

Reasoning about the elementary functions of complex analysis

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Abstract

There are many problems with the simplification of elementary functions, particularly over the complex plane. Systems tend to make “howlers” or not to simplify enough. In this paper we outline the “unwinding number” approach to such problems, and show how it can be used to systematise such simplification, even though we have not yet reduced it to a complete algorithm. The unsolved problems are probably more amenable to the techniques of artificial intelligence and theorem proving than the original problem of complex-variable analysis.

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1 Introduction

The elementary functions are traditionally thought of as log, exp and the trigonometric and hyperbolic functions (and their inverses). This should include powering (to non-integral powers) and also the n -th root. These functions are built in, to a greater or lesser extent, into many computer algebra systems (not to mention other programming languages [7, 11]), and are heavily used. However, reasoning with them is more difficult than is usually acknowledged, and all algebra systems have one, sometimes both, of the following defects:

- they make mistakes, be it the traditional schoolboy one

$$1 = \sqrt{1} = \sqrt{(-1)^2} = -1 \tag{1}$$

*This work was performed while this author held the Ontario Research Chair in Computer Algebra at the University of Western Ontario.

or more subtle ones — see footnote 5;

- they fail to perform obvious simplifications, leaving the user with an impossible mess when there “ought” to be a simpler answer. In fact, there are two possibilities here: maybe there is a simpler equivalent that the system has failed to find, but maybe there isn’t, and the simplification that the user wants is not actually valid, or is only valid outside an exceptional set. In general, the user is not informed what the simplification might have been, nor what the exceptional set is.

Throughout this paper, z and its decorations indicate a complex variable, x , y and t real variables. For the purposes of this paper, the precise definitions of the inverse elementary functions in terms of \log are those of [3]: these are reproduced in Appendix A for ease of reference.

2 The Problem

The fundamental problem is that \log is multi-valued: since $\exp(2\pi i) = 1$, its inverse is only valid up to adding any multiple of $2\pi i$. This ambiguity is traditionally resolved by making a *branch cut*: usually [1, p. 67] the branch cut $(-\infty, 0]$, and the rule (4.1.2) that

$$-\pi < \Im \log z \leq \pi. \quad (2)$$

This then completely specifies the behaviour of \log : on the branch cut it is continuous with the positive imaginary side of the cut, i.e. counter-clockwise continuous in the sense of [9].

What are the consequences of this definition¹? From the existence of branch cuts, we get the problem of a lack of continuity:

$$\lim_{y \rightarrow 0^-} \log(x + iy) \neq \log(x) : \quad (3)$$

for $x < 0$ the limit is $\log(x) - 2\pi i$. Related to this is the fact that

$$\log \bar{z} \neq \overline{\log z} \quad (4)$$

on the branch cut: instead $\log \bar{z} = \overline{\log z} + 2\pi i$ on the cut. Similarly,

$$\log\left(\frac{1}{z}\right) \neq -\log z \quad (5)$$

on the branch cut: instead $\log\left(\frac{1}{z}\right) = -\log z + 2\pi i$ on the cut.

¹Which we do not contest: it seems that few people today would support the rule one of us (JHD) was taught, viz. that $0 \leq \Im \log z < 2\pi$. The placement of the branch cut is “merely” a notational convention, but an important one. If we wanted a function that behaves like \log but with this cut, we could consider $\log(-1) - \log(-1/z)$ instead. We note that, until 1925, astronomers placed the branch cut between one day and the next at noon [6, vol. 15 p. 417].

Although not normally explained this way, the problem with (1) is a consequence of the multi-valued nature of \log : if we define (as for the purposes of this paper we do)

$$\sqrt{z} = \exp\left(\frac{1}{2} \log z\right), \quad (6)$$

then $-\frac{\pi}{2} < \Im\sqrt{z} \leq \frac{\pi}{2}$. On the real line, this leads to the traditional resolution of (1): $\sqrt{x^2} = |x|$.

Three families of solutions have been mooted to these problems.

