{-1}[4a\tau + 2a(a^2 - r) + 2a(\alpha - \beta)^2 + \partial_1\mu] \\
&\qquad\qquad\qquad\qquad - 2(\alpha + \beta)\partial_1\mu && \text{(I4, 1a}*\text{)}
\end{aligned}\right\} \quad (4.4.1)$$

as well as (LRSII). The equations are obtained in the manner depicted in figure 4.4.1.

(I4&5) $_{12}R = 0$ *and* $_{23}R = 0$ ($\beta \neq 0$)

The results here are the same as for (I4), except we also have $r = 0$. See figure 4.4.2.

(I4&6) $_{12}R = 0$ *and* $_{(0)23=0}\theta$

In addition to the system (LRSII) and the equations associated with $\beta = 0$, we also have the algebraic representation of the constraint,

$$p + \mu + 2a\dot{u} = 0 \ , \tag{I4\&6}$$

and the following propagation equation:

$$\partial_1 \mu = -4a\tau - 2a(a^2 - r) \ , \tag{I4\&6,1}$$

The steps to arrive at these equations are shown in figure 4.4.3.

(I4&5&6) $_{12}R = 0$ *and* $_{23}R = 0$ *and* $_{(0)23}\theta = 0$

Here the results are the same as for (I4&6), except that we also have $r = 0$. See figure 4.4.4.

4.5 $_{123}R = {}_{123}S = {}_{12}R = 0$ and Combinations of $_{23}R = 0$ and $_{(0)23}\theta = 0$

The constraints (I1), (I2) and (I4) are not independent. Using (4.0.1), it is easily seen that taking any two of them automatically gives the third.

$$\begin{aligned} {}_{123}R = {}_{123}S = 0 \ &\Leftrightarrow \ {}_{123}R = {}_{12}R = 0 \ \Leftrightarrow \ {}_{123}S = {}_{12}R = 0 \\ &\Leftrightarrow \ {}_{123}R = {}_{123}S = {}_{12}R = 0 \\ &\Leftrightarrow \ \partial_1 a = a^2 = r \end{aligned} \tag{I1\&2\&4}$$

Again we have two disjoint subcases, depending on whether or not β is zero.

(I1&2&4) $_{123}R = {}_{123}S = {}_{23}R = 0$ ($\beta \neq 0$)

We start by assuming all the conditions required for (I1) with $\beta \neq 0$. Then, in adding $_{123}S = 0$, we find the additional conditions necessary for this case are

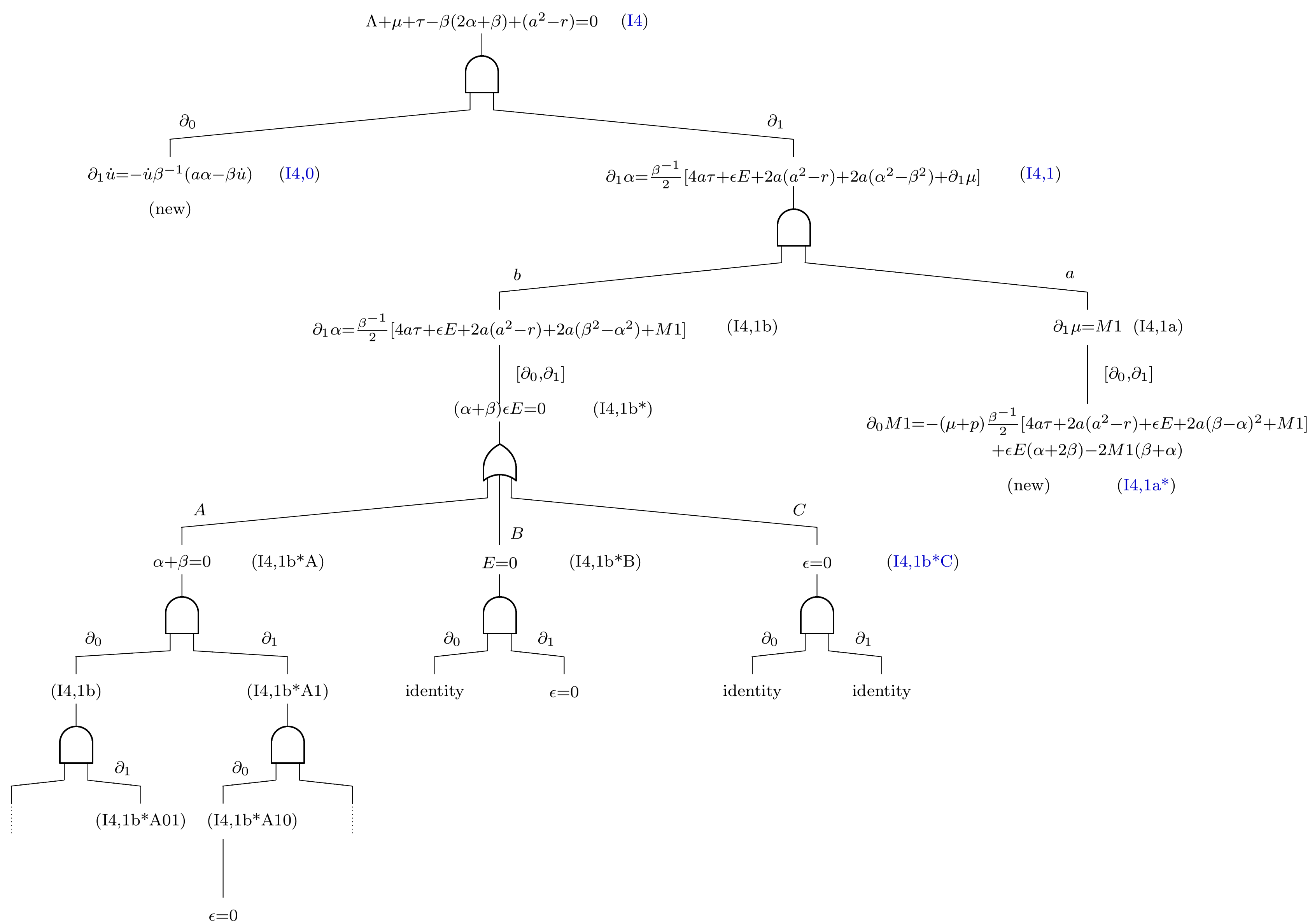

Figure 4.4.1: *I4:* ${}_{12}R$*=0 with* $\beta \neq 0$

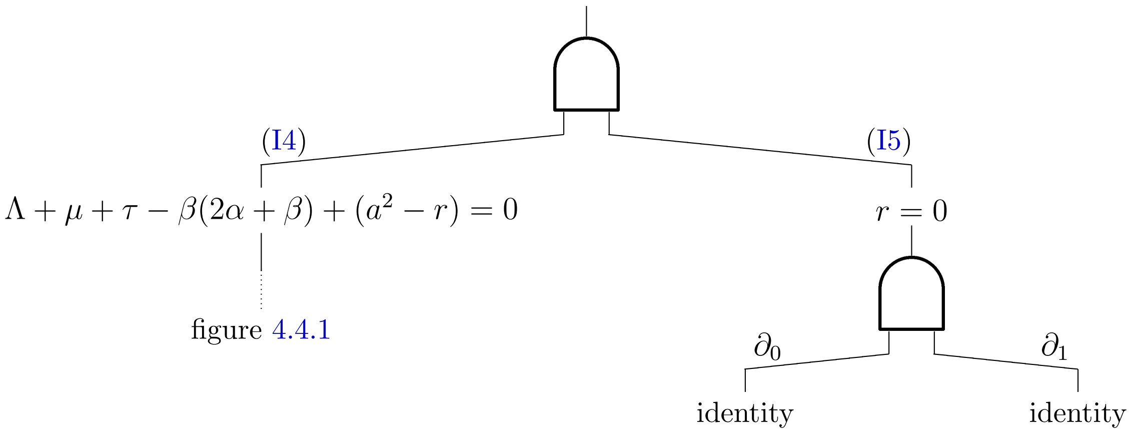

Figure 4.4.2: *I4&5:* $_{23}R$*=0 after imposing* $_{12}R$*=0 with* $\beta \neq 0$

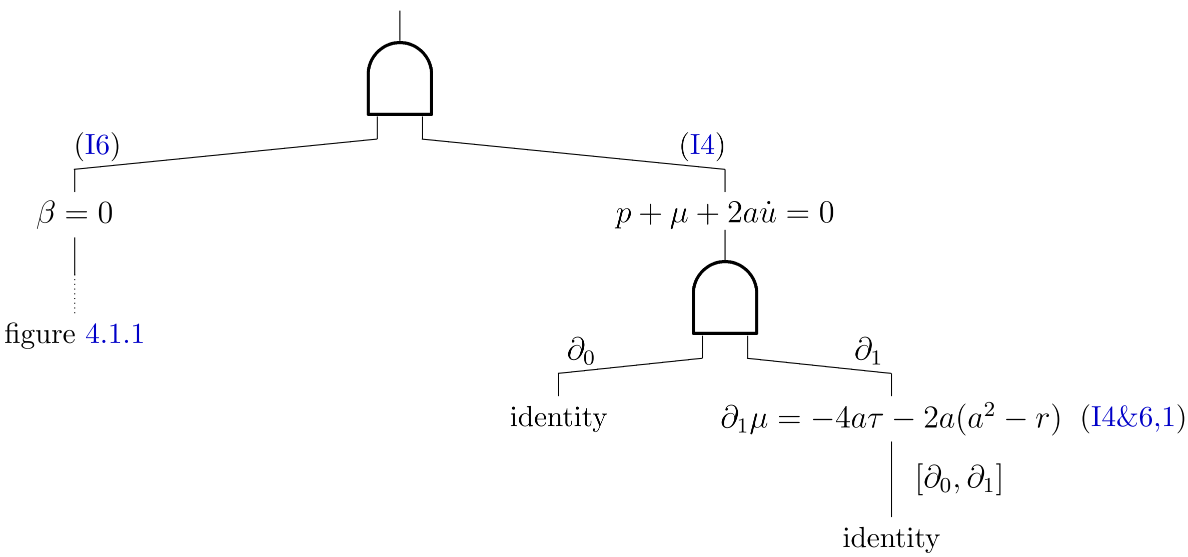

Figure 4.4.3: *I4&6:* $_{12}R$*=0 after imposing* $_{(0)23}\theta$*=0*

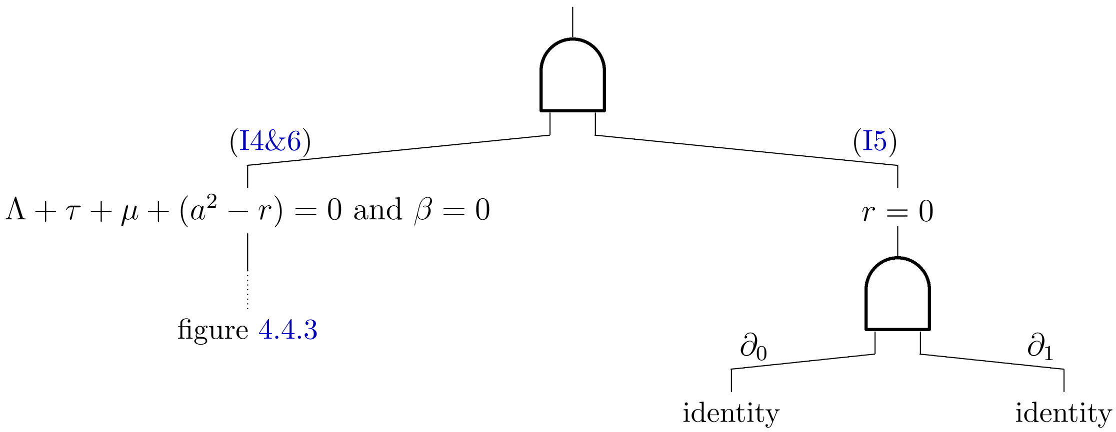

Figure 4.4.4: *I4&5&6: ${}_{23}R$=0 after imposing ${}_{12}R$=0 and ${}_{(0)23}\theta$=0*

$$a^2 - r = 0 \tag{I1\&2\&4}$$

and

$$\dot{u} = 0 \tag{I1\&2\&4,0}$$

(see figure 4.5.1). Note that (F3) now reduces to $\partial_1 a = a^2$.

(I1&2&4&5) ${}_{123}\boldsymbol{R} = {}_{123}\boldsymbol{S} = {}_{12}\boldsymbol{R} = {}_{23}\boldsymbol{R} = \mathbf{0}$ ($\boldsymbol{\beta} \neq \mathbf{0}$)

We first assume all the conditions of the case (I1&2&4). In particular note that $r = a^2$. Now, with ${}_{23}R = 0 \Leftrightarrow\ r = 0$, we may not have $\beta = 0$, since $a = \beta = 0$ has been shown to give a contradiction. However, this problem cannot arise in this case.

Setting $r = 0$ in case (I1&2&4) gives no further conditions through propagation so we have a consistent specialization. See figure 4.5.2.

(I1&2&4&6) ${}_{123}\boldsymbol{R} = {}_{123}\boldsymbol{S} = {}_{12}\boldsymbol{R} = \mathbf{0}$ *and* ${}_{(0)23}\boldsymbol{\theta} = \mathbf{0}$

We first assume all the conditions required for (I1&6). Then imposing (I2), we find that

$$a^2 - r = 0 \tag{I1\&2\&4\&6}$$

and

$$\dot{u} = 0 \tag{I1\&2\&4\&6,0}$$

are the only additional conditions necessary. This is depicted in figure 4.5.3.

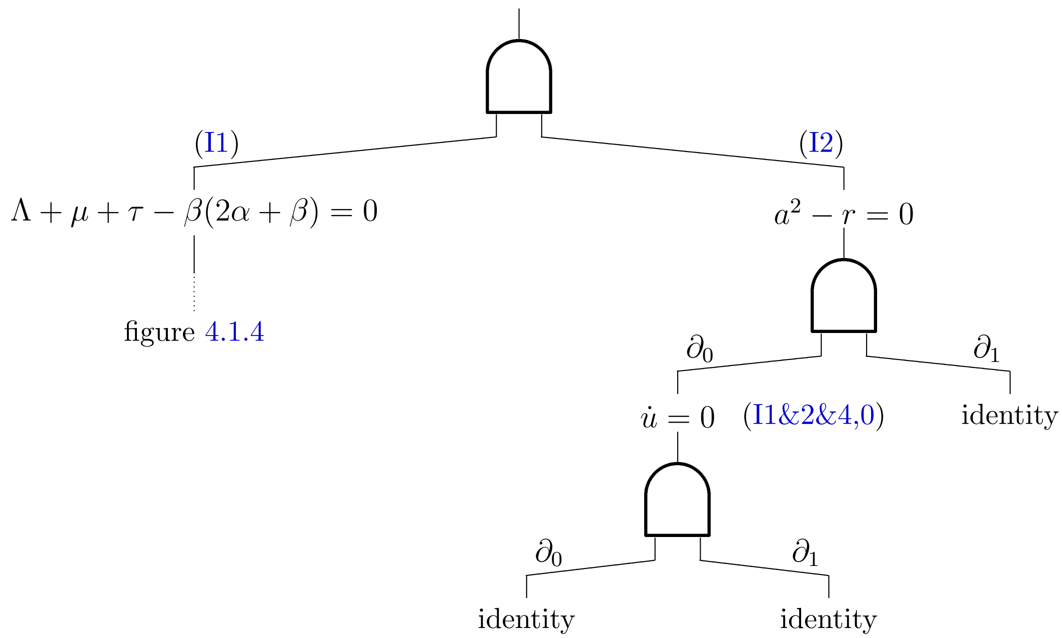

Figure 4.5.1: *I1&2&4:* ${}_{123}S=0$ *after imposing* ${}_{123}R=0$ *with* $\beta \neq 0$

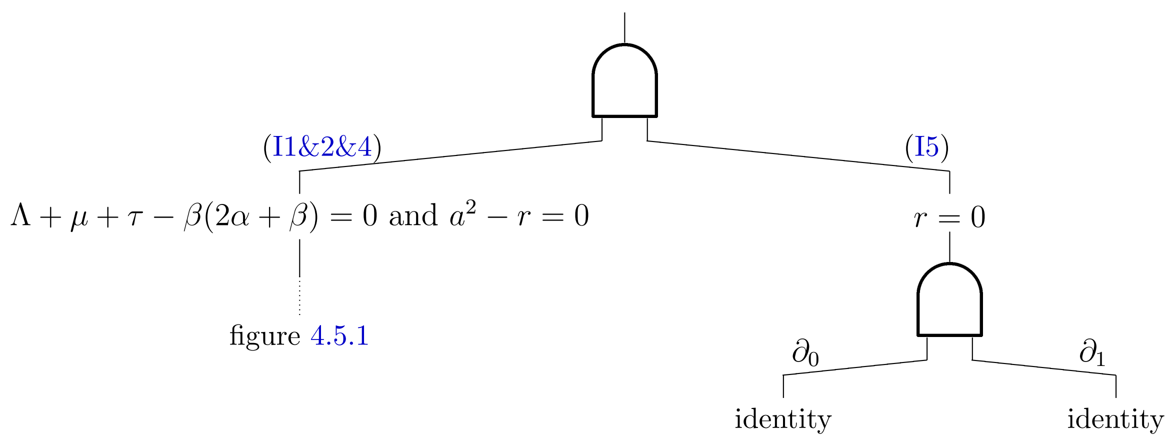

Figure 4.5.2: *I1&2&4&5:* ${}_{23}R=0$ *after imposing* ${}_{123}R={}_{123}S={}_{12}R=0$ *with* $\beta \neq 0$

(I1&2&4&5&6) ${}_{123}R = {}_{123}S = {}_{12}R = 0$ *and* ${}_{23}R = 0$ *and* ${}_{(0)23}\theta = 0$

In case (I1&2&4&6) we have $a^2 = r$ and $\beta = 0$. If we additionally impose ${}_{23}R=0$ $\Leftrightarrow$ $r = 0$, then $a = \beta = 0$ leads to a contradiction, as indicated in figure 4.5.4. This specialization of the LRS type II solutions is inconsistent and yields an empty class.

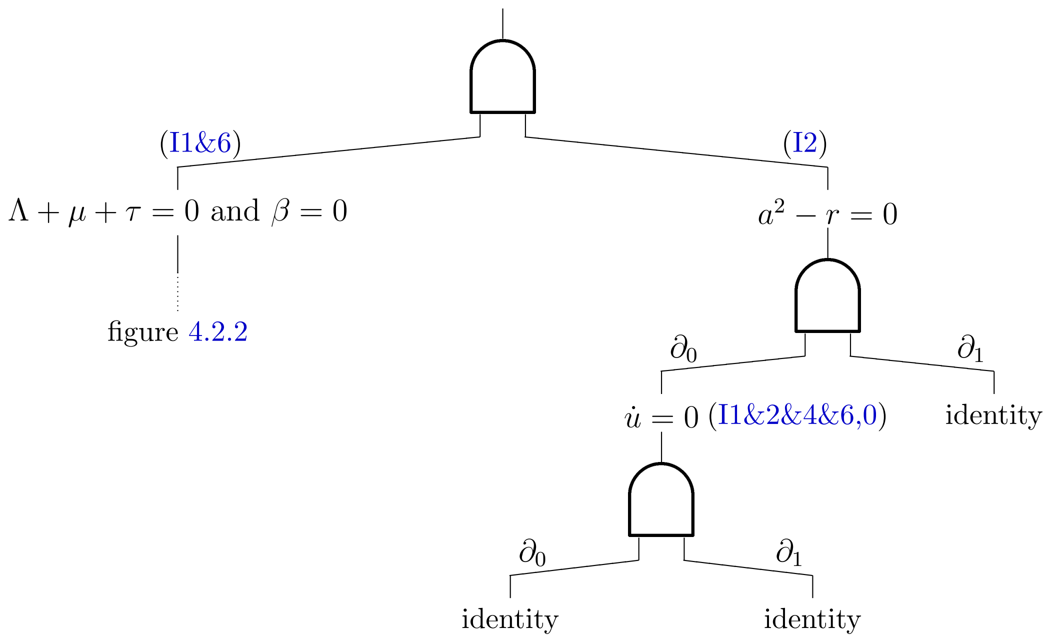

Figure 4.5.3: *I1&2&4&6:* $_{123}S=0$ *after imposing* $_{123}R=0$ *and* $_{(0)23}\theta=0$

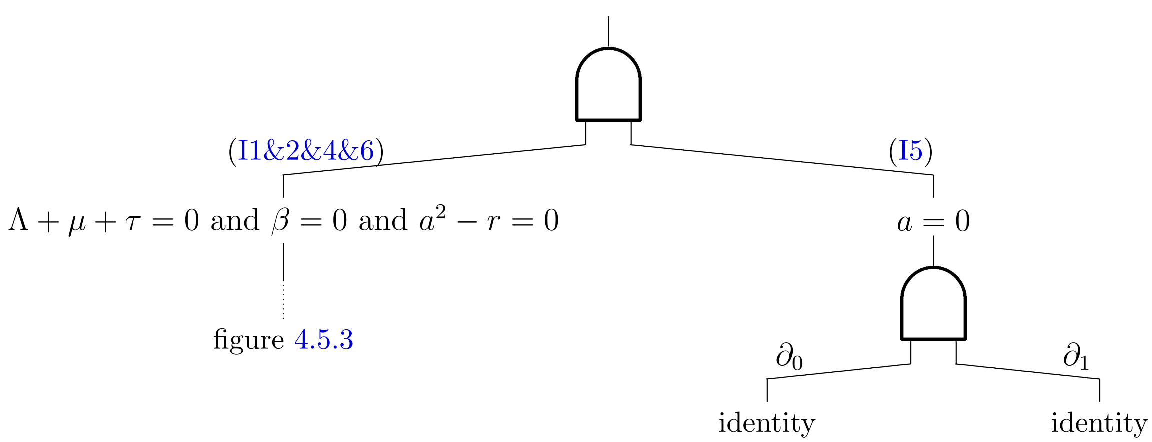

Figure 4.5.4: *I1&2&4&5&6:* $_{23}R=0$ *after imposing* $_{123}R={}_{123}S={}_{12}R=0$ *and* $_{(0)23}\theta=0$

4.6 Summary

We have now examined all possible combinations of the intrinsic symmetries given at the outset of this chapter. The inter-relation between these combinations is summarized in two specialization diagrams: figures 4.6.1 and 4.6.2. Figure 4.6.1 gives the relationship between the combinations of intrinsic symmetries in the general case when $\beta \neq 0$. For the special case $\beta = 0$, the relationship is given in figure 4.6.2. In these diagrams, each box represents the class of space-times considered in the subsection of that title and an arrow denotes the inclusion of one class in another. For example, in figure 4.6.1 the class of space-times which admit (I2) includes all space-times in the class with box labelled I2&5. Since (I3) holds identically for all LRS type II space-times, it has not been included in these diagrams.

