Report on the SNAP minisymposium at SIAM '98

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1 Background

In the essay [10], Nick Trefethen defines

> **Numerical Analysis** as "the study of algorithms for problems of continuous mathematics",

whilst in the introduction to [2], R. Loos defines

> **Computer Algebra** as "that part of computer science which designs, analyzes, implements, and applies algebraic algorithms."

Clearly the definitions are similar, and the major distinction is *continuous* versus *algebraic*. One really needs to appeal to examples to make the definitions clearer, such as the following.

Computing the approximate value of a definite integral by Gauss-Chebyshev quadrature is a Numerical Computation:

$$\begin{aligned} I &= \int_{-1}^{1} \frac{\cos x}{\sqrt{1-x^2}}\, dx \\ &\approx \frac{\pi}{4}\sum_{k=1}^{4} \cos\left[\cos\left(\frac{2k-1}{8}\pi\right)\right] \\ &= 2.403938\underline{838}\,, \end{aligned}$$

where all the non-underlined digits are correct. The study of Gaussian quadrature formulas is a problem in Numerical Analysis.

On the other hand, if your program transforms

$$\begin{aligned} I &= \int_{-1}^{1} \frac{\cos x}{\sqrt{1-x^2}}\, dx \\ &= \pi F\left(\,_{1} \middle| -\frac{1}{4}\right) \\ &= \pi J_0(1)\,, \end{aligned}$$

where F is the hypergeometric function and $J_0(x)$ is the Bessel function of zeroth order, then your program is doing Computer Algebra.

Another example would be computing the characteristic polynomial of a matrix A with 0–1 entries (that is, all the elements of A are either 0 or 1). Equivalently, the problem is to transform A to companion matrix form. This is a problem for Computer Algebra, because the coefficients of the characteristic polynomial would all be (possibly very large) exact integers. In contrast, computing the *eigenvalues* of this matrix is clearly a problem that calls for Numerical Methods, because the eigenvalues would all need to be approximated.

As an aside, it is well-known that computing the eigenvalues by first computing the companion matrix (or characteristic polynomial) is numerically unstable in floating-point, and expensive in exact arithmetic [11]. This is an example of the duality that one of the present authors (RMC) aphorizes by "Instability implies inefficiency". The point, in this case, is that computer algebra has developed tools for dealing with arithmetic with large integers, so that *if what you want is the companion matrix*, then you can find it more efficiently with computer algebra then you can with numerical methods. If, however, what you want is the eigenvalues, then you shouldn't compute the companion matrix in the first place, but rather go directly to an iterative algorithm to compute eigenvalues.

One distinction that becomes visible with these examples is that Computer Algebra is concerned with algorithms for transformation, whilst Numerical Analysis is concerned with algorithms for evaluation or approximation.

2 A non-empty intersection

But as Trefethen points out, the boundaries between fields are fuzzy and no definition is perfect. Many problems, in particular the following problems, which are representative of Symbolic-Numeric Algorithms for Polynomials (SNAP), really belong to both fields. It has been one of the most exciting developments in the past decade that ideas from both fields can be brought to bear on such problems, with success. Moreover, this synergy between the two approaches actually leads to new ideas in each field.

Problem 1. Find the GCD of two (or more) polynomials with approximately known coefficients.

Problem 2. Find the roots (solution set, variety) of $F(x) = 0$, where the F are multivariate polynomials with approximately known coefficients.

Both of these problems can be considered as problems of continuous mathematics. Yet both can also be considered algebraic, because there is an extraordinarily rich algorithmic algebraic theory for both problems in the exact arithmetic case. It is now becoming clear that proper formulation of these problems requires ideas from Numerical Analysis, but the algorithmic solution of these problems can benefit from algebraic methods such as resultants, Groebner bases, and division by an ideal.

At the end, the problems lead to the numerical approximation of the eigenvalues of large sparse commuting matrices. The fact that the matrices commute is of prime importance, and has Numerical implications as well as Algebraic implications. The interplay here leads to the synergy alluded to earlier. These developments are described in [4, 5, 6, 9]; see also the Sigsam Bulletin Volume 30, Number 4, Issue number 118, December 1996. The upcoming special issue of the Journal of Symbolic Computation, edited by H. J. Stetter and S. M. Watt, contains a significant archive of these kinds of results, and pointers to much of

the earlier literature. See also the very impressive web bibliography [8]. Here, to give the flavor only, we re-formulate problem 1 as it is studied in [1, 3, 4, 7].