- [9] points out that the concept of a “signed zero”² [8] (for clarity, we write the positive zero as 0^+ and the negative one as 0^-) can be used to solve the above problems, if we say that, for $x < 0$, $\log(x + 0^+i) = \log(x) + \pi i$ whereas $\log(x + 0^-i) = \log(x) - \pi i$. Equation (3) then becomes an equality for all x , interpreting the x on the right as $x + 0^-i$. Similarly, (4) and (5) become equalities throughout. Attractive though this proposal is, it does not answer the fundamental question as far as the designer of a computer algebra system is concerned: what to do if the user types $\log(-1)$.
- [4] points out that most “equalities” do not hold for the complex logarithm, e.g. $\log(z^2) \neq 2 \log z$ (try $z = -1$), and its generalisation

$$\log(z_1 z_2) \neq \log(z_1) + \log(z_2). \quad (7)$$

The most fundamental of all non-equalities is $z = \log \exp z$, whose most obvious violation is at $z = 2\pi i$. They therefore propose to introduce the *unwinding number* \mathcal{K} , defined³ by

$$\mathcal{K}(z) = \frac{z - \log \exp z}{2\pi i} = \left\lceil \frac{\Im z - \pi}{2\pi} \right\rceil \in \mathbf{Z} \quad (8)$$

(note that the apparently equivalent definition $\lfloor \frac{\Im z + \pi}{2\pi} \rfloor$ differs precisely on the branch cut for \log as applied to $\exp z$). (7) can then be rescued as

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) - 2\pi i \mathcal{K}(\log(z_1) + \log(z_2)). \quad (9)$$

Similarly (4) can be rescued as

$$\log \bar{z} = \overline{\log z} - 2\pi i \mathcal{K}(\overline{\log z}). \quad (10)$$

Note that, as part of the algebra of \mathcal{K} , $\mathcal{K}(\overline{\log z}) = \mathcal{K}(-\log z) \neq \mathcal{K}(\log \frac{1}{z})$. $\mathcal{K}(z)$ depends only on the imaginary part of z .

²One could ask why zero should be special and have two values. The answer is that all the branch cuts we need to consider are on either the real or imaginary axes, so the side to which the branch cut adheres depends on the sign of the imaginary or real part, including the sign of zero.

³Note that the sign convention here is the opposite to that of [4], which defined $\mathcal{K}(z)$ as $\lfloor \frac{\pi - \Im z}{2\pi} \rfloor$: the authors of [4] recanted later to keep the number of -1 s occurring in formulae to a minimum.

- Although not formally proposed in the same way in the computational community, one possible solution, often found in texts in complex analysis, is to accept the multi-valued nature of these functions (we adopt the common convention of using capital letters, e.g. Ln , to denote the multi-valued function), defining, for example

$$\text{Arcsin } z = \{y \mid \sin y = z\}.$$

This leads to $\sqrt{z^2} = \pm z$, which has the advantage that it is valid throughout **C**. Equation 7 is then rewritten as

$$\text{Ln}(z_1 z_2) = \text{Ln}(z_1) + \text{Ln}(z_2), \quad (11)$$

where addition is addition of sets ($A + B = \{a + b : a \in A, b \in B\}$) and equality is set equality⁴.

However, it seems to lead in practice to very large and confusing formulae. More fundamentally, this approach does not say what will happen when the multi-valued functions are replaced by the single-valued ones of numerical programming languages.

A further problem that has not been stressed in the past is that this approach suffers from the same aliasing problem that naïve interval arithmetic does [5]. For example,

$$\text{Ln}(z^2) = \text{Ln}(z) + \text{Ln}(z) \neq 2 \text{Ln}(z),$$

since $2 \text{Ln}(z) = \{2 \ln(z) + 4k\pi i : k \in \mathbf{Z}\}$, but $\text{Ln}(z) + \text{Ln}(z) = \{2 \ln(z) + 2k\pi i : k \in \mathbf{Z}\}$ indeed if $z = -1$, $\ln(z^2) \notin 2 \text{Ln}(z)$. Hence this method is unduly pessimistic: it may fail to prove some identities that are true.

3 The rôle of the Unwinding Number

We claim that the unwinding number provides a convenient formalism for reasoning about these problems. Introducing the unwinding number systematically allows one to make “simplifying” transformations that *are* mathematically valid. The unwinding number can be evaluated at any point, either symbolically or via guaranteed arithmetic: since we know it is an integer, in practice little accuracy is necessary. Conversely, removing unwinding numbers lets us genuinely “simplify” a result. We describe introduction and removal as separate steps, but in practice every unwinding number, once introduced by a “simplification” rule, should be eliminated as soon as possible.