The LRS type II space-times may then be characterized by division into these classes. The classes which may not contain space-times of type IIc are I1&2&4, I1&2&4&5 and I1&2&4&6, since for these classes $\dot{u} = 0$. All solutions from I1&2&4&5 will have a $= 0$ and so are of type IIa. The space-times in the class I1&2&4&6 have $\beta = 0$, which implies $a \neq 0$, so are of type IIb.

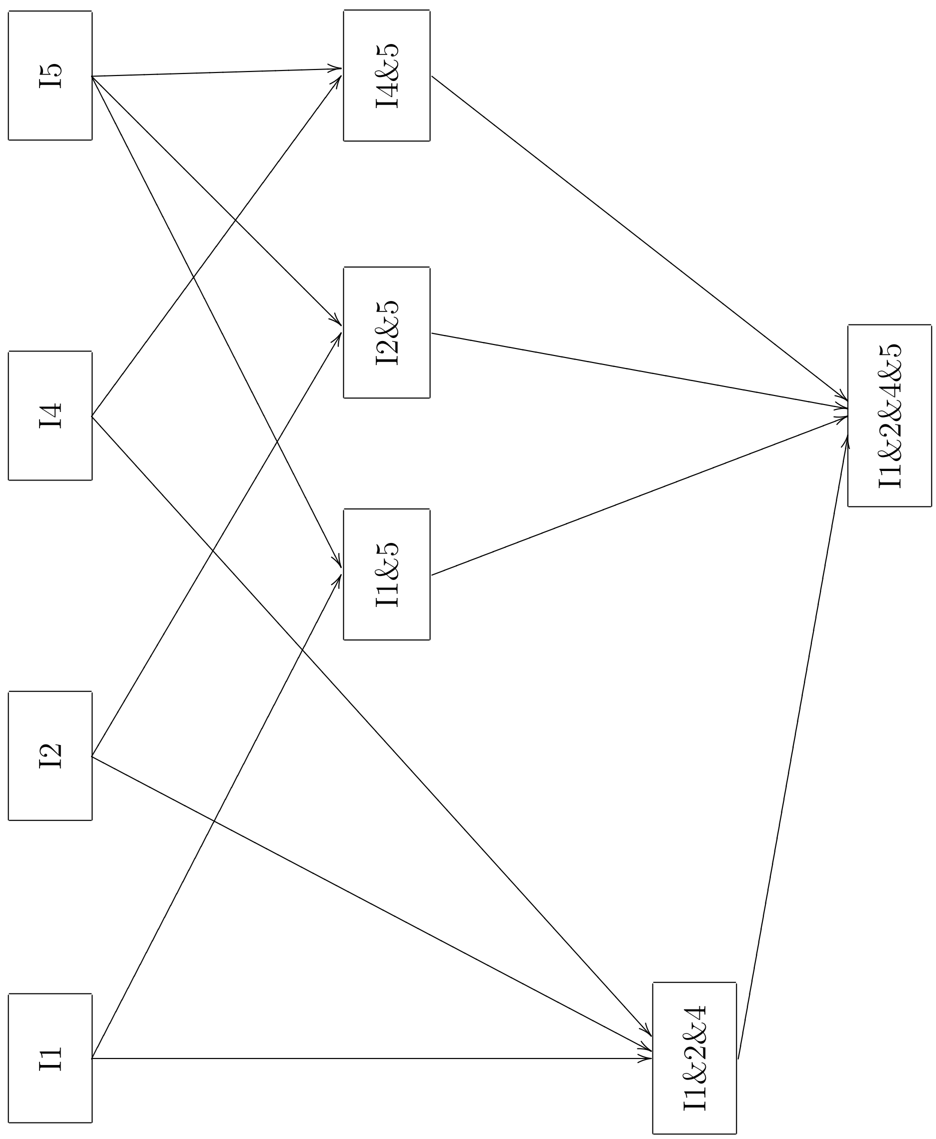

Figure 4.6.1: *Specialization Diagram with* $\beta \neq 0$

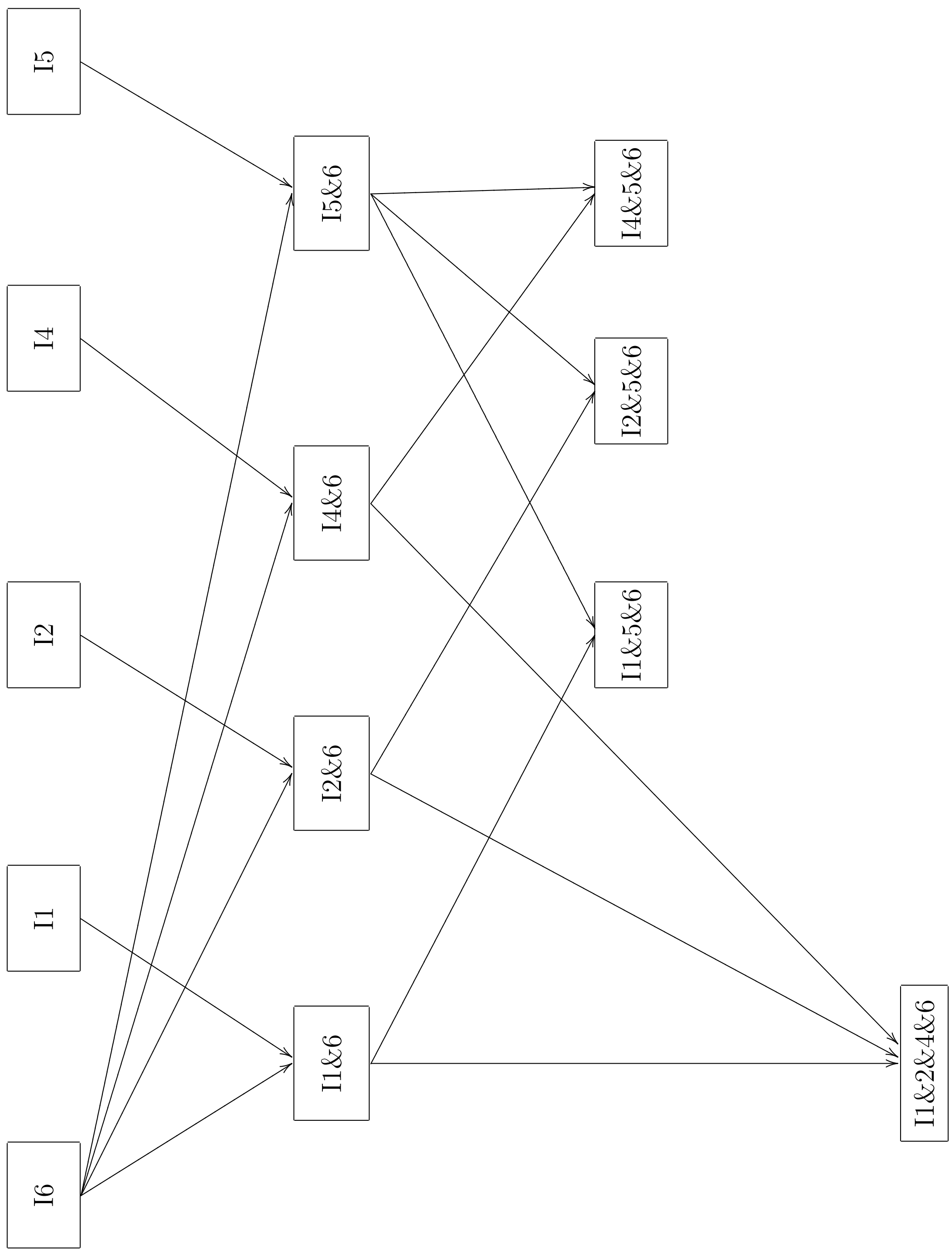

Figure 4.6.2: *Specialization Diagram with* $\beta = 0$

Chapter V

FURTHER RESULTS FOR LRS CLASS II USING INTRINSIC SYMMETRIES

In this chapter, we give some further results for the LRS class II space-times. In section 5.1 we show that the method of Chapter 2 does not always terminate, by showing that the intrinsic symmetry ${}_{01}R$=0 leads to an infinite sequence of computations. In section 5.2 we give results for intrinsic symmetries related to the congruence defined by the spacelike axis of symmetry in the fluid restspace at each point of a space-time. Finally, in section 5.3 we examine the consequences of imposing an equation of state of the form $A(p, \mu) = 0$.

5.1 A Non-terminating Process ${}_{01}R = 0$

So far, whenever we have used the procedure given in Chapter 2, the process has terminated after a finite number of steps. There are situations, however, where this does not happen. The following example, using the constraint propagation procedure for a simple partial differential equation, is such a case.

Example 5.1.1. We shall attempt to impose Laplace's equation,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 ,$$

as a constraint on the class of C^∞ real functions of two variables, x and y. The differential operators are ∂_x and ∂_y and the commutator is $[\partial_x, \partial_y] = 0$. The original system is given by

$$\partial_x f = X \qquad \partial_y f = Y$$
$$\partial_y X = A \qquad \partial_x Y = A ,$$

where we have $\partial_y X = \partial_x Y$ by $[\partial_x, \partial_y]$f $= 0$. At this stage the functions f, X ,Y and A are arbitrary apart from the four equations.

We now impose Laplace's equation as a constraint:

$$\partial_x X + \partial_y Y = 0 .$$

This equation may be split up as

$$\partial_x X = B , \quad \partial_y Y = -B .$$

Applying the commutator to X gives

$$\partial_x A = \partial_y B \;\Rightarrow\; \partial_x A = C,\; \partial_y B = C$$

and to Y gives

$$\partial_x B = -\partial_y A \;\Rightarrow\; \partial_y A = D,\; \partial_x B = -D\ .$$

Notice that the propagation equations for A and B are the same as those for X and Y with the unknowns relabelled. Therefore the application of the commutator will lead to an infinite sequence of equations, which is inconclusive from the present point of view.

When a situation such as this occurs, another method of approach may yield more information. For instance, here if we introduce

$$D := \partial_x + i\partial_y$$

then the original system plus Laplace's equation is equivalent to

$$D(X - iY) = 0\ .$$

This is a single propagation equation for X - iY using D, so we may impose it as a constraint. However, this and similar methods are unsatisfactory in that they are not systematic, as we require.

In general, the procedure of Chapter 2 may involve checking three commutators at any given stage and whether or not the process terminates depends on how the equations are coupled. The exact nature of this dependency has not been thoroughly studied by the present author.

This problem can occur in the investigation of intrisinc symmetries, even in the case of LRS type II space-times where only one commutator is involved. We shall now describe such a case.

With the usual tetrad, we have in LRS type II space-times that $\mathbf{e}_0$ and $\mathbf{e}_1$ are surface-forming. Defining ${}_{01}\mathscr{S} \in \mathscr{F}(\{\mathbf{e}_0, \mathbf{e}_1\})$, we find

$$R({}_{01}\mathscr{S}) = 2(\partial_0\alpha + \alpha^2 - \partial_1\dot{u} - \dot{u}^2)$$

and

$$S({}_{01}\mathscr{S}) = \theta({}_{01}\mathscr{S}, \mathbf{n}) = \sigma({}_{01}\mathscr{S}, \mathbf{n}) = 0$$

for all $\mathbf{n}$ such that $\mathbf{n}\cdot\mathbf{e}_0 = \mathbf{n}\cdot\mathbf{e}_1 = 0$. We shall examine the consequences of imposing $R({}_{01}\mathscr{S}) = 0$:

$$\partial_0\alpha = \partial_1\dot{u} + \dot{u}^2 - \alpha^2\ . \tag{5.1.1}$$

Using (F3), (5.1.1) reduces to

$$2\tau + \frac{p+\mu}{2} - \beta^2 + a^2 - r = 0\ . \tag{5.1.1}$$

Employing the usual methods we find the first propagation equations are

$$\partial_0 p = \beta(6\tau - 3p - \mu + 2\Lambda) + \alpha(p + \mu) \qquad (5.1.2,0)$$

and

$$\partial_1 \mu = -a(6\tau - 3p - \mu + 2\Lambda) + \dot{u}(p + \mu) - 3\epsilon E \ . \qquad (5.1.1,1)$$

The commutaion relation for p yields

$$\begin{aligned}\partial_1 \alpha = -\partial_0 \dot{u} &+ 2(3p + \mu - 6\tau - 2\Lambda)(\beta\dot{u} + \beta a + \alpha a)(\mu + p)^{-1} \\ &+ (4\epsilon E(2\beta - 2\alpha) - 24a\beta\tau)(\mu + p)^{-1} \\ &- 2a\dot{u}\end{aligned} \qquad (5.1.1,0^*)$$

and, using this, the commutation relation for μ gives

$$\beta \in E = 0 \qquad (5.1.1,1^*)$$

which in turn implies

$$\epsilon = 0 \ . \qquad (5.1.1,1^*c)$$

The pair of equations (5.1.1) and (5.1.1,0*) is of the form

$$\left.\begin{aligned}\partial_0 \alpha &= \partial_1 \dot{u} + G_1 \\ \partial_1 \alpha &= -\partial_0 \dot{u} + H \ .\end{aligned}\right\} \qquad (5.1.2)$$

This leads, in a manner similar to example 5.1.1, to an infinite sequence of equations.

When a stage such as this is reached the condition for the constraint to give a consistent specialization is that the pair of equations admit a solution. In the case of (5.1.2) the pair of equations is equivalent to a single partial differential equation, second order in $\dot{u}$ and α with over one hundred terms.

Under some circumstances, additional ground may be gained by introducing complex differential operators. This approach has not been found to be useful in dealing with (5.1.2), since both $\dot{u}$ and α appear in G and H, leading to unintelligibly lengthy expressions.

5.2 Intrinsic Symmetries Related to the Spacelike Congruence

In Chapter 4 we examined intrinsic symmetries related to the timelike congruence defined by the fluid flow in LRS type II space times. We shall now consider the congruence of curves tangent to the rest space axis of symmetry, $\mathbf{e}_1$. Examining the commutators (1.3.2) shows that for LRS type II space-times the $\mathbf{e}_1$ congruence is hypersurface orthogonal. In fact, any subset of $\{\mathbf{e}_0, \mathbf{e}_2, \mathbf{e}_3\}$ is surfaceforming. In this section, we investigate the intrinsic symmetries related to the $\mathbf{e}_1$ congruence, analogous to those studied in Chapter 4. The equations are:

$$\left.\begin{array}{llll} {}_{023}R := R({}_{023}\mathscr{S}) = 0 & \Leftrightarrow & r = -2\partial_0\beta - 3\beta^2 & \text{(I7)} \\ {}_{023}S := S({}_{023}\mathscr{S}) = 0 & \Leftrightarrow & r = \partial_0\beta & \text{(I8)} \\ {}_{023}T := T({}_{023}\mathscr{S}) = 0 & & \text{(holds identically)} & \text{(I9)} \\ {}_{02}R := R({}_{02}\mathscr{S}) = 0 & \Leftrightarrow & 0 = \partial_0\beta + \beta^2 & \text{(I10)} \\ {}_{23}R := R({}_{23}\mathscr{S}) = 0 & \Leftrightarrow & r = 0 & \text{(I5)} \\ {}_{(1)23}\theta := \theta({}_{23}\mathscr{S}, \mathbf{e}_1) = 0 & \Leftrightarrow & a = 0 & \text{(I11)} \end{array}\right\} \tag{5.2.1}$$

The subspaces are defined by

$$\begin{array}{l} {}_{023}\mathscr{S} \in \mathscr{F}_1(\{\mathbf{e}_0, \mathbf{e}_2, \mathbf{e}_3\}) \\ {}_{02}\mathscr{S} \in \mathscr{F}_2(\{\mathbf{e}_0, \mathbf{e}_2\}) \in \{\mathscr{F}_2(\{\mathbf{e}_0, \mathbf{v}\}) | \mathbf{v}\cdot\mathbf{e}_0 = \mathbf{v}\cdot\mathbf{e}_1 = 0\} \end{array}$$

and

$${}_{23}\mathscr{S} \in \mathscr{F}_2(\{\mathbf{e}_2, \mathbf{e}_3\})$$

with the usual tetrad. Each of the families $\mathscr{F}_1(\{\mathbf{e}_0, \mathbf{e}_2, \mathbf{e}_3\})$ $\{\mathscr{F}_2(\{\mathbf{e}_0, \mathbf{v}\}) | \mathbf{v}\cdot\mathbf{e}_0 = \mathbf{v}\cdot\mathbf{e}_1 = 0\}$ and $\mathscr{F}_2(\{\mathbf{e}_2, \mathbf{e}_3\})$ is geometrically well defined and $\mathscr{F}_2(\{\mathbf{e}_0, \mathbf{e}_2\})$ is a representative element of its class.

We shall summarize the results for these symmetries applied individually and in combinations. The intrinsic symmetries, (I5) and (I11), related to the group orbits will be dealt with first. Each of (I7), (I8) and (I10), then the three together will be summarized by themselves and with combinations of (I5) and (I11). The details of the calculations are given in Appendix F.

1.) *Combinations of* ${}_{23}R = 0$ *and* ${}_{(1)23}\theta = 0$ *.*

(I5) $\boldsymbol{{}_{23}R = 0}$

We have already examined

$${}_{23}R = 0 \;\Leftrightarrow\; r = 0 \tag{I5}$$

in Chapter 4 and have found that no additional conditions are necessary.

(I11) $_{(1)23}\theta = 0$

The conditions found by propagating this constraint are:

$$\left.\begin{array}{ll} a = 0 & \text{(I11)} \\ \dot{u} = 0 & \text{(I11, 0)} \\ \Lambda + \mu + \tau - \beta^2 - 2\alpha\beta - r = 0 & \text{(I11, 1)} \\ \partial_1\mu = 2\beta\partial_1\alpha & \text{(I11, 11)} \\ \epsilon = 0 & \text{(I11, 11a * C)} \\ \partial_0\partial_1\mu = -\frac{\partial_1\mu}{2\beta}(4\beta^2 + 4\alpha\beta + p + \mu) & \text{(I11, 11b*)} \end{array}\right\} \qquad (5.2.2)$$

The steps followed are outlined in figure 5.2.1.

(I5&11) $_{23}R = 0$ *and* $_{(1)23}\theta = 0$

The results here are the same as for the previous case with $r = 0$.

2.) $_{023}R$=*0 and Combinations of* $_{23}R$ = *0 and* $_{(1)23}\theta$ = *0* .

Here the main constraint is

$$_{023}R = 0 \;\Leftrightarrow\; \partial_0\beta + 3\beta^2 + r = 0 \qquad \text{(I7)}$$
$$\Leftrightarrow\; \Lambda - p + \tau - 2a\dot{u} + a^2 = 0 \;.$$

The first propagation equations are

$$2a\partial_0\dot{u} + \partial_0 p + 2\beta(a^2 - \dot{u}^2 + 2\tau) = 0 \qquad \text{(I7,0)}$$

and

$$2a\partial_1\dot{u} + (a - \dot{u})(\beta^2 + 2\sigma\beta + r) - a(2a^2 - 2\dot{u}^2 + p + \mu + 4\tau) - 2\epsilon E = 0 \;. \qquad \text{(I7,1)}$$

Examination of (I7,0) and (I7,1) shows that we need to consider two disjoint cases: one in which $a = \frac{_{(1)23}\theta}{-2}$ vanishes identically on the open set and another in which a is never zero.

We have the following cases:

(I7) $_{023}R = 0$ ($a \neq 0$)

In this specialization , we must have

$$\left.\begin{aligned}
&\Lambda + \tau - p + a(a - 2\dot{u}) = 0 && \text{(I7)}\\
&\partial_0 p = -2a\partial_0\dot{u} + 2\beta(\dot{u}^2 - a^2 - 2\tau) && \text{(I7, 0)}\\
&\partial_1\dot{u} = a^2 - \dot{u}^2 + \tfrac{p+\mu}{2} + 2\tau + \tfrac{\epsilon E}{a} + \tfrac{(\dot{u}-a)}{2a}(\beta^2 + r + 2\alpha\beta) && \text{(I7, 1)}\\
&\partial_1\partial_0\dot{u} = -\beta(3a^2 + 2a\dot{u} - \dot{u}^2 + p + \mu + 10\tau) - (a + 3\dot{u})\partial_0\dot{u} &&\\
&-\tfrac{2\beta\epsilon E}{a^2}(2a - \dot{u}) + \tfrac{a^{-1}}{2}(\beta^2 + r + 2\alpha\beta)[\tfrac{2\beta}{a}(a^2 - a\dot{u} + \dot{u}^2) + \partial_0\dot{u}] &&\\
&+\alpha(a^2 - \dot{u}^2 + 2\tau) && \text{(I7, 0a*)}
\end{aligned}\right\} \quad (5.2.3)$$

See figure 5.2.2.

