Polynomial Approximation in Handwriting Recognition

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The Pen as an Input Device

- Pen input for electronic devices is becoming important as an input modality.
- Pens can be used where keyboards can't, on very small or very large devices, in wet or dirty environments, by people with repetitive stress injuries.
- They also allow much easier 2-dimensional input, e.g. for drawings, music or mathematics.

\[ e^x = \int e^x \, dx = \sum_{i=0}^{\infty} \frac{x^i}{i!} \]
Pen-Based Math

- Input for CAS and document processing.
- 2D editing.
- Computer-mediated collaboration.
Pen-Based Math

• Different than natural language recognition:
  – 2-D layout is a combination of writing and drawing.
  – Many similar few-stroke characters.
  – Many alphabets, used idiosyncratically.
  – Many symbols, each person uses a subset.
  – No fixed dictionary for disambiguation.
Similar Few-Stroke Characters

D L D 2 3 x000
Character Ambiguities

\[ \sum_{i} i^2 \quad i + z = \sin \omega t \]
Character Ambiguities

aPa² aPaQ
Character Recognition

• A story about three statisticians
• Will concentrate on character recognition
• Several projects ignoring this problem
Digital Ink Formats

- Collected by surface digitizer or camera
- Sequence of \((X,Y)\) points sampled at some known frequency
- Possibly other info (angles, pressure, etc)
- Grouping into traces, letters, words + labelling
Ink Markup Language (InkML)

W3C Proposed Recommendation 10 May 2011

This version:
http://www.w3.org/TR/2011/PR-InkML-20110510/

Latest version:
http://www.w3.org/TR/InkML

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InkML Concepts

- Traces, trace groups
- Device information: sampling rate, resolution, etc.
- Pre-defined and application defined channels
- Trace formats, coordinate transformations
- Annotation text and XML
What the Computer Sees
What the Computer Sees
Usual Character Reco. Methods

• Smooth and re-sample data  *THEN*

• Match against $N$ models by sequence alignment  *

OR

• Identify “features”, such as
  
  – coordinate values of sample points, Number of loops, cusps, Writing direction at selected points, *etc*

Use a classification method, such as
  
  – Nearest neighbour, Subspace projection, Cluster analysis, Support Vector Machine  *THEN*

• Rank choices by consulting dictionary
Difficulties

• Having many similar characters (e.g. for math) means comparison against all possible symbol models is slow.

• Determining features from points
  – Requires many *ad hoc* parameters.
  – Replaces measured points with interpolations
  – It is not clear how many points to keep, and most methods depend on number of points
  – Device dependent

• What to do since there is no dictionary?

• New ideas are needed!
Two Thoughts

• For HWR do we need all the trace data?
  – Do we need all the points?
  – Do we need full accuracy for all the points?

• What is classification?
  – H (English aitch, Greek eta, Russian en)
  – O (zero, oh, degree, ...)
  – P, C, S (R, S, T)
Main Ideas

• Treat traces as curves, not point sets.

• Have to be able to recognize perturbed input.

• There is not always a single “right answer”.

First Axiom of HWR

\[ \forall A, \]
if a sample looks like an A, then it can be an A.
First Axiom of HWR

∀ A,
if a sample looks like an A,
then it can be an A.

Implications:
• Classification gives a set of valid possibilities.
• Must be able to classify perturbed inputs.
• Can use approximation to represent traces more conveniently.
Orthogonal Series Representation

• Main idea:
  Represent coordinate curves as truncated orthogonal series.

• Advantages:
  – **Compact** – few coefficients needed
  – **Geometric**
    – the truncation order is a property of the character set
    – gives a natural metric on the space of characters
  – **Algebraic**
    – properties of curves can be computed algebraically
      (instead of numerically using heuristic parameters)
  – **Device independent**
    – resolution of the device is not important
Inner Product and Basis Functions

- Choose a functional inner product, e.g.

\[ \langle f, g \rangle = \int_a^b f(t)g(t)w(t)\,dt \]

- This determines an orthonormal basis in the subspace of polynomials of degree \(d\).

- Determine \(\phi_i\) using GS on \(\{1, t, t^2, t^3, \ldots\}\).

- Can then approximate functions in subspaces

\[ A(t) \approx \sum_{i=0}^{d} \alpha_i \phi_i(t) \quad \alpha_i = \langle A(t), \phi_i(t) \rangle \]
First Look: Chebyshev Series

• Initially used Chebyshev series [Char+SMW ICDAR 2007].

• Found could approximate closely (small RMS error) with series of order 10.