The following section gives examples of reasoning with unwinding numbers.

⁴“The equation merely states that the sum of one of the (infinitely many) logarithms of z_1 and one of the (infinitely many) logarithms of z_2 can be found among the (infinitely many) logarithms of $z_1 z_2$, and conversely every logarithm of $z_1 z_2$ can be represented as a sum of this kind (with a suitable choice of $\text{Ln } z_1$ and $\text{Ln } z_2$).” [2, pp. 259–260] (our notation).

4 Examples of Unwinding Numbers

4.1 Forms of arccos

The following example is taken from [3], showing that two alternative definitions of arccos are in fact equal.

Lemma 1

$$\sqrt{z_1 z_2} = \sqrt{z_1} \sqrt{z_2} (-1)^{\mathcal{K}(\ln(z_1) + \ln(z_2))}. \quad (12)$$

Proof.

$$\begin{aligned} \sqrt{z_1 z_2} &= \exp\left(\frac{1}{2}(\ln(z_1 z_2))\right) \\ &= \exp\left(\frac{1}{2}(\ln(z_1) + \ln(z_2) - 2\pi i \mathcal{K}(\ln(z_1) + \ln(z_2)))\right) \\ &= \sqrt{z_1} \sqrt{z_2} \exp(-\pi i \mathcal{K}(\ln(z_1) + \ln(z_2))) \\ &= \sqrt{z_1} \sqrt{z_2} (-1)^{\mathcal{K}(\ln(z_1) + \ln(z_2))} \end{aligned}$$

Lemma 2 *Whatever the value of z ,*

$$\sqrt{1-z} \sqrt{1+z} = \sqrt{1-z^2}.$$

This is a classic example of a result that is “obvious” — the schoolboy just squares both sides, but in fact it requires proof. To show this, consider the apparently similar “result”⁵:

$$\sqrt{-i-z} \sqrt{-i+z} = \sqrt{-1-z^2}.$$

If we take $z = \frac{i}{2}$, the left-hand side becomes $\sqrt{\frac{-3i}{2}} \sqrt{\frac{-i}{2}}$: the arguments of the square roots have $\arg = \frac{-\pi}{2}$, so the square roots themselves have $\arg = \frac{-\pi}{4}$, and the product has $\arg = \frac{-\pi}{2}$, and therefore is $\frac{-\sqrt{3}}{2}i$. The right-hand side is $\sqrt{\frac{-3}{4}} = \frac{\sqrt{3}}{2}i$.

Proof. It is sufficient to show that the unwinding number term in lemma 1 is zero. Whatever the value of z , $1+z$ and $1-z$ have imaginary parts of opposite signs. Without loss of generality, assume $\Im z \geq 0$. Then $0 \leq \arg(1+z) \leq \pi$ and $-\pi < \arg(1-z) \leq 0$. Therefore their sum, which is the imaginary part of $\ln((1+z) + \ln(1-z))$, is in $(-\pi, \pi]$. Hence the unwinding number is indeed zero.

Theorem 1

$$\frac{2}{i} \ln\left(\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}}\right) = -i \ln\left(z + i\sqrt{1-z^2}\right). \quad (13)$$

⁵Maple V.5, in the absence of an explicit declaration that z is complex, will say that the two are almost never equal, with the difference being $-2i\sqrt{1-z^2}$, but in fact at $z = 2i$, the two are equal.

Proof. Now

$$\left(\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}}\right)^2 = z + i\sqrt{1-z}\sqrt{1+z} = z + 1\sqrt{1-z^2}$$

by the previous lemma. Also $2 \ln a = \ln(a^2)$ if $\mathcal{K}(2 \ln a) = 0$, so we need only show this last stipulation i.e. that

$$-\frac{\pi}{2} < \arg\left(\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}}\right) \leq \frac{\pi}{2}.$$

This is trivially true at $z = 0$. If it is false, then we have to pass through $\left|\arg\left(\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}}\right)\right| = \frac{\pi}{2}$, i.e. $\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}} = it$ for $t \in \mathbf{R}$. Squaring both sides, $z + i\sqrt{1-z^2} = -t^2$, i.e. $(z + t^2)^2 = -(1 - z^2)$. Hence $2zt^2 + t^4 = -1$, so $z = \frac{-(1+t^4)}{2t^2} \leq -1$, and in particular is real. On this half-line, the argument in question is $+\pi/2$, which is acceptable. Hence the argument never leaves the desired range, and the theorem is proved.