(I7&5) $_{023}R = 0$ *and* $_{23}R = 0$ ($a \neq 0$)

The results for this case are the same as for (I7) with r set to zero.

(I7&11) $_{023}R = 0$ *and* $_{(1)23}\theta = 0$

In addition to the system (LRSII) and (5.2.2) we must have

$$r = -\beta^2 - 2\alpha\beta + p + \mu \qquad \text{(I7\&11)}$$

and

$$\partial_0 p = -4\beta\tau\ . \qquad \text{(I7\&11,0)}$$

The steps followed are shown in figure 5.2.3.

(I7&5&11) $_{023}R = 0$ *and* $_{23}R = 0$ *and* $_{(1)23}\theta = 0$

The results for this case are the same as for (I7&5) with $r = 0$.

3.) $_{023}S$=*0 and Combinations of* $_{23}R$ = *0 and* $_{(1)23}\theta$ = *0* .

Just as with (I7) this case splits into two disjoint possibilities according as $a \neq 0$ and $a \equiv 0$.

(I8) $_{023}S = 0 \ (a \neq 0)$

Here we have

$$\left.\begin{aligned}
&\Lambda + \tau - p + a(a - 2\dot{u}) - 3(\beta^2 + r) = 0 && \text{(I8)}\\
&\partial_0 p = -2a\partial_0\dot{u} + 2\beta(\dot{u}^2 - a^2 - 2\tau) && \text{(I8, 0)}\\
&\partial_1\dot{u} = a^2 - \dot{u}^2 + \tfrac{p+\mu}{2} + 2\tau - \tfrac{(\dot{u}+2a)}{a}(\beta^2 + r - \alpha\beta) && \text{(I8, 1)}\\
&\partial_1\partial_0\dot{u} = -\beta(3a^2 + 2a\dot{u} - \dot{u}^2 + p + \mu + 10\tau) - (a + 3\dot{u})\partial_0\dot{u} &&\\
&-a^{-1}(\beta^2 + r - \alpha\beta)[\tfrac{2\beta}{a}(a^2 + 2a\dot{u} + \dot{u}^2) + \partial_0\dot{u}] &&\\
&+\alpha(a^2 - \dot{u}^2 + 2\tau) && \text{(I8, 0a}*)\\
&\epsilon = 0\ . && \text{(I8, 0b} * \text{C)}
\end{aligned}\right\} \quad (5.2.4)$$

See figure 5.2.4.

(I8&5) $_{023}S = 0$ ***and*** $_{23}R = 0 \ (a \neq 0)$

In this case the results are the same as for (I8) with r set to zero.

(I8&11) $_{023}S = 0$ ***and*** $_{(1)23}\theta = 0$

Here we have the results of (I11) and the additional relations

$$r = -\beta^2 + \alpha\beta - \frac{p + \mu}{2} \qquad \text{(I8\&11)}$$

and

$$\partial_0 p = -4\beta\tau\ . \qquad \text{(I8\&11,0)}$$

This results was achieved in exactly the same way as for (I7&11).

(I8&5&11) $_{023}S = 0$ ***and*** $_{23}R = 0$ ***and*** $_{(1)23}\theta = 0$

The results are those of (I8&11) with $r = 0$.

4.) $_{02}R$=*0 and Combinations of* $_{23}R = 0$ *and* $_{(1)23}\theta = 0$ *.*

As with (I7) and (I8) this case divides naturally depending on the vanishing or otherwise of a.

(I10) $_{02}R = 0$ $(a \neq 0)$

For this specialization we must have

$$\left.\begin{aligned}
&\Lambda + \tau - p + a(a - 2\dot{u}) - (\beta^2 + r) = 0 && \text{(I10)}\\
&\partial_0 p = -2a\partial_0\dot{u} + 2\beta(\dot{u}^2 - a^2 - 2\tau) + 2\beta(\beta^2 + r) && \text{(I10, 0)}\\
&\partial_1\dot{u} = a^2 - \dot{u}^2 + \tfrac{p+\mu}{2} + 2\tau - (\beta^2 + r) + \alpha\beta\tfrac{\dot{u}}{a} && \text{(I10, 1)}\\
&\partial_1\partial_0\dot{u} = -\beta(3a^2 + 2a\dot{u} - \dot{u}^2 + p + \mu + 10\tau) - (a + 3\dot{u})\partial_0\dot{u} &&\\
&\qquad\qquad - (\beta^2 + r)(3\beta - \alpha) + \tfrac{\alpha\beta}{a^2}(2\beta\dot{u}^2 + a\partial_0\dot{u}) &&\\
&\qquad\qquad + \alpha(a^2 - \dot{u}^2 + 2\tau) && \text{(I10, 0a*)}\\
&\epsilon = 0\ . && \text{(I10, 0b * C)}
\end{aligned}\right\} \quad (5.2.5)$$

See figure 5.2.5.

(I10&5) $_{02}R = 0$ *and* $_{23}R = 0$ $(a \neq 0)$

Here the results are the same as (I10) with $r = 0$.

(I10&11) $_{02}R = 0$ *and* $_{(1)23}\theta = 0$

In addition to the results of (I11), here we have

$$0 = \alpha\beta - \frac{\mu + p}{2} \qquad \text{(I10\&11)}$$

and

$$\partial_0 p = -4\beta\tau\ . \qquad \text{(I10\&11,0)}$$

(I10&5&11) $_{02}R = 0$ *and* $_{23}R = {}_{(1)23}\theta = 0$

The results here are the same as for (I10&11) with $r = 0$. In particular, note that we must put $r = 0$ in (5.2.2), the system of equations for (I11).

5.) ${}_{023}R$=${}_{023}S$=${}_{02}R$=*0 and Combinations of* ${}_{23}R = 0$ *and* ${}_{(1)23}\theta = 0$ *.*

Any two of (I7), (I8) and (I10) implies the third. That is,

$$\begin{aligned} {}_{023}R = {}_{023}S = 0 \;\Leftrightarrow\; & {}_{023}R = {}_{02}R = 0 \;\Leftrightarrow\; {}_{023}S = {}_{02}R = 0 \\ \Leftrightarrow\; & {}_{023}R = {}_{023}S = {}_{02}R = 0 \\ \Leftrightarrow\; & \partial_0\beta = -\beta^2 = r \ . \end{aligned} \qquad \text{(I7\&8\&10)}$$

Since each of (I7), (I8) and (I10) divide naturally into the disjoint cases $a \neq 0$ and $a \equiv 0$, so does their combination.

(I7&8&10) $\boldsymbol{{}_{023}R = {}_{023}S = {}_{02}R = 0\ (a \neq 0)}$

Imposing (I7) after (I8) we find the necessary conditions for this specialization are

$$\left.\begin{aligned} & r = -\beta^2 && \text{(I7\&8\&10)} \\ & \alpha = 0 && \text{(I7\&8\&10, 1B)} \\ & \epsilon = 0 && \end{aligned}\right\} \qquad (5.2.6)$$

in addition to the equations for (I7), (I8) or (I10). See figure 5.2.6.

(I7&8&10&5) $\boldsymbol{{}_{023}R = {}_{023}S = {}_{02}R = 0}$ ***and*** $\boldsymbol{{}_{23}R = 0\ (a \neq 0)}$

Here we impose the condition ${}_{23}R$=0 on the case (I7&8&10). If $r = 0$, then by equation (I7&8&10) $\beta = 0$. Since $a \neq 0$, we then have a consistent specialization.

(I7&8&10&11) $\boldsymbol{{}_{023}R = {}_{023}S = {}_{02}R = 0}$ ***and*** $\boldsymbol{{}_{(1)23}\theta = 0}$

Imposing (I11), (I7) and (I8), in that order, we find that we must have

$$\alpha\beta = \frac{p + \mu}{2} \qquad \text{(I7\&8\&1\&11)}$$

and

$$r = -\beta^2 \ , \qquad \text{(I7\&8\&10\&11,0)}$$

in addition to the equations (5.2.2) and (LRSII). See figure 5.2.7.

(I7&8&10&5&11) $_{023}\boldsymbol{R} = {}_{023}\boldsymbol{S} = {}_{02}\boldsymbol{R} = \mathbf{0}$ ***and*** $_{23}\boldsymbol{R} = \mathbf{0}$ ***and*** $_{(1)23}\boldsymbol{\theta} = \mathbf{0}$

We may not have $r = 0$ in (I7&8&10&11), since this would imply $\mu + p = 0$. There are therefore no LRS type II space-times with perfect fluid and, possibly, electromagnetic field that exhibit all of the intrinsic symmetries of (5.2.1).

6.) *Summary*

We have examined all combinations of the intrinsic symmetries (5.2.1). The relation between the cases considered is given in the specialization diagrams 5.2.8 and 5.2.9. As in Chapter 4, we give separate diagrams for the general case and the special case.

The classes which may contain LRS type IIc space-times are those of the general case $a \neq 0$. In the special case $a \equiv 0$, the space-times are LRS type IIa.

5.3 Equation of State with Intrinsic Symmetries

We now address the problem of an equation of state with the aid of intrinsic symmetries. We shall assume there exists a functional relationship between the quantitites p and μ. That is, we shall assume there exists a function A, of two variables such that

$$A(p, \mu) = 0 \tag{5.3.1}$$

We shall see that even with a general restriction of this sort it is possible to gain some ground using the present approach.

Equation (5.3.1) may be viewed as a constraint equation which must be propagated. Since the 2 and 3 operators yield zero for p and μ, we need only consider the ∂_0 and ∂_1 propagation equations for A:

$$\left.\begin{array}{lcl} \partial_0 A = 0 & \Leftrightarrow & \frac{\partial A}{\partial p}\partial_0 p + \frac{\partial A}{\partial \mu}\partial_0 \mu = 0 \\ \partial_1 A = 0 & \Leftrightarrow & \frac{\partial A}{\partial p}\partial_1 p + \frac{\partial A}{\partial \mu}\partial_1 \mu = 0 \end{array}\right\} \tag{5.3.2}$$

A necessary and sufficient condition for the homogeneous system (5.3.2) to have a solution (other than the trivial solution $\frac{\partial A}{\partial \mu} = \frac{\partial A}{\partial p} = 0$) is

$$\det \begin{bmatrix} \partial_0 p & \partial_0 \mu \\ \partial_1 p & \partial_1 \mu \end{bmatrix} = \partial_0 p \partial_1 \mu - \partial_1 p \partial_0 \mu = 0 \ . \tag{5.3.3}$$

Above, the second term, $\partial_1 p \partial_0 \mu$, is known from the Bianchi identities to be

$$\partial_1 p \partial_0 \mu = (\dot{u}(\mu + p) + \epsilon E)(\mu + p)(\alpha + 2\beta) \ . \tag{5.3.4}$$

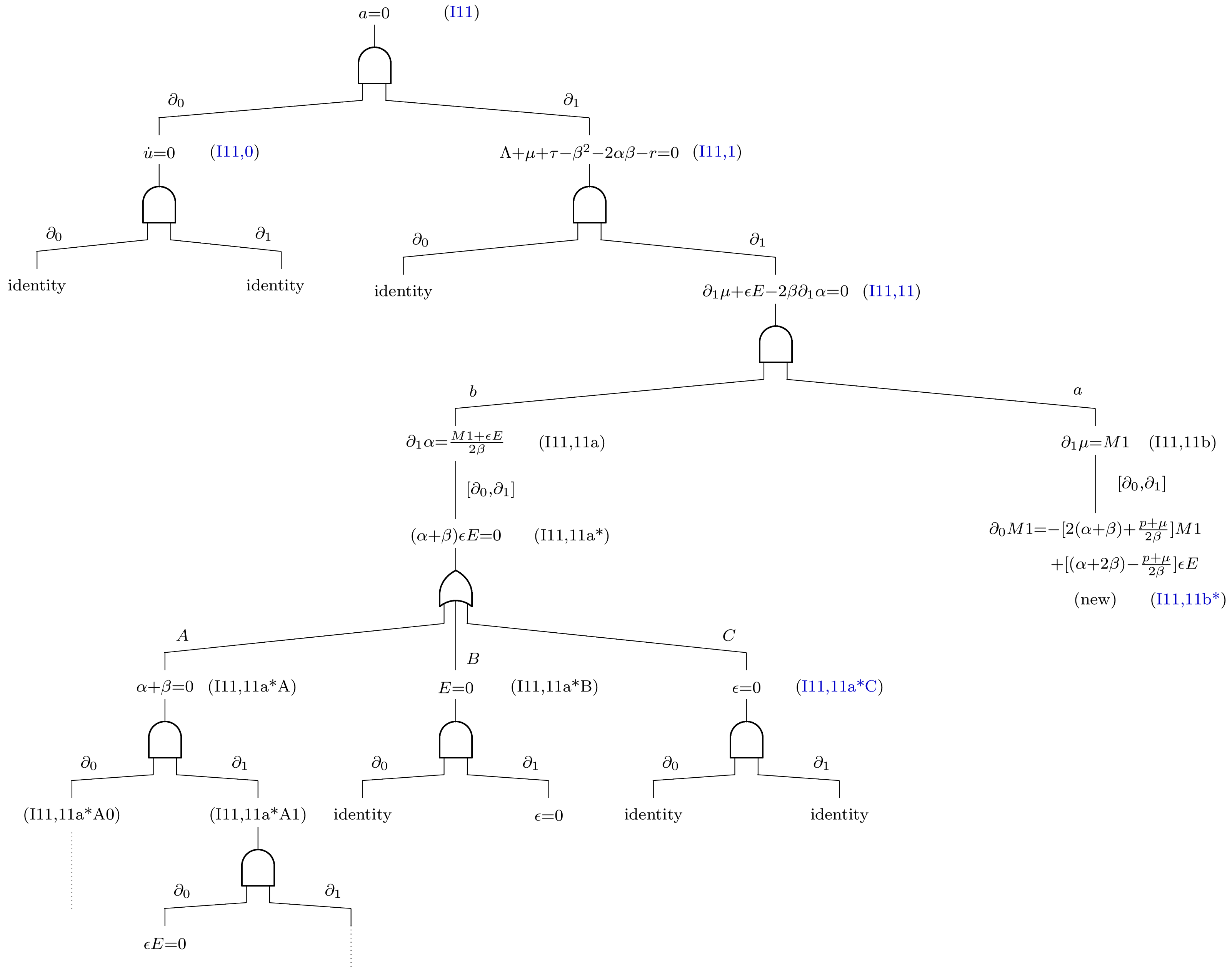

Figure 5.2.1: *I11:* $_{(1)23}\theta=0$

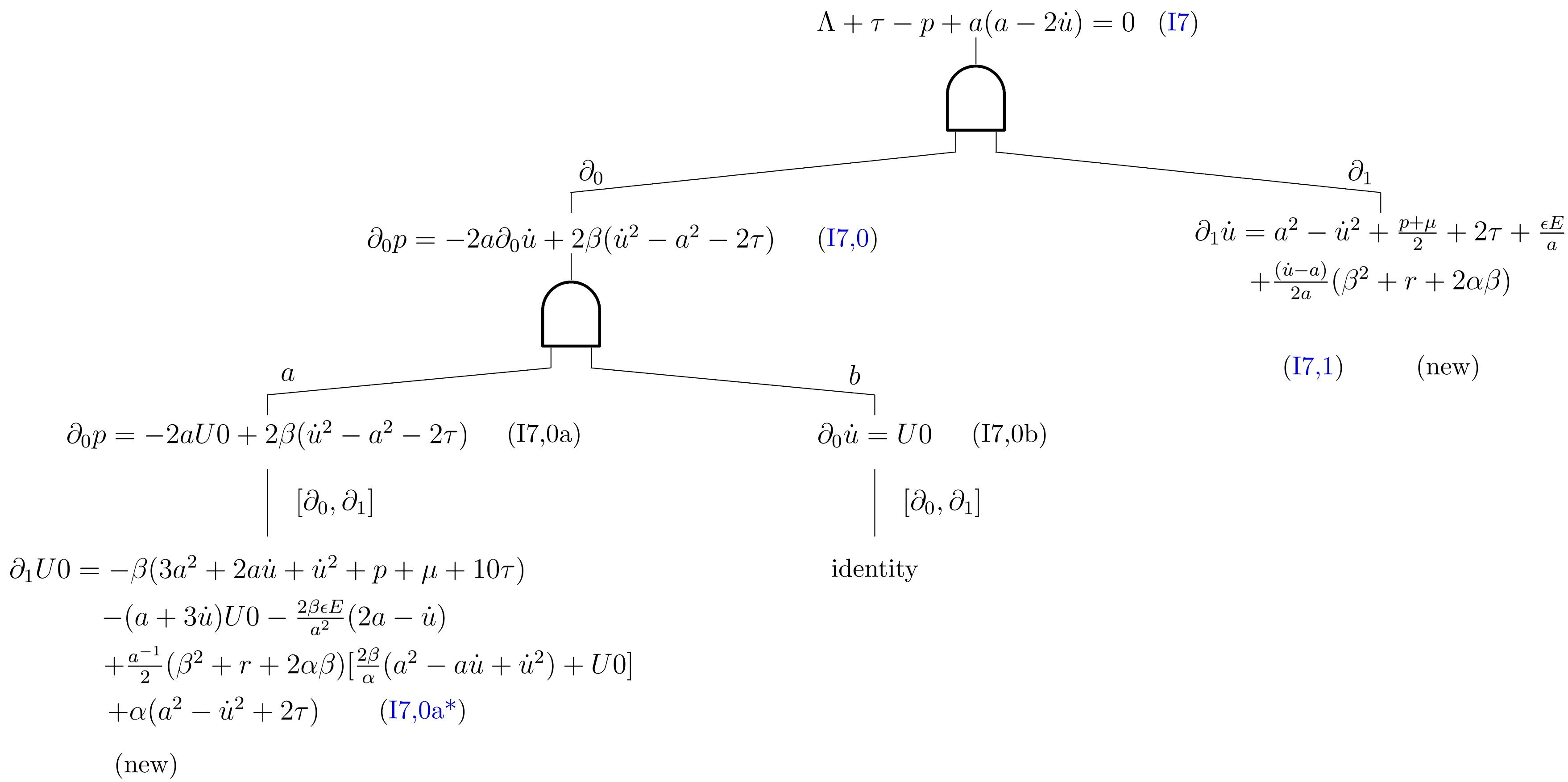

Figure 5.2.2: *I7:* ${}_{023}R=0$ with $a \neq 0$

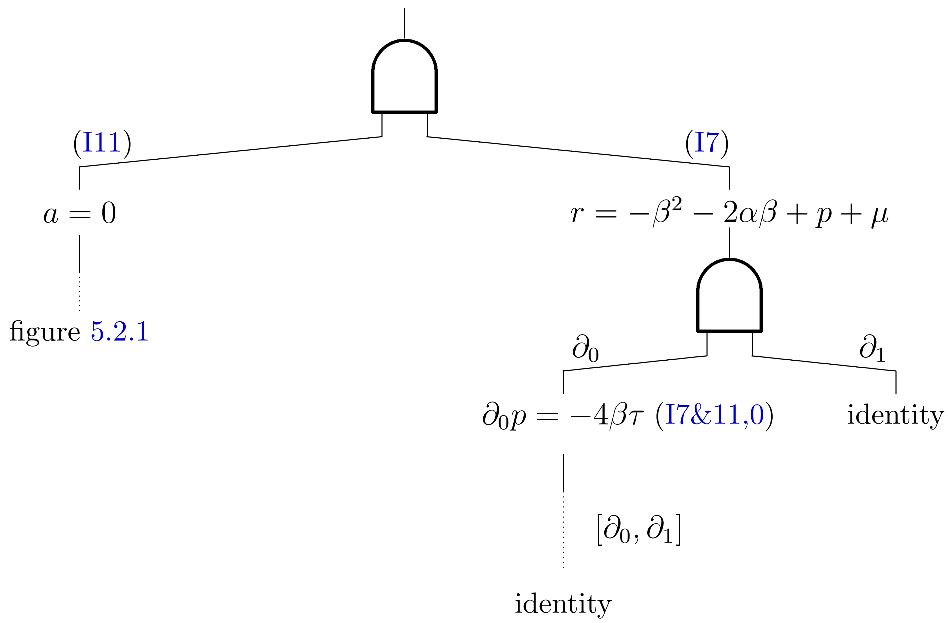

Figure 5.2.3: *I7&11:* ${}_{023}R=0$ *after* ${}_{(1)23}\theta=0$

Since we do not, in general, know the propagation equations for $\partial_0 p$ or $\partial_0 \mu$ we shall examine equation (5.3.3) with the aid of intrinsic symmetries. We shall divide the investigation into three disjoint and exhaustive cases: $\beta = 0$, $a \neq 0$; $\beta \neq 0$, $a = 0$; and $\beta \neq 0$, $a \neq 0$. (Recall $\beta = 0, a = 0$ gives a contradiction.)

