An abstract, coordinate-free, vector algebra package

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1 Statement of the problem

Abstract coordinate-free vectors are represented only by a symbol, rather than a list of components. There are two vector operations: scalar product and vector product. The problem is to implement the automatic simplification of vector expressions. For example,

$$(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{b} \wedge \mathbf{c}) \cdot (\mathbf{c} \wedge \mathbf{a}) - (\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}))^2$$

In addition to vectors, there are higher-order quantities formed as dyadic products. For example, the momentum flux in a fluid is a second order tensor written $\rho\mathbf{uu}$. Expressions containing these quantities need also to be simplified.

2 Importance of the problem

Physicists and engineers prefer to formulate their equations of motion using abstract vectors rather than components. Thus they prefer to write the velocity of a rotating body as $\omega \wedge \mathbf{r}$, rather than in some component form as

$$[\omega_2 z - \omega_3 y, \omega_3 x - \omega_1 z, \omega_1 y - \omega_2 x] \ .$$

Existing packages use components and essentially manipulate matrix arrays explicitly. One of the advances of last century was the shift from the component formulation of physics, to the vector formulation. Where a nineteenth century textbook would take a page to write down the governing equations of a problem, a vector based textbook uses 3 lines [2]. This package will supply physicists and engineers with a useful workspace for deriving equations, while at the same time providing a platform for testing and extending the abilities of Aldor to handle difficult algebraic problems.

3 Contribution to the problem

This paper described a package written in Aldor. The necessary data structures and programming types are presented, together with simplification rules.

4 Originality of the contribution

The language Aldor is strongly typed, which is important for vector analysis, because vectors can be contracted to scalars, whereupon they obey a different algebra. Therefore the first task has been to define categorical hierarchies within Aldor to serve as a framework for the package. The next step has been the identification of the algebra. Usually physics books "define" the vector operations intuitively with reference to pictures of the supposed physical action of the operation. However, for algebraic simplification, algebraic rules must be identified.

5 Non-triviality of the contribution

The algebra obeyed by vectors is very non-standard. Stoutemyer [1] listed some of the unusual properties.

- The presence of zero dividers. For example, for any vector $\mathbf{u} \wedge \mathbf{u} = 0$. If $\mathbf{p}$ is orthogonal to $\mathbf{q}$, then $\mathbf{p} \cdot \mathbf{q} = 0$.
- The absence of unit elements. Since $\mathbf{a} \cdot \mathbf{b}$ is not a vector, there is no vector $\mathbf{x}$ such that $\mathbf{a} \cdot \mathbf{x} = \mathbf{a}$. Also, since $\mathbf{a} \wedge \mathbf{b}$ is orthogonal to $\mathbf{a}$, there is no $\mathbf{x}$ such that $\mathbf{a} \wedge \mathbf{x} = \mathbf{a}$.
- Unusual distribution laws. The distribution of vector product is

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \ .$$

The difficulties of recognizing expression types prevented Stoutemyer from successfully simplifying complicated expressions. For example, the simplification of the expression in section 1 proceeds as follows.

$$(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{b} \wedge \mathbf{c}) \cdot (\mathbf{c} \wedge \mathbf{a}) = \mathbf{a} \cdot (\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c}\mathbf{b}) \wedge \mathbf{c}$$

A simplification package now must analyze all types in the expression and see that $\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c}$ is a scalar quantity that can be removed from the vector product. The complete expression then simplifies to zero.

References

[1] Stoutemyer, D. R., 1979. Symbolic computer vector analysis. Computers & Mathematics with Applications, v 5, n 1, 1979, p 1-9.

[2] Milne, E.A. 1948 Vectorial Mechanics. Methuen.