4.2 arccos and arccosh

$\cos(z) = \cosh(iz)$, so we can ask whether the corresponding relation for the inverse functions, $\operatorname{arccosh}(z) = i \operatorname{arccos}(z)$ holds. This is known in [3] as the ‘‘couthness’’ of the arccos/arccosh definitions. The problem reduces, using equations (20) and (26), to

$$2 \ln\left(\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}}\right) \stackrel{?}{=} i \left(\frac{2}{i} \ln\left(\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}}\right)\right),$$

i.e.

$$\ln\left(\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}}\right) \stackrel{?}{=} \ln\left(\sqrt{\frac{1+z}{2}} + i\sqrt{\frac{1-z}{2}}\right).$$

Since $\ln(a) = \ln(b)$ implies $a = b$, this reduces to

$$\sqrt{\frac{z-1}{2}} \stackrel{?}{=} i\sqrt{\frac{1-z}{2}} = \sqrt{-1}\sqrt{\frac{1-z}{2}}.$$

By lemma 1, the right-hand side reduces to $\sqrt{\frac{z-1}{2}}(-1)^{\mathcal{K}(\ln(-1) + \ln(\frac{z-1}{2}))}$. Hence the two are equal if, and only if, the winding number is even (and therefore zero). This will happen if, and only if, $\Im\left(\frac{z-1}{2}\right) \leq 0$, i.e. $\Im z < 0$ or $\Im z = 0$ and $z > 1$.

4.3 arcsin and arctan

The aim of this section is to prove that

$$\arcsin z = \arctan \frac{z}{\sqrt{1-z^2}} + \pi \mathcal{K}(-\ln(1+z)) - \pi \mathcal{K}(-\ln(1-z)). \quad (14)$$

We start from equations (19) and (21). Then

$$\begin{aligned}
2i \arctan \frac{z}{\sqrt{1-z^2}} &= \ln \left(1 + i \frac{z}{\sqrt{1-z^2}} \right) - \ln \left(1 - i \frac{z}{\sqrt{1-z^2}} \right) \\
&= \ln \left([1 + i \frac{z}{\sqrt{1-z^2}}] / [1 - i \frac{z}{\sqrt{1-z^2}}] \right) \\
&\quad + 2\pi i \mathcal{K} \left(\ln(1 + i \frac{z}{\sqrt{1-z^2}}) - \ln(1 - i \frac{z}{\sqrt{1-z^2}}) \right) \\
&= \ln[iz + \sqrt{1-z^2}]^2 \\
&\quad + 2\pi i \mathcal{K}(\ln(1 + i \frac{z}{\sqrt{1-z^2}}) - \ln(1 - i \frac{z}{\sqrt{1-z^2}})) \\
&= 2i \arcsin(z) \\
&\quad - 2\pi i \mathcal{K} \left(2 \ln(iz + \sqrt{1-z^2}) \right) \\
&\quad + 2\pi i \mathcal{K} \left(\ln(1 + i \frac{z}{\sqrt{1-z^2}}) - \ln(1 - i \frac{z}{\sqrt{1-z^2}}) \right)
\end{aligned}$$

The tendency for \mathcal{K} factors to proliferate is clear. To simplify we proceed as follows. Consider first the term

$$\mathcal{K}(2 \ln(iz + \sqrt{1-z^2})) .$$

For $|z| < 1$, the real part of the logarithm argument is positive and hence $\mathcal{K} = 0$. For $|z| > 1$, we solve for the critical case in which the argument of \mathcal{K} is $-\pi$ and find only $z = r \exp(i\pi)$, with $r > 1$. Therefore