• Like symbols tended to form clusters.
Raw Data for Symbol G
Coordinate fn approximations

U47.5 X All

U47.5 Y All
Chebyshev Approx to Character
RMS Error

![Graph showing RMS error with different degrees (Degree 3, Degree 4, Degree 6, Degree 8, Degree 10)]
Problems

• Want fast response – how to work while trace is being captured.

• Low RMS does not mean similar shape.
Problem 1. On-Line Ink

• The main problem:
  *In handwriting recognition, the human and the computer take turns thinking and sitting idle.*

• We ask:
  *Can we do useful work while the user is writing and thereby get the answer faster after the user stops writing?*

• We show:
  *The answer is “Yes”!*
On-Line Series Coefficients

• Use Legendre polynomials $P_i$ as basis on the interval $[-1,1]$, with weight function 1.

• Collect numerical values for $f(\lambda)$ on $[0,L]$. 
  $\lambda = \text{arc length}$. 
  $L$ is not known until the pen is lifted.

• As the sample points are collected, numerically integrate the moments $\int \lambda^i f(\lambda) d\lambda$.

• After last point, compute series coefficients for $f$ with domain and range scaled to $[-1,1]$. 
  This uses a simple linear transformation of the moments.
On-Line Series Coefficients

- Transform moments $\mu_i(f, L)$ of $f(\lambda)$ on $[0, L]$ to coefficients of $\hat{f}(\lambda) = \sum_k \hat{\alpha}_k P_k(\lambda)$ on $[-1, 1]$:

$$\hat{\alpha}_k = (-1)^k \frac{2k + 1}{L} \sum_{i=0}^{k} \left(\frac{-1}{L}\right)^i \binom{k}{i} \binom{k + i}{i} \mu_i(f, L)$$

- Normalize range of $f$:

$$\hat{\alpha}_k \frac{b - a}{f_M - f_m} + \delta_{i_0} \frac{a f_M - b f_m}{f_M - f_m}$$
On-Line Series Coefficients

• Approach works for any inner product with linear weight function.

• This is the Hausdorff moment problem (1921), shown to be unstable by Talenti (1987).

• It is just fine, however, for the dimensions we need.
An On-Line Complexity Model

• Input is a sequence of values received at a uniform rate.

• Characterize an algorithm by
  – $\Delta(n)$ complexity as $n$-th input is seen
  – $F(n)$ complexity after last input is seen

• Write on-line complexity as
  \[ \text{OL}_n[\Delta(n), F(n)] \]

• E.g., linear insertion sort requires time
  \[ \text{OL}_n[O(n), 0] \]
Complexity

- The on-line time complexity to compute coefficients for a Legendre series truncated to degree $d$ is then

\[ T_\Delta = 2(d + 2) \]

\[ T_F = \frac{3}{2} d^2 + \frac{11}{2} d + 10 \]

- The time at pen up is constant with respect to the number of points in the trace.
Problem 2. Shape vs Variation

- The corners are not in the right places.
- Work in a jet space to force coords & derivatives close.
- Use a Legendre-Sobolev inner product

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt + \mu_1 \int_a^b f'(t)g'(t)dt + \mu_2 \int_a^b f''(t)g''(t)dt + \ldots$$

- 1\textsuperscript{st} jet space $\Rightarrow$ set $\mu_i = 0$ for $i > 1$.
  - Choose $\mu_1$ experimentally to maximize reco rate.
  - Can be also done on-line.

[Golubitsky + SMW 2008, 2009]
Legendre-Sobolev Basis

\[ a = 0, b = 1, \mu = .125 \]
Life in an Inner Product Space

• With the Legendre-Sobolev inner product we have
  – Low dimensional rep for curves (10 + 10 + 1)
  – Compact rep of samples ~ 160 bits [G+W 2009]
  – >99% linear separability => convexity of classes
  – A useful notion of distance between curves that is very fast to compute
Linear Separability
Linear Separability
Linear Separability

- Can separate $N$ classes with $N(N - 1)$ SVM planes.

- Each class is then (mostly) within its own convex polyhedral cell.

- Can classify either by
  - SVM majority voting + run-off elections (96%)
  - Distance to convex hull of $k$ nearest neighbours (97.5%).
  - On-line computation.
Recognition

• Some classification methods compute the distance between the input curve and models.

  *E.g.* Elastic matching with DTW takes time up to quadratic in the number of sample points and linear in the number of models.

• Many tricks and **heuristics** to improve on this.

  *E.g.* Limit amount of dynamic time warping, pre-classify based on features, ...

• We can do substantially better.
Distance Between Curves

• Elastic matching:
  • Approximate the variation between curves by some fn of distances between sample points.
  • May be coordinate curves or curves in a jet space.