1.) $\beta = 0, a \neq 0$

This is the intrinsic symmetry (I6), examined in 4.1. There, we found $\beta = 0$ implies $\alpha = 0$ and $\partial_0 p = -2a\partial_0 \dot{u}$. Equation (5.3.3) now reads

$$-2a\partial_0 \dot{u} \partial_1 \mu = 0$$

so we have two possibilities: either $\partial_0 \dot{u} = 0$ or $\partial_1 \mu = 0$. If $\partial_0 \dot{u} = 0$, then using (I6,0) to specify $\partial_1 \dot{u}$ tne commutation relation for $\dot{u}$ is identically satisifed. If $\partial_1 \mu = 0$ then μ is a constant, since $\alpha = \beta = 0$ implies $\partial_0 \mu = 0$. Thus both of the possibilities are consistent in the LRS class II space-times.

2.) $\beta \neq 0, a = 0$

In this case we have the intrinsic symmetry (I11) from section 5.2. Among the conditions necessary for $a = 0$, it was found that we must have $\dot{u} = \epsilon = 0$ and $\partial_1 \mu = 2\beta \partial_1 \alpha$.

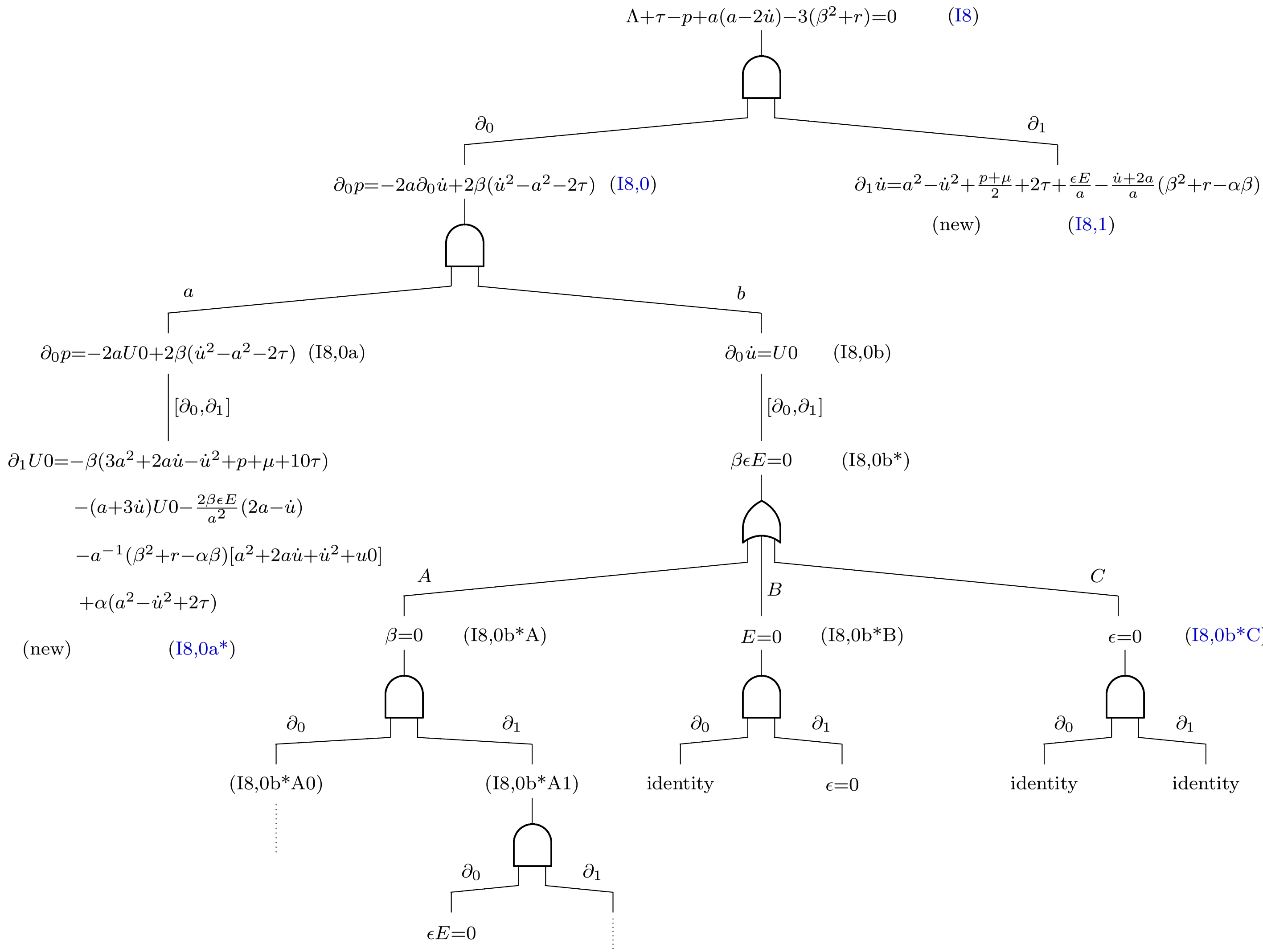

Figure 5.2.4: *I8:* ${}_{023}S=0$ *with* $a \neq 0$

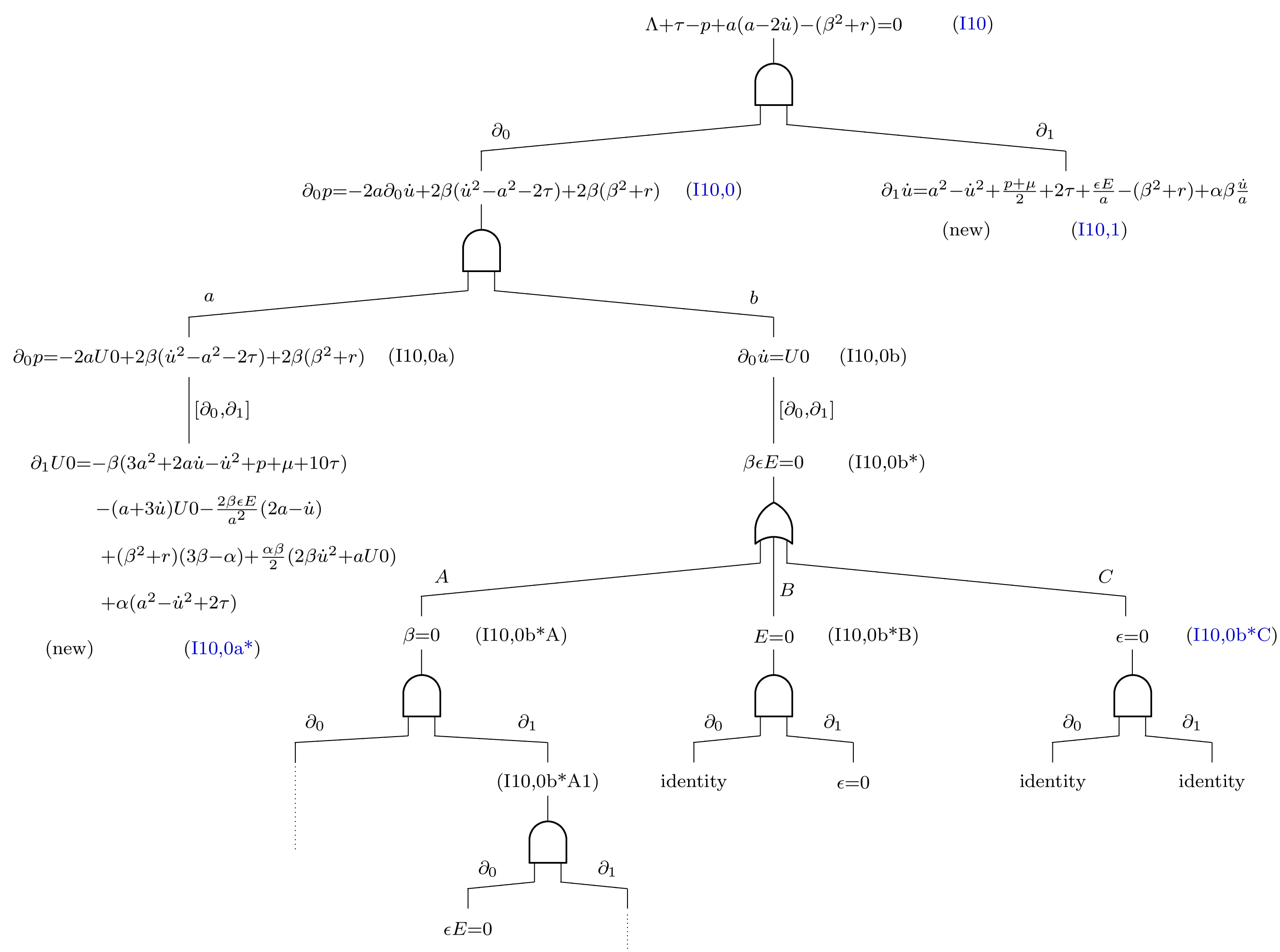

Figure 5.2.5: *I10:* $_{02}R=0$ *with* $a \neq 0$

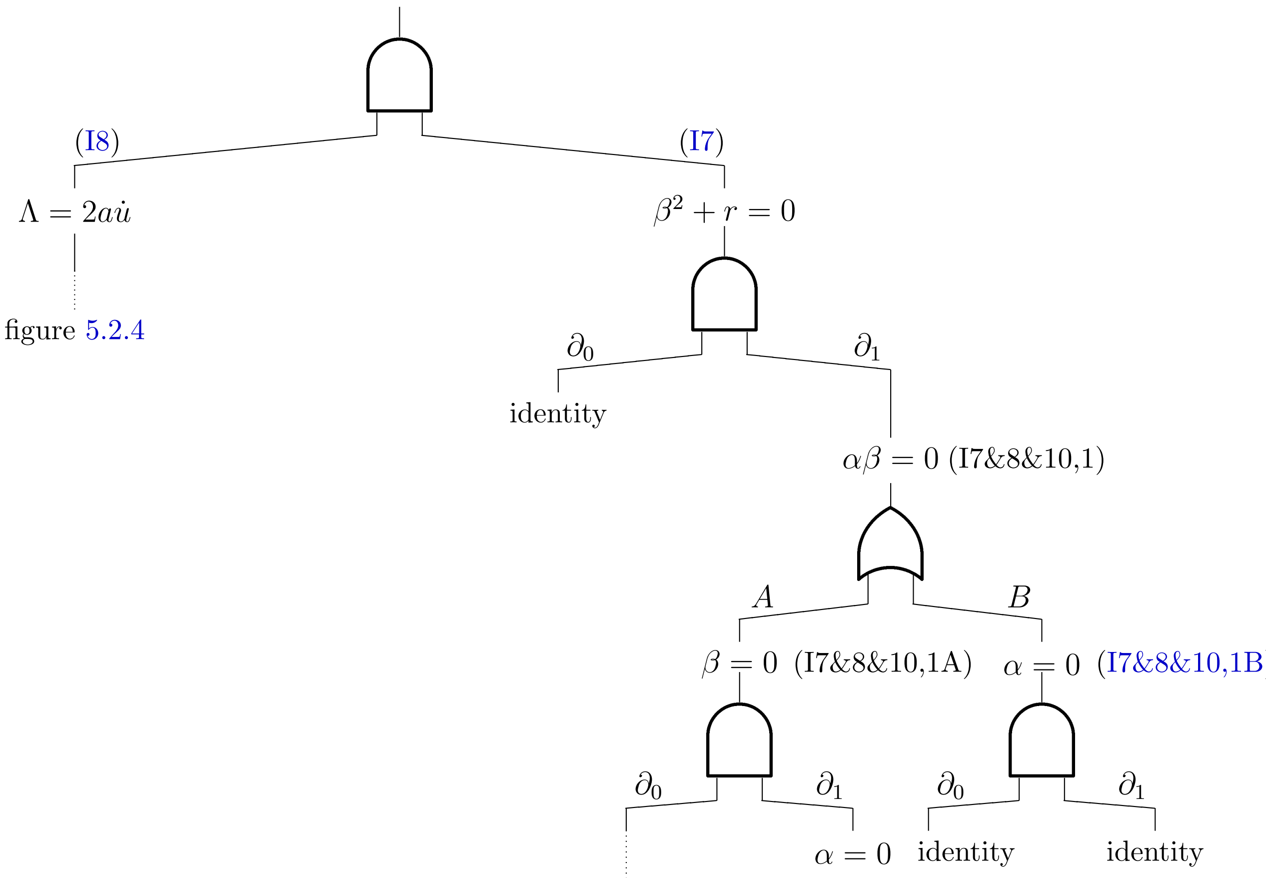

Figure 5.2.6: *I7&8&10: ${}_{023}R$=0 after ${}_{023}S$=0 with* $a \neq 0$

Since $\dot{u}$ and ϵ vanish we have $\partial_1 p = 0$ so equation (5.3.3) becomes

$$2\beta\partial_0 p\partial_1\alpha = 0 \ .$$

If $\partial_0 p = 0$ then p is a constant and trivially satisfies the integrability condition. If $\partial_0\alpha = 0$ then, using (F1) for $\partial_0\alpha$, we see that α would also satisfy the commutation relation. Therefore both of these possibilities are consistent in the LRS class II space-times.

3.) $\beta \neq 0, a \neq 0$

In this case we divide the investigation into two parts according as $\partial_1\mu$ is zero or nonzero.

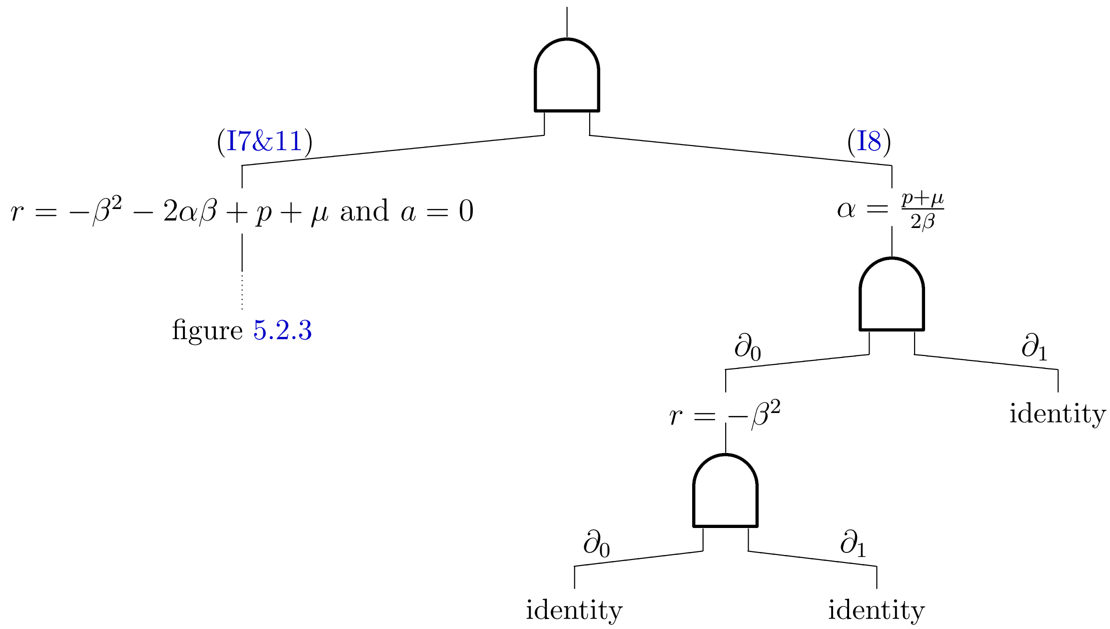

Figure 5.2.7: *I7&8&10&11:* ${}_{023}R={}_{023}S={}_{02}R=0$ *and* ${}_{(1)23}\theta=0$

(3a) $\partial_1\mu = 0$

From (5.3.3), (5.3.4) and $\mu + p \neq 0$, we have

$$(\alpha + 2\beta)(\dot{u}(\mu + p) + \epsilon E) = 0 \ .$$

Since at least one factor must vanish we have two main subcases, which we shall examine in turn.

(3ai) $\alpha + 2\beta = 0$

We must check the consistency of the constraint

$$\alpha + 2\beta = 0 \tag{5.3.5}$$

propagation along $\mathbf{e}_0$ yields

$$\partial_1\dot{u} = \dot{u}(2a - \dot{u}) + \tau - \Lambda + 6\beta^2 + \frac{\mu + 3p}{2} \ , \tag{5.3.5,0}$$

which is the first propagation equation for $\dot{u}$. Using this, propagation of (5.3.5) along $\mathbf{e}_1$ gives

$$\partial_1\alpha = -6a\beta \ . \tag{5.3.5,1}$$

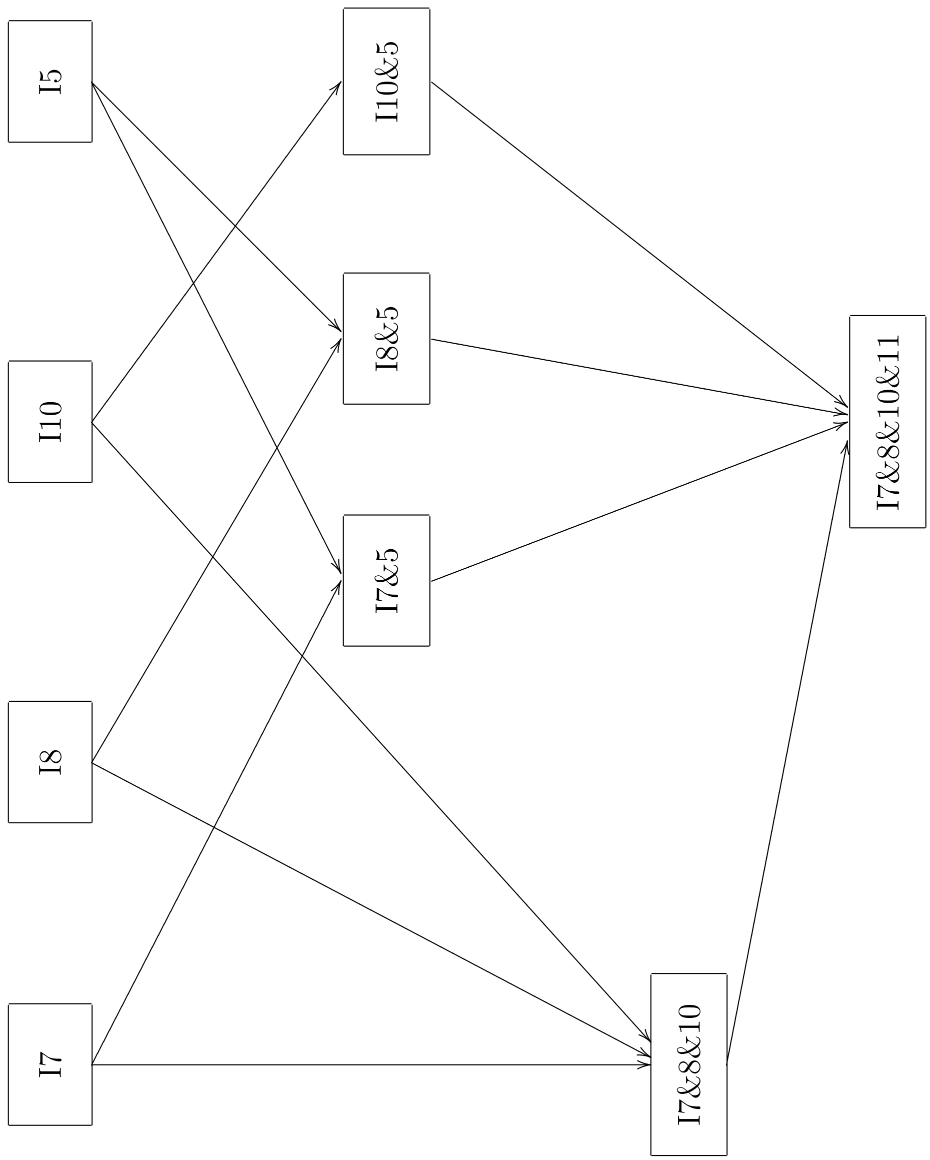

Figure 5.2.8: *Specialization Diagram with* $a \neq 0$

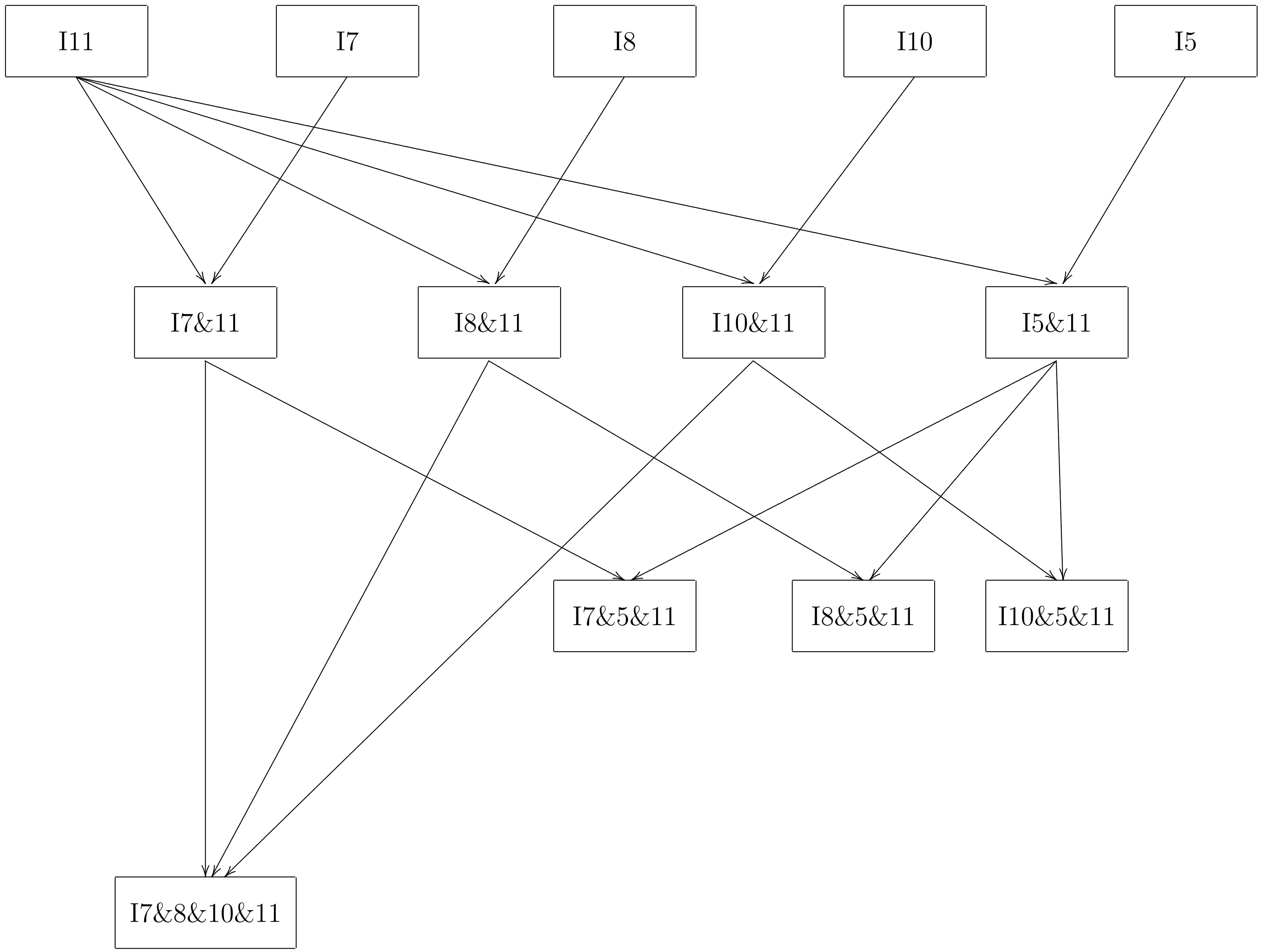

Figure 5.2.9: *Specialization Diagram with* $a = 0$

Checking the commutation relation for α gives $\epsilon E = 0$ which implies

$$\epsilon = 0 \tag{5.3.5,1*}$$

and there are no futher constraints to satisfy.