Problem $1'$.

a) Given polynomials p and q, and a positive integer k, find Δp, Δq such that deg $\mathrm{GCD}(p + \Delta p, q + \Delta q) = k$ and $||(\Delta p, \Delta q)||$ is smallest.

b) Given polynomials p and q, find Δp, Δq such that deg $\mathrm{GCD}(p + \Delta p, q + \Delta q) > 0$ and $||(\Delta p, \Delta q)||$ is smallest.

c) Given polynomials p and q, and a positive number ε, prove that $\mathrm{GCD}(p + \Delta p, q + \Delta q) = 1$ for all Δp, Δq with $||(\Delta p, \Delta q)|| < \varepsilon$.(See [1] for a solution to this problem)

3 The SNAP minisymposium

The program for the SNAP minisymposium at SIAM this year in Toronto was as follows.

This minisymposium was intended to present a timely update to the emerging understanding of problems involving multivariate polynomials with inexactly-known coefficients. Such polynomial problems arise, for example, when physical measurements or numerical computations are used to specify a polynomial system. In particular, applications to computer-aided design are important already, and one expects a very wide array of applications to become important in the future.

One of the goals of holding this minisymposium was to stimulate interest from the numerical analysis community. While there is widespread activity in the computer algebra community, there are as yet only a handful of numerical analysts involved. Since the discovery that some important multivariate polynomial problems can and should be phrased instead as eigenvalue problems for nearly-commuting families of very large sparse matrices, it has become clear that there is a significant role for numerical analysis to play.

Some of these talks had been given already at various meetings of the computer algebra community, but repetition was deliberately encouraged here in an attempt to provide an opportunity for the numerical analysis community to participate more fully in these developments.

Session I

- George Labahn, University of Waterloo, Canada, *When Are Two Numeric Polynomials Relatively Prime?* (Joint work with Bernard Beckerman)
- Paulina Chin, Wilfred Laurier University, Canada *Optimization strategies for the approximate GCD problem* (Joint work with Rob Corless and George Corliss)
- Stephen Watt, University of Western Ontario, Canada *On the Factorization of Approximate Multivariate Polynomials* (joint work with André Galligo)
- Shankar Krishnan, ATT Labs, *Solving Algebraic Systems Using Multi-Polynomial Resultants: Algorithms, Implementations, and Applications* (joint work with Dinesh Manocha)

Session II

- Victor Pan, Lehman College, City University of New York, *Structured Matrices, Polynomial Zeros, Polynomial GCDs and Polynomial Systems of Equations*
- Rob Corless, University of Western Ontario, *A Reordered Schur Factorization Method for Zero-Dimensional Polynomial Systems with Multiple Roots* (joint work with Barry Trager and Patrizia Gianni)
- Douglas B. Meade, University of South Carolina, Columbia *A Test for the Irreducibility of Lacunary 0-1 Polynomials* (joint work with Michael Filaseta)
- Dario Bini, University of Pisa *Fast solution of polynomials*

4 Open problems in SNAP

At the moment, there are great many open problems in SNAP—too many to list here. Here we give only a few of our favorites. Some of them may be solved already by the time you read this.

1. Is there an efficient norm-independent (coordinate independent) global optimization approach to the approximate GCD problem? If not, which norm leads to the cheapest solution? Which norms are the most useful in practice?
2. Is multivariate division by a reduced ideal well conditioned? Find a stable algorithm if so.
3. Characterize the term orderings for which reduction to a Groebner basis is numerically stable. Examples are well-known where the reduction is unstable. However, it would be useful to know, before the computation was done, if the reduction is likely to be stable.

5 Invitation

This paper is being printed in the hopes that people will be attracted to work on some of the problems in this area. A significant body of expertise exists in the numerical analysis world, which we would like to take advantage of. These problems are similar enough (to some of numerical linear algebra) that one expects immediate application of some major ideas.

Short notices and extended abstracts of such work are invited to be sent to Rob Corless at the address above, for nearly immediate publication in the Sigsam Bulletin; longer articles are welcome for the formally reviewed articles section of the Sigsam Bulletin. Other venues of course also exist, such as the Journal of Symbolic Computation and the Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC), which are particularly relevant.

Watch this space for announcements of further SNAP workshops.

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