$$\mathcal{K}(2 \ln(iz + \sqrt{1-z^2})) = \mathcal{K}(-\ln(1+z)) .$$

Repeating the procedure with

$$\mathcal{K}(\ln(1 + iz/\sqrt{1-z^2}) - \ln(1 - iz/\sqrt{1-z^2}))$$

shows that $\mathcal{K} \neq 0$ only for $z > 1$. Therefore

$$\mathcal{K}(\ln(1 + iz/\sqrt{1-z^2}) - \ln(1 - iz/\sqrt{1-z^2})) = \mathcal{K}(-\ln(1-z))$$

and so finally we get

$$\arctan \frac{z}{\sqrt{1-z^2}} = \arcsin(z) - \pi \mathcal{K}(-\ln(1+z)) + \pi \mathcal{K}(-\ln(1-z)) , \quad (15)$$

and this cannot be simplified further.

5 The Unwinding Number: Introduction

Unwinding numbers are normally introduced by use of equation (9) and its converse:

$$\log \left(\frac{z_1}{z_2} \right) = \log(z_1) - \log(z_2) - 2\pi \mathcal{K}(\log(z_1) - \log(z_2)) . \quad (16)$$

Equation (10) may also be used, as may its close relative (also a special case of (16))

$$\log\left(\frac{1}{z}\right) = -\log(z) - 2\pi\mathcal{K}(-\log(z)). \quad (17)$$

In practice, results such as lemma 1 would also be built in to a simplifier.

The definition of K gives us

$$\log(\exp(z)) = z - 2\pi iK(z), \quad (18)$$

which is another mechanism for introducing unwinding numbers while “simplifying”. The formulae for other inverse functions are given in appendix B.

6 The Unwinding Number: Removal

It is clearly easier to introduce unwinding numbers than to remove them. There are various possibilities for the values of unwinding numbers.

- A winding number may be identically zero. This is the case in lemma 2 and theorem 1. The aim is then to prove this.
- An unwinding number may be zero everywhere except on certain branch cuts in the complex plane. This is the case in equation (10), and its relative $\log\frac{1}{z} = -\log z - 2\pi i\mathcal{K}(-\log(z))$. A less trivial case of this can be seen in equation (14). Derive has a different definition of arctan to eliminate this, so that, for Derive, $\arcsin(z) = \underbrace{\arctan}_{\text{Derive}} \frac{z}{\sqrt{1-z^2}}$. This definition can be related to ours either via unwinding numbers or via $\underbrace{\arctan}_{\text{Derive}}(z) = \overline{\arctan \bar{z}}$.

It is often possible to disguise this sort of unwinding number, which can often be recognised by being of the form $\mathcal{K}(-\ln(\dots))$ or $\mathcal{K}(\ln z)$ by resorting to such a “double conjugate” expression, though as yet we have no algorithm for this. Equally, we have no algorithm as yet for the sort of simplification we see in section 14.

- An unwinding number may divide the complex plane into two regions, one where it is non-zero and one where it is zero. A typical case of this is given in section 4.2. Here the proof methodology consists in examining the critical case, i.e. when the argument to \mathcal{K} has imaginary part $\pm\pi$.
- An unwinding number may correspond to the usual $+n\pi$: $n \in \mathbb{Z}$ of many trigonometric identities: examples of this are given in appendix B.

7 Conclusion

Unwinding number introduction allows combination of logarithms, square roots etc., as well as canceling functions and their inverses, while retaining mathematical correctness. This can be done completely algorithmically.

Unwinding number removal, where it is possible, then simplifies these results to the expected form. This is not a process that can currently be done algorithmically, but it is much better suited to current artificial intelligence techniques than the general problems of complex analysis.

When the unwinding numbers cannot be eliminated, they can often be converted into a case analysis that, while not ideal, is at least comprehensible while being mathematically correct.

More generally, we have reduced the analytic difficulties of simplifying these functions to more algebraic ones, in areas where we hope that artificial intelligence and theorem proving stand a better chance of contributing to the problem.