(3aii) $\dot{u}(\mu + p) + \epsilon E = 0$

In this case we may write

$$\dot{u} = \frac{-\epsilon E}{(\mu + p)} \tag{5.3.6}$$

so these space-times are accelerating if and only if $\epsilon \neq 0$. Introducing two new quantities

$$\partial_1 \epsilon = Q1$$
$$\partial_0 p = P0$$

the first propagation equations of (5.3.6) are

$$\partial_0 \dot{u} = \frac{[P0 + 2(p + \mu)]\epsilon E}{(p + \mu)^2} \tag{5.3.6,0}$$

and

$$\partial_1 \dot{u} = \frac{(p + \mu)(\epsilon E + \epsilon^2 + 2a\epsilon E + EQ1) + (\epsilon E)^2}{(p + \mu)^2} \tag{5.3.6,1}$$

The commutators for ϵ and p yield, respectively,

$$\partial_0 Q1 = -2(\alpha + \beta)Q1$$
$$\partial_1 P0 = \frac{\epsilon E}{\mu + p}[P0 + 4\beta(\mu + p)]$$

and the commutator for $\dot{u}$ is satisfield identically.

Since we have specified $\partial_1 \mu = 0$ we must check the integrability condition for μ. Doing this yields

$$\partial_1 \alpha = 2a(\alpha - \beta) + \frac{\epsilon E(\alpha + 2\beta)}{\mu + p} \ .$$

Checking the commutator for a yields a lengthy propagation equation for $\partial_1 Q1$ and checking the commutation relation for Q1 yields an even lengthier propagation equation for $\partial_1 P0$. The unknown P0 then satisifes the integrability condition if and only if an algebraic constraint with approximately 200 terms holds.

The consistency of imposing the equation of state has not been checked in full for this case since, propagating the 200 term constraint along $\mathbf{e}_0$, we obtain an equation with nearly 19,000 terms and we do not feel it worthwhile to pursue this case further. Imposing some additional constraints would be appropriate in a case such as this. However, since most of the intrinsic symmetries we have examined require $\epsilon = 0$, imposing one of them reduces this case to checking the consistency of $\dot{u} = 0$, which is trivial.

(3b) $\boldsymbol{\partial_1 \mu \neq 0}$

In this case we may use (5.3.3) to obtain

$$\partial_0 p = \frac{[\dot{u}(\mu + p) + \epsilon E](\mu + p)(\alpha + 2\beta)}{M1} ,$$

where we have defined

$$\partial_1 \mu = M1 .$$

We have found this too complicated to check for consistency in the full generality and have restricted our attention to the case where ${}_{123}R$=0. (The investigation would be quite similar if another intrinsic symmetry had been used instead.) With this restriction, we have

$$\partial_0 p = \frac{\dot{u}(\mu + p)^2(\alpha + 2\beta)}{M1} . \tag{5.3.7}$$

checking the commutation relation for p we obtain a propagation equation for $\partial_0 \dot{u}$ involving $\partial_1 M1$. Splitting this propagation equation by defining

$$\partial_1 M1 = M11 ,$$

we must check the commutation relations for both $\dot{u}$ and M1, since $\partial_1 \dot{u}$ and $\partial_0 \mu$ have been found in the investigation of ${}_{123}R$=0. Checking the commutator for $\dot{u}$, we obtain an algebraic constraint, if $\alpha + 2\beta = 0$, or a lengthy propagation equation for $\partial_1 M11$, if $\alpha + 2\beta \neq 0$. Checking the commutator for M1 yields a propagation equation for $\partial_0 M11$. Therefore this case divides naturally, depending on whether the expansion is zero or non-zero.

(3bi) $\boldsymbol{\alpha + 2\beta = 0}$

In this case $\partial_0 \mu = \partial_0 p = 0$, and we must propagate the constraint

$$\alpha = -2\beta . \tag{5.3.8}$$

We shall show that (5.3.8) leads to a contradiction. Propagating the constraint along $\mathbf{e}_0$ we obtain

$$6a\dot{u} + 4\tau + 18\beta^2 - 3a^2 + 3(p+\mu) + 3r = 0 \tag{5.3.8,0}$$

Differentiating (5.3.8,0), we obtain

$$3\dot{u}^2 - 12a\tau - 18\beta^2 - 3(p+\mu) + 8\tau = 0 \tag{5.3.8,00}$$

and

$$\dot{u}^3 - 3a\dot{u}^2 + (6\beta^2 - 4a^2 - p - \mu)\dot{u} + a(30\beta^2 - p - \mu) = 0\ . \tag{5.3.8,01}$$

Then propagation of (5.3.8,00) along $\mathbf{e}_0$ yields

$$-9\dot{u}^2 + 8a\dot{u} + 6\beta^2 + \mu + p = 0\ . \tag{5.3.8,000}$$

Another propagation along $\mathbf{e}_0$, then comparing the result with (5.3.8,00) gives

$$\dot{u} = \frac{5a}{17}\ . \tag{5.3.8,0000}$$

Substituting this back into (5.3.8,01) and using (5.3.8,000) gives

$$a(3468\beta^2 + 55a^2) = 0\ .$$

This holds if and only if $a = 0$, which is a contradiction.

We have shown that the LRS class II space-times with ${}_{123}R$=0 and $a\beta\partial_1\mu \neq 0$ do not admit an equation of state of the form $A(p,\mu) = 0$, if the expansion vanishes.

(3bii) $\boldsymbol{\alpha + 2\beta \neq 0}$

In this case, we have propagation equations for both $\partial_0 M11$ and $\partial_1 M11$ so we must check the commutator of M11. Doing this yields an equation with approximately 13,000 terms so this case remains intractable unless further conditions are imposed.

We have now completely examined all of the cases except (3aii) and (3b). The case (3bi) has been examined in full with the additional condition ${}_{123}R$=0. Even with this additional condition, the case (3bii) remains too complicated to handle unless further restrictions are imposed. All of the cases we have examined completely contain solutions with an equation of state $A(p,\mu) = 0$. The case (3bi) does not admit an equation of state of this form if ${}_{123}R$=0. The cases which have not been examined in full remain inconclusive.

Finally, we note that in cases (2) and (3) we may have space-times of LRS class IIc. In case (3bii) we may not use the intrinsic symmetries ${}_{123}S$=0 or ${}_{12}R$=0 to complete the investigation if we wish to obtain solutions in class IIc, since we have shown that any two of (I1), (I2) and (I4) imply $\dot{u} = 0$.

Chapter VI
AUTOMATING THE CONSTRAINT-CHECKING PROCEDURE

We now revisit the topic of Chapter 2 — the method for checking whether a constraint may be imposed consistently on a class of solutions. Although the procedure does not always terminate, we have seen that in many cases it does and that in these cases it is a useful computational method. There are at least two reasons why we might consider writing a computer program for the constraint-checking procedure. One reason is simply that such a program could prove useful in investigating problems similar to those with which we have dealt. Another reason is that to write such a program we must describe in detail all aspects of the problem and would therefore be forced to consider fine points which might otherwise be overlooked. In this chapter we discuss some aspects of implementing our procedure as a computer program.

6.1 General Considerations

In the calculation of previous chapters, we have encountered many possible situations, using the method for checking the consistency of a constraint equation. Often we have come to a stage where any one of a number of new constraint equations is sufficient for the equation under consideration to hold. Sometimes when this happens the investigation is divided into cases, as with the vanishing or otheriwse of β while examining ${}_{123}R = 0$. At other times one of the conditions includes the others as special cases so only the general case need be considered. This occurs, for example, when $\epsilon E = 0$, because $E = 0$ implies $\epsilon = 0$ so only the latter case need be examined.

Another possibility is that we have a number of new conditions, all of which must hold. This happens whenever both propagations yield non-trivial equations that must both be checked. A second situation where this would happen occurs when we split an equation with more than one derivative into separate propagation equations, which must all hold.

We shall abstract away from the specifics of propagation equations and integrability conditions and deal with the problem from a more general point of view.

Suppose we have a constraint which implies that at least one of a number of further conditions hold. Then this divides the investigation into a number of cases in which the consistency of each of the new condtions is checked. If there was more than one case to start with, then each of the new conditions must be checked with each of the old ones, giving a new set of cases in which the constraint may hold.

If we have a constraint which implies that a number of new conditions all hold, then one way of checking for consistency is to apply the new constraints one after another and demand that all of the generated conditions hold. When there is more than one case to start with, then this should be done for each case and the original constraint may hold only if it may hold in at least one of these cases.

The methods for handling these two general situations may be given as the following two MACSYMA procedures:

```
CHECK%OR(CONSTRAINT%LIST, CASE%LIST) :=

   BLOCK([NEW%CASE%LIST, THIS%CASE%LIST],
         NEW%CASE%LIST : [],
         FOR CASE IN CASE%LIST DO
            FOR CONSTRAINT IN CONSTRAINT%LIST DO
               (THIS%CASE%LIST : CHECK%ONE(CONSTRAINT, CASE),
                IF THIS%CASE%LIST # CONTRADICTION THEN
                   NEW%CASE%LIST : APPEND(NEW%CASE%LIST,
                                          THIS%CASE%LIST)),
         RETURN(IF NEW%CASE%LIST = [] THEN
                   CONTRADICTION
                ELSE
                   NEW%CASE%LIST))$

CHECK%AND(CONSTRAINT%LIST, CASE%LIST) :=

   BLOCK([NEW%CASE%LIST],
         NEW%CASE%LIST : CHECK%OR([FIRST(CONSTRAINT%LIST)],
                                  CASE%LIST),
         FOR CONSTRAINT IN REST(CONSTRAINT%LIST)
         WHILE NEW%CASE%LIST # CONTRADICTION DO
            NEW%CASE%LIST : CHECK%OR([CONSTRAINT], NEW%CASE%LIST),
         RETURN(NEW%CASE%LIST))$
```

In these procedures, `CHECK%ONE` is the function used to check the consistency of a single constraint in a single case, returning either a list of cases or the atom `CONTRADICTION`.

If the above procedures had been used in performing the calculations in the preceding chapters, then the use of `CHECK%OR` would correspond to the OR gates (⌓) in the diagrams and the use of `CHECK%AND` would correspond to the AND gates (⌂).

6.2 A Prototype Computer Program

In this section we shall discuss a prototype program which can be used for simple problems. The purpose of this program is not to implement the full constraint-checking procedure, but rather to illustrate the feasibility of doing so. A listing of the MACSYMA procedures and two example runs are given in Appendix G.

Before describing the operation of the functions, we define a few data types to facilitate the discussion. First, we define an "algebraic rule" to be an equation of the form *variable = expression*, where *expression* is free of derivatives. We shall define a "propagation rule" to be a list of the form *[variable, variable, expression]*. This shall be interpreted as an argument list to the function `GRADEF`. Finally, we define a "case" to be a two element list of the form *[list of algebraic rules, list of propagation equations]*. The main procedure of the constraint-checking program is `CHECK%ONE`, which is a recursive function of two arguments. The first argument is an equation, which represents the constraint, and the second argument is a case, which should contain algebraic and propagation rules for the basic equations of the class of space-times. If the constraint equation may not be imposed consistently for the given case, then the atom `CONTRADICTION` is returned. When the constraint may be applied consistently, then the list of cases for which the constraint may hold is returned. Each case in the list would contain the rules for the basic equations, the constraint and additional necessary conditions.

A number of simplifying assumptions are made about the problems with which `CHECK%ONE` deals. First, it is assumed that the space-time is spatially homogeneous so that only one propagation equation need be checked for a given algebraic constraint. Second, it is assumed that an equation which contains derivatives which are not known is always a propagation equation. That is, it is assumed that such an equation does not necessitate checking two cases in which a coefficient is zero or non-zero, accordingly. Also, it is assumed that the expressions with which the procedures deal are rational functions of the unknowns and their derivatives.

Interinally, `CHECK%ONE` performs the following actions. First local `GRADEF`s are set up using the propagation rules from the given case and a list of algebraic rules, to be used in simplifications, is taken from the first element of the case. These initializations are performed by the function `INITIALIZE%LEVEL`. The function `CLEAN` is then used to recast the equation in a form in which it will be easier to divide the investigation into spacial cases if necessary. `CLEAN` first collects all terms on the right-hand side of the equation and then performs an evaluation, using the local `GRADEF`s and the list of algebraic rules. The resulting expression is then factored and returned to `CHECK%ONE`. The constraint equation is then examined to see what type it is. If it is an identity for the current case, then no additional conditions are necessary and a list of cases is returned which contains only the one input case. If the equation is recognized as a contradiction then the atom `CONTRADICTION` is returned. If the equation does not fall into either of these categories, we check whether it may be a propagation equation by looking for first derivatives. At this point one of PROPN%PROCESS or ALG%PROCESS is called according to the result of the test. The function called returns a list of cases (or CONTRADICTION) and this result is returned by CHECK%ONE.

With the assumptions we have made, whenever PROPN%PROCESS is called nothing further need be done than record the equation and return the current case.

When ALG%PROCESS is invoked, the constraint equation is reshaped to give an algebraic rule. In fact a number of cases may need to be investigated if the constraint equation gives rise to a product as zero or if the equation is not linear in any of its unknowns. The propagation equation(s) are then checked recursively and the result is returned. (Note that for the spatially homogeneous case CHECK%AND(DIFF%LSTI, [CASE]) is equivalent to CHECK%ONE(D%OP(1, eqn), CASE)).

There are a number of possible directions in which this prototype may be improved. First, it would be desirable to implement a more intelligent version of MAY%NOT%ZERO%PRED, which remembered the answers to the prompts for each case, and used them in future deductions. It would be useful to extend the scope of the program to include spatially inhomogeneous space-times. This would be achieved mainly by rewriting PROPN%PROCESS, using CHECK%OR and CHECK%AND to combine the results of division in cases and checking commutators.

There is one final point we shall note. When these calculations are performed by hand and a stage is reached where a number of conditions must all hold, it is easiest to perform a "breadth first" investigation. That is, one would normally compute all the first propagation equations for the conditions before proceeding on to second and further propagations. A procedure such as CHECK%AND, on the other hand, performs a "depth first" investigation, completely investigating one constraint before going on to the rest. This is done to take advantage of the recursion stack to simplify the specification of the procedures.

Chapter VII
CONCLUDING REMARKS

We have used our method of imposing constraints both for intrinsic symmetries and for an equation of state, with LRS Class II space-times.

The intrinsic symmetries we have examined are related to the $\mathbf{u}$ and $\mathbf{e}_1$ congruences and the subspaces orthogonal to them. Imposing any one of the intrinsic symmetries leads to a number of additional relationships between the unknowns. Using this method, a number of identities have been obtained for Class IIa by imposing $a = 0$ as the intrinsic symmetry ${}_{(1)23}\theta = 0$ on Class II. By examining all combinations of intrinsic symmetries associated with each of the $\mathbf{u}$ and $\mathbf{e}_1$ congruences, we have shown which may hold simultaneously and have obtained the related necessary conditions. With the intrinsic symmetries we have examined, the simplification of the field equations has not yet led to the discovery of new solutions. This is a question which the author intends to investigate further.

We have examined the restriction that the equation of state be of the general form $A(p, \mu) = 0$. Dividing the investigation into a number of disjoint cases, we have completely examined most of the cases and determined the additional conditions which must hold. In the most general cases we have been forced to impose additional conditions, otherwise the calculations become impractical even using a large system for symbolic computation.

In the last chapter we have shown how the procedure for checking the consistency of a constraint may be implemented on a computer. Even though the general theory of imposing constraints in this way is still incomplete, the method presented could well be useful in other areas besides general relativity.

Appendix A BASIC EQUATIONS FOR LRS CLASS II

For the tetrad of section 1.3 with LRS class II space-times, the commutation relations, Jacobi identities, Maxwell equations, field equations and Bianchi identities reduce to the following formulæ (Steward & Ellis [1968]).

Commutation Relations:

$$[\mathbf{e}_0, \mathbf{e}_1] = \dot{u}\mathbf{e}_0 - \alpha\mathbf{e}_1 \tag{CR1}$$
$$[\mathbf{e}_0, \mathbf{e}_2] = -\beta\mathbf{e}_2 \tag{CR2}$$
$$[\mathbf{e}_0, \mathbf{e}_3] = -\beta\mathbf{e}_3 \tag{CR3}$$
$$[\mathbf{e}_2, \mathbf{e}_3] = s\mathbf{e}_3 \tag{CR4}$$
$$[\mathbf{e}_3, \mathbf{e}_1] = -a\mathbf{e}_3 \tag{CR5}$$
$$[\mathbf{e}_1, \mathbf{e}_2] = a\mathbf{e}_2 \tag{CR6}$$

Jacobi Identities:

$$\begin{pmatrix} 3 \\ 123 \end{pmatrix} \Leftrightarrow \quad \partial_1 s = as \tag{J1}$$
$$\begin{pmatrix} 2 \\ 012 \end{pmatrix} \Leftrightarrow \quad \partial_1 \beta + \partial_0 a = -\beta\dot{u} - \alpha a \tag{J2}$$
$$\begin{pmatrix} 3 \\ 023 \end{pmatrix} \Leftrightarrow \quad \partial_0 s = -\beta s \tag{J3}$$

Maxwell Equations:

$$\partial_1 E = 2aE + \epsilon \tag{M1}$$
$$\partial_1 H = 2aH \tag{M2}$$
$$\partial_0 E = -2\beta E \tag{M3}$$
$$\partial_0 H = -2\beta H \tag{M4}$$

Field Equations:

$$
\left.\begin{array}{l}(00)\\(11)\\(22)\end{array}\right\}\Leftrightarrow\left\{\begin{array}{lcl}
\partial_0\alpha &=& -\frac{1}{2}(\mu+p)+\beta^2-\alpha^2-a^2+r+\partial_1\dot{u}+\dot{u}^2-2\tau \qquad (F1)\\
\partial_0\beta &=& \frac{1}{2}\left(\Lambda-p-3\beta^2+a^2-r-2a\dot{u}+\tau\right) \qquad (F2)\\
\partial_1 a &=& \frac{1}{2}\left(\Lambda+\mu-\beta^2-2a\beta+3a^2-r+\tau\right) \qquad (F3)
\end{array}\right.
$$

$$
(01) \quad \Leftrightarrow \quad \partial_1\beta = a(\beta-\alpha) \qquad (F4)
$$

Bianchi Identities:

$$
(0)\Leftrightarrow \qquad \partial_0\mu = -(\mu+p)(\alpha+2\beta) \qquad \text{(BI1)}
$$

$$
(1)\Leftrightarrow \qquad \partial_1 p = -(\mu+p)\dot{u}-\epsilon E \qquad \text{(BI2)}
$$

From the Jacobi identities, we obtain further useful relations. Since $\partial_1\beta$ is known from (F4), the identity (J2) gives a propagation equation for $\partial_0 a$:

$$
\partial_0 a = -\beta(a+\dot{u}). \qquad \text{(J2')}
$$

From the definition $r := \partial_2 s - s^2$, the identities (J3) and (J1) imply, respectively,

$$
\partial_0 r = -2\beta r \qquad \text{(J4)}
$$

$$
\partial_1 r = 2ar \qquad \text{(J5)}
$$

Using Maxwell's equations, three additional propagation equations may be obtained. The definition $\tau = \frac{1}{2}\left(E^2+H^2\right)$ implies

$$
\partial_0\tau = -4\beta\tau \qquad \text{(M5)}
$$

and

$$
\partial_1\tau = 4a\tau + \epsilon E. \qquad \text{(M6)}
$$

Applying (CR1) to E and simplifying using (M1), (M3) and (J2) yields

$$
\partial_0\epsilon = -(\alpha+2\beta)\epsilon. \qquad \text{(M7)}
$$

Finally, let

$$
\text{LRSII} = \left\{\text{CR1},\ldots,\text{CR6},\text{J1},\text{J2}',\text{J3},\text{J4},\text{J5},\text{M1},\ldots,\text{M7},\text{F1},\ldots,\text{F4},\text{BI1},\text{BI2}\right\}.
$$

Then, for any space-time of LRS class II, each equation of the basic set (LRSII) holds.