• Sequence alignment
• Interpolation (“resampling”)

• Why not just calculate the area?
• This is very fast in ortho. series representation.
Distance Between Curves

\[
\bar{x}(t) = x(t) + \xi(t) \quad \xi(t) = \sum_{i=0}^{\infty} \xi_i \phi_i(t), \quad \phi_i \text{ ortho on } [a, b] \text{ with } w(t) = 1.
\]

\[
\bar{y}(t) = y(t) + \eta(t) \quad \eta(t) = \sum_{i=0}^{\infty} \eta_i \phi_i(t)
\]

\[
\rho^2(C, \bar{C}) = \int_{a}^{b} \left[ (x(t) - \bar{x}(t))^2 + (y(t) - \bar{y}(t))^2 \right] dt
\]

\[
= \int_{a}^{b} \left[ \xi(t)^2 + \eta(t)^2 \right] dt
\]

\[
\approx \int_{a}^{b} \left[ \sum_{i=0}^{d} \xi_i^2 \phi_i^2(t) + \text{cross terms} + \sum_{i=0}^{d} \eta_i^2 \phi_i^2(t) + \text{cross terms} \right] dt
\]

\[
= \sum_{i=0}^{d} \xi_i^2 + \sum_{i=0}^{d} \eta_i^2
\]
Comparison of Candidate to Models

• Use Euclidean distance in the coefficient space.
  
• *Just as accurate* as elastic matching.

• *Much less expensive.*

• Linear in $d$, the degree of the approximation.
  
  $< 3 \ d$ machine instructions (30ns) vs several thousand!

• Can trace through SVM-induced cells incrementally.

• Normed space for characters gives other advantages.
Choosing between Alternatives

Red class or blue class?
Choosing between Alternatives

The nearest $k$ samples are blue.
The Joy of Convex

- Convexity ⇒ Linear homotopies stay within a class

\[ C = (1 - t) A + t B \]

- Can compute distance of a sample to this line
- Distance to convex hull of nearest neighbors in class gives best recognition [Golubitsky+SMW 2009, 2010]
Choosing between Alternatives

The nearest convex hull of neighbors is red.
Training

• Using CHKNN allows training with relatively few samples. (Dozens vs Thousands per class)
Recognition Summary

- Database of samples ⇒ set of LS points
- Character to recognize ⇒ Integrate moments as being written
  - Lin. trans. to obtain one point in LS space
  - Classify by distance to convex hull of $k$-NN.
Error Rates as Fn of Distance

- Error rate as fn of distance gives confidence measure for classifiers [MKM – Golubitsky + SMW 2009]
Combining with Statistical Info

• Empirical confidence on classifiers allows geometric recognition of isolated symbols to be combined with statistical methods.

• Domain-specific $n$-gram information:
  – Research mathematics – 20,000 articles from arXiv [MKM -- So+SMW 2005]
  – 2nd year engineering math – most popular textbooks [DAS -- SMW 2008]
  – Inverse problem – identifying area via $n$-gram freq! [DML -- SMW 2008]
Deciding with Confidence Measure

Symbol $X$ in an Expression $E$

Symbol Recognizer:
$X \in \text{Class}_1 \textbf{with Conf } \xi_1$

Context-Based Predictor:
$X \in \text{Class}_2 \textbf{ with Conf } \xi_2$

$X \in \text{Class}_i \iff \xi_i = \max(\xi_1, \xi_2)$
Orientation and Shear

- Reco when writing at an angle, or with slanted chars.

- Instead of taking ortho series of coord fns $x(\lambda)$ and $y(\lambda)$, use ortho series of integral invariants of these.
  
  [Golubitsky, Mazalov, SMW 2009 rotn, 2010 shear]

\[
I_0(\lambda) = \text{radius} \quad \quad I_1(\lambda) = \text{area}
\]

\[
I_{k>1}(\lambda) = \text{more complicated integrals}
\]
SNC for Features
Sensible Critical Points

- Functional approx uses non-local information
- Puts critical points where they should be.
- Univ. polynomial root finding.

\[
\frac{dx}{d\lambda} = 0 \quad \frac{dy}{d\lambda} = 0
\]
Representations
SNC Problems

• Want small perturbations wrt LS norm.
• Transformation between LS basis and monomial basis ill-conditioned.
• Want to compute resultants, etc, without transforming to monomial basis.
• Can use degree-grading to push some arguments through. How far can we take this?
Conclusions

• Ask what are we really trying to do.
• Work with ink traces as curves, rather than as collections of sample points.

• Admits powerful analytic tools.
• Have useful geometry on space of curves.

• Gives device/resolution independence.
• Gives faster algorithms.
• Gives useful insights.
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