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A Definition of the Elementary Inverse Functions

These definitions are taken from [3]. They agree with [1, ninth printing], but are more precise on the branch cuts, and agree with Maple with the exception of arccot , for the reasons explained in [3].

$$\arcsin z = -i \ln \left(\sqrt{1 - z^2} + iz \right). \quad (19)$$

$$\arccos(z) = \frac{\pi}{2} - \arcsin(z) = \frac{2}{i} \ln \left(\sqrt{\frac{1+z}{2}} + i \sqrt{\frac{1-z}{2}} \right). \quad (20)$$

$$\arctan(z) = \frac{1}{2i} (\ln(1+iz) - \ln(1-iz)). \quad (21)$$

$$\operatorname{arccot} z = \frac{1}{2i} \ln \left(\frac{z+i}{z-i} \right) = \arctan \left(\frac{1}{z} \right). \quad (22)$$

$$\operatorname{arcsec}(z) = \arccos(1/z) = -i \ln(1/z + i\sqrt{1-1/z^2}), \quad (23)$$

with $\operatorname{arcsec}(0) = \frac{\pi}{2}$.

$$\operatorname{arccsc}(z) = \arcsin(1/z) = -i \ln(i/z + \sqrt{1-1/z^2}), \quad (24)$$

with $\operatorname{arccsc}(0) = 0$.

$$\operatorname{arcsinh}(z) = \ln \left(z + \sqrt{1+z^2} \right). \quad (25)$$

$$\operatorname{arccosh}(z) = 2 \ln \left(\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}} \right). \quad (26)$$

$$\operatorname{arctanh}(z) = \frac{1}{2} (\ln(1+z) - \ln(1-z)). \quad (27)$$

$$\operatorname{arccoth}(z) = \frac{1}{2} (\ln(-1-z) - \ln(1-z)). \quad (28)$$

$$\operatorname{arcsech}(z) = 2 \ln \left(\sqrt{\frac{z+1}{2z}} + \sqrt{\frac{1-z}{2z}} \right). \quad (29)$$

$$\operatorname{arccsch}(z) = \ln \left(\frac{1}{z} + \sqrt{1 + \left(\frac{1}{z} \right)^2} \right), \quad (30)$$

B Formulae for inverse functions

These formulae are taken from [10]. They make use of the function csgn , defined by

$$\text{csgn}(z) = (-1)^{\mathcal{K}(2\ln(z))} = \begin{cases} +1 & \Re(z) > 0 \text{ or } \Re(z) = 0; \Im(z) \geq 0 \\ -1 & \Re(z) < 0 \text{ or } \Re(z) = 0; \Im(z) < 0 \end{cases}.$$

$$\arcsin(\sin(z)) = \begin{cases} z - 2\pi\mathcal{K}(zi) & \text{csgn}(\cos z) = 1 \\ \pi - z - 2\pi\mathcal{K}(i(\pi - z)) & \text{csgn}(\cos z) = -1 \end{cases}. \quad (31)$$

$$\arccos(\cos z) = \begin{cases} z - 2\pi\mathcal{K}(zi) & \text{csgn}(\sin z) = 1 \\ -z - 2\pi\mathcal{K}(-zi) & \text{csgn}(\sin z) = -1 \end{cases}. \quad (32)$$

$$\arctan(\tan z) = z + \pi (\mathcal{K}(-zi - \ln \cos z) - \mathcal{K}(zi - \ln \cos z)) \quad (33)$$

provided $z \neq \frac{\pi}{2} + n\pi$: $n \in \mathbf{Z}$.

$$\operatorname{arcsinh}(\sinh(z)) = \begin{cases} z - 2\pi i\mathcal{K}(z) & \text{csgn}(\cosh z) = 1 \\ i\pi - z - 2\pi i\mathcal{K}(i\pi - z) & \text{csgn}(\cosh z) = -1 \end{cases}. \quad (34)$$

$$\operatorname{arccosh}(\cosh z) = \begin{cases} z - 2\pi\mathcal{K}(z) & \text{csgn}(\sinh z) \cos(n\pi) = 1 \\ -z - 2\pi i\mathcal{K}(-z) & \text{csgn}(\sinh z) \cos(n\pi) = -1 \end{cases}. \quad (35)$$

where $n = \mathcal{K}(\ln(\cosh(z) - 1) + \ln(\cosh(z) + 1))$.

$$\operatorname{arctanh}(\tanh z) = z + i\pi (\mathcal{K}(z - \ln \cosh z) - \mathcal{K}(z - \ln \cosh z)) \quad (36)$$

provided $z \neq \frac{\pi}{2}i + in\pi$: $n \in \mathbf{Z}$.