Appendix B TWO RESULTS

In this appendix, we prove two results, the first referred to in section 1.3 and the other in section 3.1.

Result 1. If a space time containing perfect fluid and electromagnetic field is LRS in the region of interest, then in a tetrad with $\mathbf{e}_0 = \mathbf{u}$ and $\mathbf{e}_1$ an axis of symmetry, the electromagnetic stress-energy tensor has $\tau_{01} = 0$ if and only if $\mathbf{E} = E\mathbf{e}_1$ and $\mathbf{B} = B\mathbf{e}_1$.

Proof. We shall use the following formulæ:

$$\left.\begin{aligned} \tau_{00} &= \tfrac{1}{8\pi}\left(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}\right) \\ \tau_{0\alpha} &= -\tfrac{1}{4\pi}\left(\mathbf{E}\times\mathbf{B}\right)_\alpha \\ \tau_{\alpha\beta} &= \tfrac{1}{4\pi}\left[-\left(E_\alpha E_\beta + B_\alpha B_\beta\right) + \tfrac{1}{2}g_{\alpha\beta}\left(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}\right)\right] \end{aligned}\right\} \tag{B.1}$$

and by LRS we have

$$\tau_{ab} = \begin{bmatrix} \tau_{00} & \tau_{01} & 0 & 0 \\ \tau_{10} & \tau_{11} & 0 & 0 \\ 0 & 0 & \tau_{22} & 0 \\ 0 & 0 & 0 & \tau_{33} \end{bmatrix}$$

where $\tau_{22} = \tau_{33}$ as well as $\tau_{01} = \tau_{10}$ and $\tau^a{}_a = 0$.

Suppose $\tau_{01} = 0$. Then, examinging the components of τ_{ab}, we see

$$\tau_{22} = \tau_{33} \quad \Rightarrow \quad (E_2)^2 + (B_2)^2 = (E_3)^2 + (B_3)^2 \tag{B.2a}$$

$$\tau_{23} = 0 \quad \Rightarrow \quad E_2E_3 = -B_2B_3 \tag{B.2b}$$

$$\tau_{01} = 0 \quad \Rightarrow \quad E_2B_3 = B_2E_3 \tag{B.2c}$$

Without loss of generality, assume $E_2 \geq 0$. If $E_2 = 0$ then either $B_2 = 0$, in which case E_3 and B_3 are zero by (B.2a), or $B_2 \neq 0$, in which case $B_3 = E_3 = 0$ by (B.2b) and (B.2c) and (B.2a) gives a contradiction. If $E_2 > 0$ then (B.2b) $\cdot\, B_3$ and (B.2c) $\cdot\, E_3$ combine to give

$$-B_2(B_3)^2 = B_2(E_3)^2.$$

Then if $B_2 \neq 0$ we have $B_3 = E_3 = 0$ and (B.2a) gives a contradiction. Otherwise $B_2 = 0$ so (B.2b) and (B.2c) imply $B_3 = E_3 = 0$ and (B.2a) again gives a contradiction. The only case which does not lead to a contradiction is $E_2 = E_3 = B_2 = B_3 = 0$. Therefore we have shown

$$\tau_{01} = 0 \quad \Rightarrow \quad \mathbf{E} = E\mathbf{e}_1 \quad \text{and} \quad \mathbf{B} = B\mathbf{e}_1.$$

Now suppose $\mathbf{E} = E\mathbf{e}_1$ and $\mathbf{B} = B\mathbf{e}_1$. Substituting the tetrad component values of $\mathbf{E}$ and $\mathbf{B}$ into (B.1) gives $\tau_{01} = 0$. □

Assuming the cosmological constant is zero, we have

Result 2. In a space-time with perfect fluid admiting a homothetic vector field $\mathbf{h}$ such that $\mathbf{h} \cdot \mathbf{u} = 0$, we have either $p = \mu$ or $\mathbf{h}$ is a Killing vector (McIntosh [1976]). If in addition there is an electromagnetic field, with stress energy-tesnsor τ_{ab}, this result remains true only if the additional condotion

$$\tau^{ab}{}_{;\,a} h_b = \frac{1}{3}\left[\left(\tau_{ab}u^a u^b\right)_{,\,c} h^c + 2b\tau_{ab}u^a u^b\right]$$

holds.

Proof. Since $\mathbf{h}$ is a homothetic vector we have

$$\mathscr{L}_{\mathbf{h}} g_{ab} = 2b g_{ab},$$

for some constant b. We may define $f_{ab} = f_{[ab]}$ and j^a such that $j^a{}_{;\,a} = 0$ by

$$h_{a;\,b} = b g_{ab} + f_{ab}$$

and

$$j^a = f^{ab}{}_{;\,b} = R^{ab} h_b$$

(c.f. McIntosh [1976]).

The field equations give

$$R^{ab} = (\mu + p)u^a u^b + \frac{1}{2}(\mu - p)g^{ab} + \tau^{ab}.$$

We have $\mathbf{h} \cdot \mathbf{u} = 0$ by hypothesis, so contracting the above equation with $\mathbf{h}$ gives

$$j^a = \frac{1}{2}(\mu - p)h^a + \tau^{ab} h_b$$

and $j^a{}_{;\,a} = 0$ implies

$$\begin{aligned} 0 &= \frac{1}{2}(\mu - p)_{,\,a} h^a + \frac{1}{2}(\mu - p)h^a{}_{;\,a} + \tau^{ab} h_{b;\,a} + \tau^{ab}{}_{;\,a} h_b \\ &= \frac{1}{2}\mathscr{L}_{\mathbf{h}}(\mu - p) + 2b(\mu - p) + \tau^{ab}{}_{;\,a} h_b \end{aligned} \tag{B.3}$$

using the fact that τ_{ab} is symmetric and trace-free to deduce that $\tau^{ab}h_{b;\,a} = 0$. To evaluate $\mathscr{L}_{\mathbf{h}}(\mu - p)$, we use the relations

$$\left.\begin{array}{rcl} \mathscr{L}_{\mathbf{h}}\left(T_{ab}u^a u^b\right) & = & -2bT_{ab}u^a u^b \\ \mathscr{L}_{\mathbf{h}}\left(T^a{}_a\right) & = & -2bT^a{}_a \end{array}\right\} \tag{B.4}$$

(Eardley [1974]). From these we have

$$\mathscr{L}_{\mathbf{h}}\left(\mu + \tau_{ab}u^a u^b\right) = -2b\left(\mu + \tau_{ab}u^a u^b\right)$$
$$\mathscr{L}_{\mathbf{h}}\left(-\mu + 3p\right) = -2b\left(\mu + 3p\right)$$

so

$$\mathscr{L}_{\mathbf{h}}\left(\mu - p\right) = -2b\left(\mu - p\right) - \frac{2}{3}\left[\left(\tau_{ab}u^a u^b\right)_{,\,c} h^c + 2b\tau_{ab}u^a u^b\right].$$

Using this result, (B.3) becomes

$$b(\mu - p) = \frac{1}{3}\left[\left(\tau_{ab}u^a u^b\right)_{,\,c} h^c + 2b\tau_{ab}u^a u^b\right] - \tau^{ab}{}_{;\,a}h_b.$$

Finally, if $p = \mu$ or $\mathbf{h}$ is a Killing vector then the left-hand side of the above equation vanishes and

$$\tau^{ab}{}_{;\,a}h_b = \frac{1}{3}\left(\tau_{ab}u^a u^b\right)_{,\,c} h^c + 2b\tau_{ab}u^a u^b. \qquad \square$$

Remark. Note that if the perfect fluid and electromagnetic field are non-interacting, then

$$\tau^{ab}{}_{;\,a} = 0.$$

If it is also assumed that

$$\mathscr{L}_{\mathbf{h}}\left(\tau_{ab}u^a u^b\right) = -2b\tau_{ab}u^a u^b,$$

thent he condition of Result 2 is satisfied and therefore $p = \mu$ or $\mathbf{h}$ is a Killing vector, as in the perfect fluid case.

Appendix C SOME POSSIBLE INTRINSIC SYMMETRIES FOR LRS CLASS II

With space-times from LRS class II all combinations of the basis vectors given in 1.3 are surface-forming. This gives several geometrically well defined families of submanifo1ds with which we may use intrinsic symmetries. The combinations of basis vectors lead us to consider the following surfaces:

$$
\begin{aligned}
{}_{123}\mathscr{S} &\in \mathscr{F}_1(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}) \\
{}_{023}\mathscr{S} &\in \mathscr{F}_1(\{\mathbf{e}_0, \mathbf{e}_2, \mathbf{e}_3\}) \\
{}_{012}\mathscr{S} &\in \mathscr{F}_1(\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\})^\dagger \in \{\mathscr{F}_1(\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{v}\}) | \mathbf{v}\cdot\mathbf{e}_0 = \mathbf{v}\cdot\mathbf{e}_1 = 0\}
\end{aligned}
$$

$$
\begin{aligned}
{}_{01}\mathscr{S} &\in \mathscr{F}_2(\{\mathbf{e}_0, \mathbf{e}_1\}) \\
{}_{02}\mathscr{S} &\in \mathscr{F}_2(\{\mathbf{e}_0, \mathbf{e}_2\})^\dagger \in \{\mathscr{F}_2(\{\mathbf{e}_0, \mathbf{v}\}) | \mathbf{v}\cdot\mathbf{e}_0 = \mathbf{v}\cdot\mathbf{e}_1 = 0\} \\
{}_{12}\mathscr{S} &\in \mathscr{F}_2(\{\mathbf{e}_1, \mathbf{e}_2\}) \in \{\mathscr{F}_2(\{\mathbf{e}_1, \mathbf{v}\}) | \mathbf{v}\cdot\mathbf{e}_0 = \mathbf{v}\cdot\mathbf{e}_1 = 0\} \\
{}_{23}\mathscr{S} &\in \mathscr{F}_2(\{\mathbf{e}_2, \mathbf{e}_3\})
\end{aligned}
$$

Letting

$$a\ldots b^{R_{de}} := R_{de}({}_{a\ldots b}\mathscr{S})$$

and

$$(c)a\ldots b^{\theta_{de}} := \theta_{de}({}_{a\ldots b}\mathscr{S}, \mathbf{e}_c)\ ,$$

we may consider the intrinsic symmetries listed below.

${}_{123}\mathscr{S}$:

$$
\begin{aligned}
&{}_{123}R = 2r + 4\partial_1 a - 6a^2 = 0 && {}_{(0)123}\theta = \alpha + 2\beta = 0 \\
&{}_{123}S^2 = \tfrac{(\partial_1 a - r)^2}{3} = 0 && {}_{(0)123}\sigma^2 = \tfrac{(\alpha-\beta)}{3} = 0 \\
&{}_{123}T^6 = 0 \text{ (holds identically)} && {}_{(0)23}\tau^6 = 0 \text{ holds identically}
\end{aligned}
$$

${}_{023}\mathscr{S}$:

$$
\begin{aligned}
&{}_{023}R = 2r + 4\partial_0\beta + 6\beta^2 = 0 && {}_{(1)023}\theta = \dot{u} - 2a = 0 \\
&{}_{023}S^2 = \tfrac{(\partial_0\beta - r)^2}{3} = 0 && {}_{(1)023}\sigma^2 = \tfrac{(\dot{u}+a)^2}{3} = 0 \\
&{}_{123}T^6 = 0 \text{ (holds identically)} && {}_{(1)023}\tau^6 = 0 \text{ (holds identically)}
\end{aligned}
$$

†These families are not defined uniquely but are representative of their classes.

${}_{012}\mathscr{S}$:

$$ {}_{012}R = 0 $$
$$ {}_{012}S^2 = 0 $$
$$ {}_{012}T^6 = 0 $$

${}_{01}\mathscr{S}$:

$$ {}_{01}R = -2\partial_1 \dot{u} - 2\dot{u}^2 + 2\partial_0 \alpha + 2\alpha^2 = 0 $$
$$ {}_{01}S^2 = 0 \text{ (holds identically)} $$

${}_{02}\mathscr{S}$:

$$ {}_{02}R = 2\partial_0 \beta + 2\beta^2 = 0 \qquad {}_{(1)02}\theta = \dot{u} - a = 0 $$
$$ {}_{02}S^2 = 0 \qquad {}_{(1)02}\sigma^2 = \tfrac{(\dot{u}+a)^2}{4} = 0 $$

${}_{12}\mathscr{S}$:

$$ {}_{12}R = 2\partial_1 a - 2a^2 = 0 \qquad {}_{(0)12}\theta = \alpha + \beta = 0 $$
$$ {}_{12}S^2 = 0 \qquad {}_{(0)12}\sigma^2 = \tfrac{(\alpha-\beta)^2}{4} = 0 $$

${}_{23}\mathscr{S}$:

$$ {}_{23}R = 2r = 0 \qquad {}_{(0)23}\theta = 2\beta = 0 $$
$$ {}_{23}S^2 = 0 \qquad {}_{(0)23}\sigma^2 = 0 \text{ (holds identically)} $$
$$ {}_{(1)23}\theta = -2a = 0 $$
$$ {}_{(1)23}\sigma^2 = 0 \text{ (holds identically)} $$

Here we have not included the equations for the extrinsic curvature of a surface with normal $\mathbf{e}_2$ or $\mathbf{e}_3$, since $\gamma^3{}_{23} = s$ and the equations depend on the choice of tetrad. Also, we have omited the explicit formulae for the intrinsic symmetries of ${}_{012}\mathscr{S}$ because their subspace is quite general and the expressions are lenghty compared to the other cases.

Appendix D INITIALIZATION FILE FOR LRS CLASS II

```
/*****************************************************************/
/*                                                               */
/*      This file defines the basic equations for LRS Class II    */
/*      space-times. First a few control variables are reset      */
/*      and the unknowns are defined. Then the propagation        */
/*      equations are implemented as GRADEF's.                    */
/*                                                               */
/*****************************************************************/

                /* TAILOR THE MACSYMA ENVIRONMENT */

LINEL : 65$                        /* Set output line length. */
BOTHCASES : TRUE$                  /* Distinguish upper and lower case. */
DECLARE("&", ALPHABETIC)$          /* Treat & as a..z,A..Z and %. */
DERIVABBREV : TRUE$                /* Use subscriptt notation. */
PROGRAMMODE : TRUE$                /* Make SOLVE return evaluated list. */
INFEVAL : TRUE$                    /* Make EV use INFEVAL mode. */
LOADPRINT : FALSE$                 /* . Suppress loading messages. */

                       /* DEFINE UNKNOWNS */

ORDERGREAT(F, X)$                         /* Change default ordering so */
ORDERLESS(mu, p, a, r, alf, bet)$         /* expressions group nicely. */
UNKNOWNS: [F, X, Y, s, a, r, udt, alf, bet, E, H, tau, eps, mu, p]$
COORDS : [t, x]$
DEPENDS (UNKNOWNS,' COORDS, LAM, []) $

           /* DEFINE DIFFERENTIAL OPERATORS AND THE COMMUTATOR */

e0(ARG) := DIFF(ARG, t)*F$
e1(ARG) := DIFF(ARG, x)/X$
CR(ARG) := 0 = e0(e1(ARG)) - e1(e0(ARG)) - udt*e0(ARG) + alf*el(ARG)$

                  /* GIVE THE BASIC SYSTEM OF EQUATIONS */

                        /* Jacobi identities: */

/* J1 */       GRADEF(s,x, a*s*X)$
/* J2'*/       GRADEF(a,t, -bet*(a+ udt)/F)$
/* J3 */       GRADEF(s, t, -bet*s/F)$
/* J4 */       GRADEF(r,t, -2*bet*r/F)$
/* J5 */       GRADEF(r,x, 2*a*r*X)$

                  /* Maxwell's equations: */

/* M1 */       GRADEF(E, x, (2*a*E + eps)*X)$ GRADEF(H, x, 2*a*H*X)$
/* M2 */       GRADEF(H, x, 2*a*H*X)$
/* M3 */       GRADEF(E, t, -2*bet*E/F)$
/* M4 */       GRADEF(H, t, -2*bet*H/F)$
/* M5 */       GRADEF(tau, t, -4*bet*tau/F)$
/* M6 */       GRADEF(tau, x, (4*a*tau + eps*E)*X)$
/* M7 */       GRADEF(eps, t, -eps*(alf + 2*bet)/F)$

                     /* Field equations: */
```

```
/* F1 */        GRADEF(alf, t, (-mu/2 - p/2 + bet^2 - alf^2 - a^2 + r
                                + e1(udt) + udt^2 - 2*tau)/F )$
/* F2 */        GRADEF(bet, t, (LAM - p - 3*bet^2 + a^2 - r - 2*a*udt
                                + tau)/(2*F) )$
/* F3 */        GRADEF(a, x, (LAM + mu - bet^2 - 2*alf*bet + 3*a^2 - r
                                + tau) *X/2 ) $
/* F4 */        GRADEF(bet, x, a*(bet - alf)*X )$

                    /* Bianchi identities: */

/* BI1 */       GRADEF(mu, t, -emu + p)*(alf + 2*bet}/F )$
/* BI2 */       GRADEF(p, x, (-(mu + p)*udt - eps*E)*X )$

                       /* END OF FILE */
```

Appendix E DETAILS OF CALCULATIONS FOR CHAPTER IV

The following pages give the details of the calculations, in the form of a MACSYMA session, for the intrinsic symmetries examined in Chapter 4. The symmetry (I2) has been omitted since the MACSYMA session was discussed in the text.

I1: $_{123}R = 0$ *with* $\beta \neq 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1)                  DONE
(C2) BATCH("MC:SWATT\;CASE I1");
(C3) I1 : SOLVE (r = -2*e1 (a) + 3*a^2, LAM) [1];

                                  2
(D3)   LAM = - tau + bet + 2 alf bet - mu

(C4) I1\,0: SOLVE(EV(e0('I1), I1), DIFF(udt, x))[1];

(D4) udt = - (2 bet udt + (- 2 a bet - 2 a alf) udt
        x
                   2         2
          + (r - a) bet + (a - r) alf) X/(2 bet)

(C5) GRADEF(udt, x, RHS(I1\,0))$
(C6) GRADEF(alf, t, EV(DIFF(alf, t), DIFF))$
(C7) DEPENDS(M1, COORDS)$
(C8) I1\,1a : DIFF(mu, x) = M1*X$
(C9) GRADEF(mu, x, RHS(I1\,1a))$
(C10) I1\,1b : SOLVE(EV(e1('I1), I1), DIFF(alf, x)) [1];

                                          2         2
           (4 a tau + E eps + M1 - 2 a bet + 2 a alf ) X
(D10) alf = -----------------------------------------------
         x                      2 bet

(C11) GRADEF(alf, x, RHS(I1\,lb))$
(C12) I1\,1a\*: SOLVE(EV(CR(mu), I1), DIFF(M1, t))[1];

                                                  2
(D12) M1 = - ((4 p + 4 mu) a tau + (- 4 bet - 2 alf bet + p + mu) E eps
        t
                  2                                             2
           + (4 bet + 4 alf bet + p + mu)M1 + (2 p + 2 mu) a bet
                                                         2
           + (- 4 p - 4 mu) a alf bet + (2 p + 2 mu) a alf)/(2 bet F)

(C13) GRADEF(M1, t, RHS(I1\,1a\*))$
(C14) I1\,1b\* : RATSIMP(EV(CR(alf), I1));

         (bet + alf) E eps
(D14) 0 = -----------------
                bet

(C15) I1\,1b\*A : bet = -alf$
(C16) I1\,1b\*A0 : SOLVE(EV(e0('I1\,1b\*A), I1, I1\,1b\*A), mu) [1];

                                 2
(D16)      - 2 a udt + 4 tau + 4 alf + r - a + 2p
(D16) mu=   -------------------------------------
                             2

(C17) I1\,1b\*A1 : SOLVE(EV(e1('I1\,1b\*A), I1, I1\,1b\*A), M1)[1];
```

```
                                                         2
(D17) M1 = - 4 a tau - E eps + 2 a alf bet - 2 a alf

(C18) I1\,1b\*A01 :
RATSIMP(EV(e1(RHS('I1\,1b\*A0) - LHS('I1\,1b\*A0) = 0),
                I1, I1\,1b\*A, I1\,1b\*A0, I1\,1b\*A1));

                       2    2                        2
(D18) -(4 tau + 4 alf + 2a)udt + 8 a tau + 8 a alf + ar - a3 = 0

(C19) I1\,1b\*A10 :
RATSIMP(EV(e0 (RHS ('I1\,1b\*A1) - LHS('I1\,1b\*A1) = 0),
I1, I1\,1b\*A, I1\,1b\*A0, I1\,1b\*A1));

                            3     2
(D19) (- 4 alf tau - 4 alf - 2 a alf) udt - 8 a alf tau
                                3    3
       - 2 alf E eps - 8 a alf + (a - ar)alf = 0

(C20) RATSIMP(I1\,1b\*A01 - I1\,1b\*A10/(2*alf));
(D20) E eps = 0
(C21) I1\,1b\*B : E = 0$
(C22) EV(e0('I1\,1b\*B), I1\,1b\*B);
(D22) a = 0
(C23) EV(e1('I1\,1b\*B), I1\,1b\*B);
(D23) eps = 0
(C24) I1\,1b\*C : eps = as
(C25) EV(e0('I1\,1b\*C), I1\,1b\*C);
(D25) 0 = 0
(C26) I1LIST:[I1, I1\,1b\*C];
(D26) [LAM = - tau + bet + 2 alf bet - mu, eps = 0]
(D27) BATCH DONE
(C28) KILL(ALL)$
```

I1&6: ${}_{123}R = 0$ *After Imposing* ${}_{(0)23}\theta = 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I1&6");
(C3) BATCHLOAD("MC:SWATT\; CASE I6");
(D3) DONE
(C4) I1&6 : EV(r = -2*e1(a) + 3*a^2, I6LIST, EXPAND);

                           2
(D4) r = - 2 a udt + a  - p - mu

(C5) EV(e0('I1&6), I6LIST, EXPAND);
(D5) 0 = 0
(C6) I1&6\,1 : SOLVE(EV(e1('I1&6), I6LIST), DIFF(mu, x)) [1];

(D6) mu = - 4 a tau X
       x

(C7) GRADEF(mu, x, RHS(I1&6\,1))$
(C8) EV(CR(mu), I6LIST, I1&6, EXPAND};
(DS) 0 = 0
(D9)  BATCH DONE
(C10) KILL (ALL) $
```

I4: $_{12}R = 0$ *with* $\beta \neq 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I4");
(C3) I4 : SOLVE(0 = e1(a) - a^2, LAM) [1];

                         2                   2
(D3) LAM = - tau + bet + 2 alf bet + r - a - mu
(C4) I4\,0 : SOLVE(EV(e0('I4), I4), DIFF(udt, x))[1];

                         2
               (bet udt - a alf udt) X
(D4) udt = -  ------------------------
        x               bet

(C5) GRADEF(udt, x, RHS(I4\,0))$
(C6) GRADEF(alf, t, EV(DIFF(alf, t), DIFF))$
(C7) DEPENDS(M1, COORDS)$
(C8) I4\,1a : DIFF(mu, x) = M1*X$
(C9) GRADEF(mu, x, RHS(I4\,1a))$
(C10) I4\,1b : SOLVE(EV(e1('I4), I4), DIFF(alf, x))[1];

                                             2         2          3
(D10) alf =   (4 a tau + E eps + M1 - 2 a bet + 2 a alf - 2ar + 2a ) X
         x   -----------------------------------------------------------
                                           2 bet

(C11) GRADEF(alf, x, RHS(I4\,1b))$
(C12) I4\,1a\* : SOLVE(EV(CR(mu), I4), DIFF(M1, t))[1];

                                                 2
(D12) M1 = - ((4 p + 4 au) a tau + (- 4 bet - 2 alf bet + p + mu) E eps
        t
                    2                                                 2
             + (4 bet + 4 alf bet + p + mu) M1 + (2 p + 2 mu) a bet

                                                           2
             + (- 4p - 4 mu) a alf bet + (2 p + 2 mu) a alf

                                                3
             + (- 2p - 2 mu) a r + (2 p + 2 mu) a )/(2 bet F)

(C13) GRADEF(M1, t, RHS(I4\,1a\*))$
(C14) I4\,1b\* : RATSIMP(EV(CR(alf), I4));

          (bet + alf) E Eps
(D14) 0 =  -----------------
                bet

(C15) I4\,1b\*A : bet = -alf$
(C16) I4\,1b\*A0 : SOLVE(EV(e0('I4\,1b\*A), I4, I4\,1b\*A), mu) [1];

                                     2       2
(D16) mu = -2 a udt - 2 tau - 2 alf + r - a - p

(C17) I4\,1b\*A1 : SOLVE(EV(e1('I4\,1b\*A), I4, I4\,Ib\*A), M1) [1];
```

```
                                                   2            3
(D17) M1 = - 4 a tau - E eps + 2 a alf bet - 2 a alf+ 2 a r - 2 a

(C18) I4\,1b\*A01 :
RATSIMP(EV(e1(RHS('I4\,1b\*A0) - LHS('I4\,1b\*A0) = 0),
        I4, I4\,1b\*A, I4\,1b\*A0, I4\,1b\*A1));

                      2       2                        2
(D18) (- 2 tau - 2 alf + r - a) udt - 4 a tau - 4 a alf = 0

(C19) I4\,1b\*A10 :
RATS1MP(EV(e0(RHS('I4\,1b\*A1) - LHS('I4\,1b\*A1) = 0),
        I4, I4\,1b\*A, I4\,1b\*A0, I4\,1b\*A1));

                         3            2
(D19) (-4 alf tau - 4 alf + (2 r - 2 a) alf) udt - 8 a alf tau

                              3
       - 2 alf E eps - 8 a alf = 0

(C20) RATSIMP(I4\,1b\*A01 - I4\,1b\*A10/(2*alf));
(D20) E eps = 0
(C21) I4\,1b\*B : E = 0$
(C22) EV(e0('I4\,1b\*B), I4\,1b\*B);
(D22) 0 = 0
(C23) EV(e1('I4\,1b\*B), I4\,1b\*B);
(D23) eps = 0
(C24) I4\,1b\*C : eps = 0$
(C25) EV(e0('I4\,1b\*C), I4\,1b\*C);
(D25) 0 = 0
(C26) I4LIST: [I4, I4\,1b\*C];

                        2                   2
(D26) [LAM = - tau + bet + 2 alf bet + r - a - mu, eps = 0]

(D27) BATCH DONE
(C28) KILL(ALL)$
```

I4&6: $_{12}R = 0$ *After Imposing* $_{(0)23}\theta = 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I4&6");
(C3) BATCHLOAD("MC:SWATT\;CASE I6");
(D3) DONE
(C4) I4&6 : EV(r = -2*e1(a) + 3*a^2, I6LIST, EXPAND);

                             2
(D4) r = - 2 a udt + a - p - mu

(C5) EV(e0('I4&6), I6LIST, EXPAND);
(D5) 0 = 0
(C6) I4&6\,1 : SOLVE(EV(e1('I4&6), I6LIST), DIFF(mu, x))[1];
(D6) mu = - 4 a tau X
(C7) GRADEF(mu, x, RHS(I4&6\,1))$
(C8) EV(CR(mu), I6LIST, I4&6, EXPAND);
(D8) 0 = 0
(D9) BATCH DONE
(C10) KILL(ALL)$
```

I5: $_{23}R = 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I5");
(C3) I5 : r = 0$
(C4) EV(e0 ('I5), I5);
(D4) 0 = 0
(C5) EV(e1('I5), I5);
(D5) 0 = 0
(D6) BATCH DONE
(C7) KILL(ALL)$
```

I6: $_{(0)23}\theta = 0$

```
(C1) BATCHLOAD("MC:SWATT\:LRSII SETUP"):
(D1) DONE
(C2) BATCH("MC:SWATT\:CASE I6");
(C3) I6 : bet = 0$
(C4) I6LIST : [I6]$
(C5) I6\,0 : SOLVE(eV(e0('I6), I6LIST), LAM) [1];

                                  2
(D5) LAM = 2 a udt - tau + r - a + p

(C6) I6LIST : ENDCONS(I6\,0 , I6LIST)$
(C7) /* Show that a = 0 is a contradiction */
CONTR : a = 0 $

(C8) EV(e1(2*'CONTR), I6LIST, CONTR, EXPAND);
(D8) p + mu = 0
(C9) KILL(CONTR)$
(C10) /* Continue with the propagation of bet=0 */
I6\,1 : EV(e1(-'I6), I6LIST)/a:
(D10) alf = 0
(C11) I6LIST : ENDCONS(I6\,1 , I6LIST)$
(C12) DEPENDS(U0, COORDS)$
(C13) GRADEF(udt, t, U0/F)$
(C14) I6\,00: SOLVE(EV(e0('I6\,0), I6LIST), DIFF(p, t))[1];

          2 a U0
(D14) p = ------
       t    F

(C15) GRADEF(p, t, RHS(I6\,00))$
(C16) I6\,00\*: FACTOR(SOLVE(EV(CR(p), I6LIST), DIFF(U0, x))[1]);

(D16) U0 = -U0(3 ud t + a) X
        x

(C17) GRADEF(U0, x, RHS(I6\,00\*))$
(C18) I6\,01 : FACTOR(SOLVE(EV(e1('I6\,0), I6LIST), DIFF(udt, x))[1]);
(D18) udt  =
         x

               2                                3
   (2 a -udt - 4 a tau - 2 E eps + 2 a r - 2 a - p a - mu a) X
 - -----------------------------------------------------------
                                   2 a

(C19) GRADEF(udt, x, RHS(I6\,01))$
(C20) GRADEF(alf, t, EV(DIFF(alf, t), DIFF))$
(C21) EV(CR(udt), I6LIST, EXPAND);
(D21) 0 = 0
(C22) I6\,10 : EV(e0('I6\,1), I6LIST, EXPAND)*a;
(D22) E eps = 0
(C23) I6\,10A : E = 0$
(C24) I6\,10B : eps = 0$
(C25) EV(e0('I6\,10B), I6\,10B);
(D25) 0 = 0
```

```
(C26) I6LIST : ENDCONS(I6\,10B , I6LIST)$
(C27) I6\,11 : e1(I6\,1);

                 alf
                    x
(D27)           ----- = 0
                  X

(C28) GRADEF(alf, x, RHS(I6\,11))$
(C29) EV(CR(alf), I6LIST, EXPAND);
(D29) 0 = 0
(C30) I6LIST;

                                                   2
(D30) [bet = 0, LAM = 2 a udt - tau + r - a + p, alf = 0,

                                                                eps = 0]

(D31) BATCH DONE
(C32) KILL(ALL)$
```

Appendix F DETAILS OF CALCULATIONS FOR CHAPTER V

This appendix gives MACSYMA sessions for the intrinsic symmetries examined in Chapter 5. The intrinsic symmetry (I5) is not included since it has been given in Appendix E.

I7: $_{023}R = 0$ *with* $a \neq 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I7");
(C3) I7:EV(SOLVE(e0(bet)=-(3*bet^2+r)/2,LAM) [1]);

                                2
(D3) LAM = 2 a udt - tau - a + p
(C4) DEPENDS(U0,[t,x])$
(C5) GRADEF(udt,t,U0/F)$
(C6) I7\,0:EV(SOLVE(EV(e0('I7), I7),DIFF(p,t))[1]);

                       2                            2
            2 bet udt - 4 bet tau - 2 a U0 - 2 a bet
(D6)    p = ------------------------------------------
         t                     F

(C7) GRADEF(p,t,RHS(I7\,0))$
(C8) I7\,1:EV(SOLVE(EV(e1('I7), I7),DIFF(udt,x))[1]);

                       2       2
(D8) udt = - (2 a udt + (- bet - 2 alf bet - r) udt - 4 a tau
        x

                            2                          3
          - 2 E eps + a bet + 2 a alf bet + a r - 2 a + (- p - mu) a) X
/(2 a)
(C9) GRADEF(udt,x,RHS(I7\,1))$
(C10) GRADEF(alf,t,RATSIMP(EV(DIFF(alf,t),DIFF)))$
(C11) I7\,0a\* :EV(SOLVE(EV(CR(p), I7) ,DIFF(U0,x)) [1]);

                  3           2            2           2
(D11) U0 = ((2 bet + 4 alf bet + (2 r + 2 a ) bet - 2 a alf)
        x

   2                    2            3             2
udt + (4 bet E eps - 6 a U0 - 2 a bet - 4 a alf bet

                3                2           2
+ (- 2 a r - 4 a ) bet) udt + (4 a alf - 20 a bet) tau

                        2                          3
- 8 a bet E eps + (a bet + 2 a alf bet + a r - 2 a ) U0

     2   3     2        2      2         4
+ 2 a bet + 4 a alf bet + (2 a 4 r - 6 a

                 2            4         2
+ (-2p - 2 mu) a ) bet + 2 a  alf) X/(2 a )
(C12) GRADEF(U0,x,RHS(I7\,0a\*))$
(C13) I7\,0b\*:RATSIMP(EV(CR(udt) , I7));
(D13) 0 = 0
(C14) I7LIST : [I7];
                                2
(D14) [LAM = 2 a udt - tau - a + p]
(D15) BATCH DONE
(C16) KILL(ALL)$
```

I7&11: ${}_{023}R = 0$ *After Imposing* ${}_{(1)23}\theta = 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I7&11");
(C3) BATCHLOAD("MC:SWATT\;CASE I11");
(D3) DONE
(C4) I7&11 : SOLVE (EV (e0 (bet)=-(3 *bet^2+r) /2, I11LIST) ,mu) [1];

                 2
(D4) mu = bet  + 2 alf bet + r - p

(C5) I7&11\,0:SOLVE(EV(e0(I7&11) ,I11L1ST) ,DIFF(p, t)) [1];

              4 bet tau
(D5) p   =  - -----------
      t            F

(C6) GRADEF(p,t,RHS(I7&11\,0))$
(C7) RATSIMP(EV(e1('I7&11) ,I11LIST);
(D7) M1 = M1
(C8) EV(CR(p),I11LIST,I7&11,EXPAND);
(D8) 0 = 0
(D9) BATCH DONE
(C10) KILL (ALL)$
```

I8: ${}_{023}S = 0$ *with* $a \neq 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I8");
(C3) I8:EV(SOLVE(e0(bet)=r,LAM) [1];

                                  2          2
(D3) LAM = 2 a udt - tau + 3 bet  + 3 r - a  + p

(C4) DEPENDS(U0,[t,x])$
(C5) GRADEF(udt,t,U0/F)$
(C6) I8\,0:EV(SOLVE(EV(e0('I8), I8), DIFF(p,t)) [1]);

                        2                           2
             2 bet udt  - 4 bet tau - 2 a U0 - 2 a bet
(D6)    p  = -----------------------------------------
         t                      F

(C7) GRADEF(p,t,RHS(I8\,0))$
(C8) I8\,1:EV(SOLVE (EV(e1 ('I8), I8) ,DIFF(udt,x)) [1]);

                         2        2
(D8) udt  = - (2 a udt  + (2 bet  - 2 alf bet + 2 r) udt
        x

                                  2                         3
- 4 a tau - 2 E eps + 4 a bet  - 4 a alf bet + 4 a r - 2 a

 + (- p - mu) a) X/(2 a)

(C9) GRADEF(udt,x,RHS(I8\,1))$
(C10) GRADEF(alf,t,RATSIMP(EV(DIFF(alf,t) ,DIFF)))$
(C11) I8\,0a\*:EV(SOLVE(EV(CR(p) , I8),DIFF(U0,x)) [1]);

                    3            2          2         2
(D11) U0  = - ((2 bet  - 2 alf bet  + (2 r - a ) bet + a  alf)
        x

   2                      2             3             2
udt  + (- 2 bet E eps + 3 a  U0 + 4 a bet  - 4 a alf bet

                  3             2          2
+ (4 a r + 2 a ) bet) udt + (10 a bet - 2 a  alf) tau

                        2                          3          2   3
+ 4 a bet E eps + (a bet  - a alf bet + a r + a ) U0 + 2 a  bet

      2       2       2        4             2         4
- 2 a  alf bet  + (2 a  r + 3 a  + (p + mu) a ) bet - a  alf) X

  2
/a

(C12) GRADEF(U0,x,RHS(I8\,0a\*))$
(C13) I8\,0b\* :RATSIMP (EV(CR (udt) , I8));
```

```
         3 bet E eps
(D13) 0 = ----------
               a

(C14) I8\,Ob\*A:bet = 0;
(D14) bet = 0
(C15) I8\,Ob\*AO:RATSIMP(EV(e0('I8\,Ob\*A), I8\,Ob\*A, I8));
(D15) r = 0
(C16) I8\,Ob\*A1:EV(e1('I8\,Ob\*A), I8\,Ob\*A)/(-a);
(D16) alf = 0
(CI7) RATSIMP (EV(e0 ('I8\,Ob\*A1) , I8\,Ob\*A, I8\,Ob\*A0, I8\,Ob\*A1));

      E eps
(D17) ----- = 0
        a

(C18) I8\,Ob\*B:E = 0;
(D18) E = 0
(C19) EV(e1('I8\,Ob\*B), I8\,Ob\*B);
(D19) eps = 0
(C20) I8\,Ob\*C:eps = 0;
(D20) eps = 0
(C21) EV(e0 ('I8\,Ob\*C), I8\,Ob\*C);
(D21) 0 = 0
(C22) I8LIST : [I8, I8\,Ob\*C];

                                     2         2
(D22) [LAM = 2 a udt - tau + 3 bet + 3 r - a + p, eps = 0]

(D23) BATCH DONE
(C24) KILL (ALL)$
```

I8&11: ${}_{023}S = 0$ *After Imposing* ${}_{(1)23}\theta = 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP"):
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I8&11");
(C3) BATCHLOAD("MC:SWATT\;CASE I11");
(D3) DONE
(C4) I8&11 : SOLVE (EV(eO (bet)=r ,I11LIST) ,mu) [1];

                     2
(D4) mu = - 2 bet + 2 alf bet - 2 r - p.

(C5) I8&11\,0:SOLVE(EV(e0('I8&11) ,I11LIST, I8&11),DIFF(p, t)) [1]:

             4 bet tau
(D5) p =  - -----------
      t          F

(C6) GRADEF(p,t,RHS(I8&11\,0))$
(C7) RATSIMP(EV(e1('I8&11),I11LIST));
(D7) M1 = M1
(C8) EV(CR(p) ,I11LIST,I8&11,EXPAND);
(DS) 0 = 0
(D9) BATCH DONE
(C10) KILL(ALL)$
```

I10: $_{02}R = 0$ *with* $a \neq 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I10");
(C3) I10 :EV(SOLVE (e0 (bet) =-bet^2, LAM) [1]) ;

                                   2       2
(D3) LAM = 2 a udt - tau + bet + r - a + p

(C4) DEPENDS(U0,[t,x])$
(C5) GRADEF(udt,t,U0/F)$
(C6) I10\,0:EV(SOLVE(EV(e0 ('I10), I10) ,DIFF(p,t)) [1]);

(D6) p =
      t

                 2                             3            2
        2 bet udt - 4 bet tau - 2 a U0 + 2 bet + (2 r - 2 a ) bet
        ----------------------------------------------------------
                                      F

(C7) GRADEF(p,t,RHS(I10\,0))$
(C8) I10\,1:EV(SOLVE(EV(e1('I10), I10),DIFF(udt,x))[1]);

                    2
(D8) udt = - (2 a udt - 2 alf bet udt - 4 a tau - 2 E eps
        x

                  2          3
          + 2 a bet + 2ar - 2a + (- P - mu) a) X/(2 a)

(C9) GRADEF(udt,x,RHS(I10\,1))$
(C10) GRADEF(alf,t,RATSIMP(EV(DIFF(alf,t) ,DIFF)))$
(C11) I10\,0a\*:EV(SOLVE(EV(CR(p) , I10),DIFF(U0,x))[1]);.

                        2   2       2        2
(D11) U0 = ((2 alf bet + a bet - a alf) udt
        x

                  2          3
+ (2 bet E eps - 3 a - U0 - 2 a bet) udt

     2          2
+ (2 a alf - 10 a bet) tau - 4 a bet E eps

                3          2   3   2       2
+ (a alf bet - a ) U0 + 3 a bet - a alf bet

     2       4              2          4   2           2
+ (3 a r - 3 a + (- p - mu) a ) bet + (a - a r) alf) X/a

(C12) GRADEF(U0,x,RHS(I10\,0a\*))$
(C13) I10\,0b\*:RATSIMP(EV(CR(udt) , I10));
```

```
         bet E eps
(D13) 0 = -----------
              a

(C14) I10\,0b\*A:bet = 0;
(D14) bet = 0
(C15) I10\,0b\*A1:EV(e1('I10\,0b\*A), I10\,0b\*A)/(-a);
(D15) alf = 0
(C16) RATSIMP(EV(e0('I10\,0b\*A1), I10\,0b\*A,I10\,0b\*A1));

       E eps
(D16) ------- = 0
         a

(C17) I10\,0b\*B:E = 0;
(D17) E = 0;
(C18) EV(e1('I10\,0b\*B), I10\,0b\*B);
(D18) eps = 0
(C19) I10\,0b\*C:eps = 0;
(D19) eps = 0
(C20) EV(e0('I10\,0b\*C), I10\,0b\*C);
(D20) 0 = 0
(C21) I10LIST : [I10, I10\, 0b\ *C];

                                       2        2
(D21) [LAM = 2 a udt - tau + bet + r - a + p, eps = 0]

(D22) BATCH DONE
(C23) KILL(ALL)$
```

I10&11: $_{02}R = 0$ *After Imposing* $_{(1)23}\theta = 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C2) BATCH("MC:SWATT\;CASE I10&11");
(C3) BATCHLOAD("MC:SWATT\;CASE I11");
(D3) DONE
(C4) I10&11:SOLVE(EV(e0(bet)=-bet^2,I11LIST) ,mu) [1];
(D4) mu = 2 alf bet - p
(C5) I10&11\,0:SOLVE(EV(e0('I10&11) ,I11LIST, I10&11),DIFF(p,t))[1];

                                 3
           4 bet tau - 2 bet - 2 r bet
(D5) p = - --------------------------
      t                  F

(C6) GRADEF(p,t,RHS(I10&11\,0))$
(C7) RATSIMP(EV(e1('I10&11) ,I11LIST));
(D7) M1 = M1
(C8) EV(CR(p) ,I11LIST,I10&11,EXPAND);
(D8) 0 = 0
(D9) BATCH DONE
(C10) KILL(ALL)$
```

I11: $_{(1)23}\theta = 0$

```
(C1) BATCHLOAD("MC:SWATT\;LRSII SETUP");
(D1) DONE
(C3) I11:a = 0;
(C2) BATCH("MC:SWATT\;CASE I11");
(C3) I11:a = 0;
(D3) a = 0
(C4) I11\,0:-EV(e0('I11),I11)/bet;
(D4) udt = 0
(C5) I11LIST:(I11,I11\,0]$
(C6) I11\,1:SOLVE(EV(e1('I11),I11LIST) ,LAM) (1];

                             2
(D6) LAM = - tau + bet + 2 alf bet + r - mu

(C7) I11LIST:ENDCONS(I11\,1,I11LIST)$
(C8) RATSIMP(EV(e0('I11\,1),I11LIST));
(D8) 0 = 0
(C9) DEPENDS(M1,COORDS)$
(C10) GRADEF(mu,x,M1*X)$
(C11) I11\,11a:SOLVE(EV(e1('I11\,1),I11LIST),DIFF(alf,x)) [1];

             (E eps + M1) X
(D11) alf = ---------------
         x       2 bet

(C12) GRADEF(alf,x,RHS(I11\,11a))$
(C13) I11\,11b\*:SOLVE(EV(CR(mu) ,I11LIST) ,DIFF(M1,t)) [1];

                         2
(D13) M1 = ((4 bet + 2 alf bet - p - mu) E eps
        t

          2
+ (- 4 bet - 4 alf bet - p - mu) M1)/(2 bet F)

(C14) GRADEF(M1,t,RHS(I11\,11b\*))$
(C15) I11\,11a\*:RATSIMP(EV(CR(alf) ,I11LIST));

           (bet + alf) E eps
(D15) 0 = --------------------
                 bet

(C16) I11\,11a\*A:alf = -bet;
(D16) alf = - bet
(C17) I11\,11a\*A0:SOLVE(EV(e0('I11\,11a\*A),I11LIST,I11\,11a\*A) ,mu) [1];

                             2
(D17) mu = - 2 tau - 2 bet + r - p

(C18) ALIST:[I11\,11a\*A,I11\,11a\*A0]$
(C19) I11\,11a\*A1:SOLVE(EV(e1('I11\,11a\*A),I11LIST,ALIST) ,M1) [1];
(D19) M1 = - E eps
(C20) ALIST:ENDCONS(%,ALIST)$
(C21) RATSIMP(EV(e0(I11\,11a\*A1) ,ALIST,I11LIST));
(D21) bet E eps = 3 bet E eps
```

```
(C22) KILL(ALIST)$
(C23) I11\,11a\*B:E = 0;
(D23) E = 0
(C24) EV(e0('I11\,11a\*B) ,I11\,11a\*B);
(D24) 0 = 0
(C25) EV(e1('I11\,11a\*B) ,I11\,11a\*B);
(D25) eps = 0
(C26) I11\,11a\*C:eps = 0;
(D26) eps = 0
(C27) EV(e0('I11\,11a\*C) , I11\,11a\*C) ;
(D27) 0 = 0
(C28) I11LIST:ENDCONS(I11\,11a\*C,I11LIST);

                                                 2
(D28) [a = 0, udt = 0, LAM = - tau + bet + 2 alf bet + r - mu, eps = 0]

(D29) BATCH DONE
(C30) KILL(ALL)$
```

Appendix G A PROTOTYPE PROGRAM

This appendix contains a listing of the proqram discussed in Chapter 6. Two sample runs, corresponding to examples 2.2.1 and 2.2.2, are included after the listing.

Program Listing

```
/**********************************************************************/
/*      This file contains a collection of functions to check the      */
/*      consistency of a constraint with aset of equations. To do      */
/*      this the function CHECK%ONE is invoked at command level and    */
/*      questions regarding whether or not certain quantities are      */
/*      zero aust be answered. The result returned is either a list    */
/*      of possible cases or the atom CONTRADICTION. The method used   */
/*      is discussed in Chapter 6.                                     */
/**********************************************************************/

/* These two names shall be used as mnemonic indices to case lists. */

(ALG : 1, PROPN : 2)$

CHECK%ONE(CONSTRAINT, CASE) :=
  BLOCK(/* This procedure checks whether the given constraint
           may hold in the given case. The result is returned
           as either a list of possible cases or as the
           atom CONTRADICTION. */

           [ALG\LIST PROPN\LIST],
           LOCAL(UNKNOWNS),
           INITIALIZE%LEVEL() ,
           CONSTRAINT : CLEAN(CONSTRAINT),
           IF IDENT%PRED(CONSTRAINT) THEN
   RETURN([CASE])
           ELSE IF ALG%CONTRADICT%PRED(CONSTRAINT)
   RETURN(CONTRADICTION)
           ELSE
           IF PROPN%PRED(CONSTRAINT) THEN
           RETURN(PROPN%PROCESS(CONSTRAINT))
           ELSE
           RETURN(ALG%PROCESS(CONSTRAINT)))$
```

```
CHECK%OR(CONSTRAINT%LIST, CASE%LIST):=
  BLOCK(/* This procedure checks each of the list of
           constraints against each of a list the cases.
           The list constructed by concatenating all the resulting
           constraint lists is returned. */

           [NEW%CASE%LIST, THIS%CASE%LIST],
           NEW%CASE%LIST : [],
           FOR CASE IN CASE%LIST DO
            FOR CONSTRAINT IN CONSTRAINT\LIST DO
              (THIS%CASE%LIST : CHECK%ONE(CONSTRAINT, CASE),
               IF THIS%CASE%LIST # CONTRADICTION THEN
               NEW%CASE%LIST : APPEND(NEW%CASE%LIST,
                                            THIS%CASE%LIST)),
           RETURN(IF NEW%CASE%LIST = []
                     CONTRADICTION
                  ELSE
                     NEW\CASE%LIST))$

CHECK%AND(CONSTRAINT%LIST, CASE%LIST) :=
  BLOCK(/* This procedure checks whether all of the constraints
           in CONSTRAINT%LIST hold for any of the cases in CASE%LIST.
           This is achieved by iteratively checking the results of
           one constraint using the next. The resulting list
           of cases is returned. */

           [NEW%CASE%LIST],
            NEW\CASE%LIST : CHECK%OR([FIRST(CONSTRAINT%LIST)],
                                                   CASE%LIST),
            FOR CONSTRAINT IN REST(CONSTRAINT%LIST)
            WHILE NEW%CASE%LIST # CONTRADICTION DO
               NEW\CASE%LIST : CHECK%OR([CONSTRAINT], NEW%CASE%LIST),
            RETURN(NEW%CASE%LIST))$


INITIALIZE%LEVEL() :=
           (/* Define the list of equations to be used with EV and
            the local GRADEFs for this level. */
            ALG%LIST : CASE(ALG],
            PROPN%LIST : CASE(PROPN],
            FOR GRAD%SPEC IN PROPN%LIST DO
               APPLY('GRADEF,GRAD%SPEC) ,
            DONE)$

ADD(KIND, WHAT%TO%ADD, CASE) : =
            /* This function is to add new results to the collection of
            old results stored in the variable CASE. */
            SUBSTPART(ENDCONS(WHAT%TO%ADD,CASE(KIND]), CASE, KIND)$
```

```
CLEAN(EQN) :=
   BLOCK(/* Put the equation in a form amenable to spotting
         special cases. */

        [RHSIDE1,
         RHSIDE : FACTOR(EV(RHS(EQN)-LHS(EQN),ALG%LIST,DIFF,INFEVAL)),
         IF (NOT ATOM(RHSIDE)) AND (PART(RHSIDE,0)="-") THEN
            RHSIDE : -RHSIDE,
         RETURN(0=RHSIDE))$


IDENT%PRED(EQN) :=
        /* This predicate checks for a syntactic identity. */

        IS(0 = RHS(EQN))$


ALG%CONTRADICT%PRED(EQN) :=
        /* Check if the equation gives a contradiction by
        seeing if the RHS may never be zero. */

        MAY%NOT%ZERO%PRED(RHS(EQN))$


PROPN%PRED(EQN) :=
   BLOCK(/* See if the equation is a propagation equation
         by looking for first derivatives. */
         [EXPR, VARIABLE%LIST, RESULT],
         EXPR : RHS(EQN),
         VARIABLE%LIST : SORT(LISTOFVARS(EXPR)),
         RESULT: FALSE,
         FOR COORD IN COORDINATES UNLESS RESULT DO
            FOR VAR IN VARIABLE%LIST UNLESS RESULT DO
               IF 1 = DERIVDEGREE(EXPR, VAR, COORD) THEN
                  RESULT: TRUE,
         RETURN(RESULT))$


PROPN%PROCESS(EQN) :=
         /* Nothing need be done in the spatially homogeneous case. */
         PRINT("Base of recursion reached checking",
               "the propagation equation:",
                EQN)$
```

```
ALG%PROCESS(EQN) :=
   BLOCK(/* In this procedure we first solve the constraint equation for one
         of the variables so that the result may be used to simplify any
         further propagation equations.  If the equation is not linear in
         all of the variables, then we have more than one possible solution.
         We check each one of the possibilities by calling CHECK%OR with
         the solution list as an argument.  Otherwise, there is only one
         solution so we check the propagation(s) of that constraint. */
          [SOLN%LIST, DIFF%LIST, N],
          PRINT("Checking algebraic constraint:", EQN),
          SOLN%LIST : RESHAPE(EQN),
          N : LENGTH (SOLN%LIST) ,
          IF N > 1 THEN
             RETURN(CHECK%OR(SOLN%LIST,[CASE]))
          ELSE IF N = 1 THEN
            (CASE : ADD(ALG, SOLN%LIST[1], CASE),
             DIFF%LIST : [],
             FOR COORDNO : 1 THRU LENGTH(COORDINATES) DO
                DIFF%LIST : ENDCONS(CLEAN(D%OP(COORDNO, SOLN%LIST[1])),
                                           DIFF%LIST) ,
             RETURN (CHECK%AND(DIFF%LIST,[CASE])))
          ELSE IF N = 0 THEN
             RETURN (CONTRADICTION)
          ELSE
             ERROR("ALG%PROCESS: internal error"))$

MAY%NOT%ZERO%PRED(EXPR) :=
   BLOCK(/* This function checks whether an expression may not be zero,
          by asking about appropriate parts. */
         [PIECE, RESULT],
          EXPR : NUM(FACTOR(EXPR)),
          IF ATOM(EXPR) THEN
             IF NUMBERP(EXPR) THEN
                RESULT: IS(0 # EXPR)
             ELSE
                RESULT: IS(NO = MAY%ZERO%PROMPT(EXPR))
          ELSE
            (PART(EXPR, 0),
             IF PIECE = "-" THEN
                RESULT : MAY%NOT%ZERO%PRED(-EXPR)
             ELSE IF PIECE = "*" THE~
                RESULT : MAY%NOT%ZERO%PRED(FIRST(EXPR)) AND
                         MAY%NOT%ZERO%PRED(REST(EXPR))
             ELSE IF PIECE = "^" THEN
                RESULT : MAY%NOT%ZERO%PRED(FIRST(EXPR))
             ELSE
                RESULT: IS (NO = MAY%ZERO%PROMPT(EXPR))),
          RETURN(RESULT))$
```

```
MAY%ZERO%PROMPT(EXPR) :=
         (PRINT("May the following expression ever assume",
                "the value zero?", EXPR),
          READ("Enter YES\; or NO\; or DONTKNOW\; ."))$

D%OP(COORDNO, EXPR) :=
          /* This is the differentiation operator.
          By specifying the list of coefficients or by
          giving an alternative function, we may choose the
          directional derivative operators as we wish. */

          D%OP%COEFF%LIST[COORDNO]*DIFF(EXPR, COORDINATES[COORDNO])$

RESHAPE(EQN) :=
   BLOCK (/* From the input equation, this procedure constructs
          a list of solutions for the same varable. First the
          procedure checks for a linear varable so as to
          return a one element list if possible. If the equation
          is not linear in any of the variables,
          then the varaible of the lowest degree is sought. */

          [EXPR,VARIABLE%LIST, RESHAPED, MIN%DEGREE,
           MIN%DEGREE%VAR, RESULT],
           EXPR : RHS(EQN), /* The input is factored. */
           IF (NOT ATOM(EXPR)) AND (PART (EXPR, 0) = "*") THEN
              (RESULT: [],
               FOR FACTOR IN EXPR DO
                  IF NOT MAY%NOT%ZERO%PRED(FACTOR) THEN
                     RESULT: APPEND(RESHAPE(0=FACTOR), RESULT))
           ELSE
              (EXPR : RAT (EXPR) ,
               VARIABLE%LIST : SORT(LISTOFVARS(EXPR)),
               RESHAPED: FALSE,
               FOR VAR IN VARIABLE%LIST UNLESS RESHAPED DO
                  IF HIPOW(EXPR, VAR) = 1 AND LOPOW(EXPR, VAR) >=0 THEN
                     IF MAY%NOT%ZERO%PRED(RATCOEF(EXPR, VAR, 1)) THEN
                       (RESULT: SOLVE (EQN, VAR),
                        RESHAPED: TRUE),
               IF NOT RESHAPED THEN
                  (MIN\DEGREE : HIPOW(EXPR, FIRST(VARIABLE%LIST)),
                   MIN%DEGREE%VAR : FIRST(VARIABLE%LIST),
                   FOR VAR IN REST(VARIABLE%LIST) DO
                      IF HIPOW(EXPR, VAR) < MIN%DEGREE THEN
                         (MIN%DEGREE : HIPOW(EXPR,VAR),
                          MIN%DEGREE%VAR : VAR),
                   RESULT: SOLVE (EQN, MIN%DEGREE%VAR))),
               RETURN(RESULT))$
```

Sample Runs

Bianchi-Behr type $VI_{h=0}$: $n_1 = 0,\ n_2 > 0,\ n_3 < 0$

```
(C1) BATCHLOAD("MC:SWATT\;CHECK V0")$
(C2) BATCHLOAD("MC:SWATT\;TEST BIAN6H")$
(C3) CHECK%ONE(N1+N2+N3=0, CASE)$

May the following expression ever assume the value zero? N3 + N2
Enter YES; or NO; or DONTKNOW; .
DONTKNOW;

Checking algebraic constraint: 0 = N3 + N2
May the following expression ever assume the value zero? N3
Enter YES; or NO; or DONTKNOW; .
NO;

May the following expression ever assume the value zero? TH3 - TH2
Enter YES; or NO; or DONTKNOW; .
DONTKNOW;

Checking algebraic constraint: 0 = 2 N3 (TH3 - TH2)
May the following expression ever assume the value zero? N3
Enter YES; or NO; or DONTKNOW; .
NO;

May the following expression ever assume the value zero? TH3 - TH2
Enter YES; or NO; or DONTKNOW; .
DONTKNOW;

(C4) GRIND(D3);

[[[THETA = TH3+TH2+TH1,N1 = 0,N2 = -N3,TH2 = TH3],
[[N1,T,N1*{2*TH1-THETA)],(N2,T,N2*(2*TH2-THETA)],(N3,T,N3*C2*TH3-THETA)],
 [THETA,T,-TH3^2-TH2^2-TH1^2-(3*P+MU)/2+LAM],
 [TH1,T,-TH1*THETA+(MU-P)/2+(N2-N3)^2/2-N1^2/2+LAM],
 [TH2,T,-TH2*THETA+(MU-P)/2+(N3-N1)^2/2-N2^2/2+LAM],
 [TH3,T,-TH3*THETA+(MU-P)/2-N3^2/2+(N1-N2)^2/2+LAM],(MU,T,-(P+MU)*THETA]]]$
(D4) DONE
(C5) KILL(ALL)$
```

Bianchi-Behr type VIII: $n_1 > 0,\ n_2 > 0,\ n_3 < 0$

```
(C1) BATCHLOAD("MC:SWATT\;CHECK VO")$
(C2) BATCHLOAD("MC:SWATT\;TEST BIAN8")$
(C3) CHECK%ONE na+N2+N3=O, CASE);

May the following expression ever assume the value zero? N2 + Nl + N3
Enter YES; or NO; or DONTKNOW; .
DONTKNOW;

Checking algebraic constraint: 0 = N2 + N1 + N3
May the following expression ever assume the value zero?

N2 TH2 + Nl THl - TH3 N2 - TH3 NI
Enter YES; or NO; or DONTKNOW; .
DONTKNOW;

Checking algebraic constraint: 0 = 2 (N2 TH2 + N1 TH1 - TH3 N2 - TH3 N1)
May the following expression ever assume the value zero?

N2 TH2 + N1 TH1 - TH3 N2 - TH3 N1
Enter YES; or NO; or DONTKNOW; .
DONTKNOW;

May the following expression ever assume the value zero? N2 + N1
Enter YES; or NO; or DONTKNOW; .
NO;

May the fol1owing expression ever assume the value zero? N1
Enter YES; or NO; or DONTKNOW; .
NO;

May the following expression ever assume the value zero? N2
Enter YES; or NO; or DONTKNOW; .
NO;

May the following expression ever assume the value zero?

   2                 2     2              2
TH2 - 2 TH1 TH2 + TH1 + 3N2 + 6N1 N2 +3N1

Enter YES; or NO; or DONTKNOW; .
NO;

(D3) CONTRADICTION
(C4) KILL(ALL)$
```

Input files for Sample Runs

MC:SWATT;TEST BIAN6H

```
UNKNOWNS: [N1,N2,N3,TH1,TH2,TH3,P,MU,THETA]$
COORDINATES: [T]$
D%OP%COEFF%LIST : [1] $
DEPENDS (UNKNOWNS, COORDINATES)S
ALGL: [THETA = TH1+TH2+TH3, N1 = 0]$
PRL: [[N1,T,N1*(2*TH1-THETA)],
      [N2,T,N2*(2*TH2-THETA)],
      [N3,T,N3*(2*TH3-THETA)],
      [THETA,T,-(TH1^2+TH2^2+TH3^2)-(MU+3*P)/2+LAM] ,
      [TH1,T,-THETA*TH1-N1^2/2+(N2-N3)A2/2+(MU-P)/2+LAM],
      [TH2,T,-THETA*TH2-N2^2/2T(N3-N1)A2/2+(MU-P)/2+LAM],
      [TH3,T,-THETA*TH3-N3^2/2+(NI-N2)A2/2+(MU-P)/2+LAM],
      [MU,T,-(MU+P)*THETA] ]$
CASE: [ALGL, PRL];
```

MC:SWATT;TEST BIAN8

```
ORDERLESS(N3,TH3)$
UNKNOWNS : [N1, N2, N3, TH1, TH2, TH3, P ,MU, THETA]$
COORDINATES: [T] $
D%OP%COEFF%LIST : [1] $
DEPENDS (UNKNOWNS, COORDINATES)S
ALGL: [THETA = TH1+TH2+TH3]$
PRL : [[N1,T,N1* (2*TH1-THETA)],
       [N2,T,N2* (2*TH2-THETA)],
       [N3,T,N3* (2*TH3-THETA)],
       [THETA,T,-(TH1^2+TH2^2+TH3^2)-(MU+3*P)/2+LAM] ,
       [TH1,T,-THETA*TH1-N1^2/2+(N2-N3)^2/2+(MU-P)/2+LAM],
       [TH2,T,-THETA*TH2-N2^2/2+(N3-N1)^2/2+(MU-P)/2+LAM],
       [TH3,T,-THETA*TH3-N3^2/2+(N1-N2)A2/2+(MU-P)/2+LAM],
       [MU,T,-(MU+P)*THETA] ]$
CASE : [ALGL, PRL];
```

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Colophon

This is a typeset version of the original work that was prepared using an IBM Selectric^{TM} typewriter, in multiple scripts, with *xiii* + 190 pages. In this typeset version, the following conventions have been used:

- text originally underlined has been typeset in italics;
- symbols originally marked with an undertilde have been displayed in bold;
- bibliographic entries have been renumbered using the year of publication and a unique letter, if necessary, to facilitate automatic processing.

Errata

The following typographical corrections have been applied:

- page xi: ... mnemonic equation labels [was "tables"] ...
- page 3: ... $T^a{}_b\,\mathbf{e}_a \otimes \mathbf{e}^b$ [was "$T^a{}_b\,\mathbf{e}_a \otimes \mathbf{e}_b$"] ...
- page 5: ... be two nearby [was "nearly"] world lines ...
- page 6: ... $\mathbf{e}_\alpha \cdot \nabla_{\mathbf{u}}\mathbf{e}_\beta$ [was "$\mathbf{e}_\alpha \cdot \nabla_u \mathbf{e}_\beta$"] ...
- page 7: ... for observers [was "observer's"] with velocity $\mathbf{u}$, [was "$\mathbf{u}_a$"] ...
- page 15: ... the original system and the commutators (1.2.10) [was (1.2.29)].
- page 50: ... `bet=−alf$` [was "`bet=`$-\alpha$`$`"]
- page 96: ... and electromagnetic field [inserted "field"] are non-interacting, ...

September 2006