Topic 17

Analysis of Algorithms

Analysis of Algorithms- Review

- ***Efficiency*** of an algorithm can be measured in terms of :
  - ***Time complexity***: a measure of the amount of time required to execute an algorithm
  - ***Space complexity***: amount of memory required
- Which measure is more important?
  - It often depends on the limitations of the technology available at time of analysis (e.g. processor speed vs memory space)

Time Complexity Analysis

- ***Objectives*** of time complexity analysis:
  - To determine the feasibility of an algorithm by estimating an ***upper bound*** on the amount of work performed
  - To compare different algorithms before deciding which one to implement
- Time complexity analysis for an algorithm is ***independent*** of the programming language and the machine used

Time Complexity Analysis

- Time complexity expresses the relationship between
  - the ***size of the input***
  - and the ***execution time*** for the algorithm
- Usually expressed as a ***proportionality***, rather than an exact function

Time Complexity Measurement

- Essentially based on the number of ***basic operations*** in an algorithm:
  - Number of arithmetic operations performed
  - Number of comparisons
  - Number of times through a critical loop
  - Number of array elements accessed
  - etc.
- Think of this as the ***work*** done

Example: Polynomial Evaluation

Consider the polynomial

$$P(x) = 4x^4 + 7x^3 - 2x^2 + 3x^1 + 6$$

Suppose that exponentiation is carried out using multiplications. Two ways to evaluate this polynomial are:

Brute force method:

$$P(x) = 4*x*x*x*x + 7*x*x*x - 2*x*x + 3*x + 6$$

Horner's method:

$$P(x) = (((4*x + 7) * x - 2) * x + 3) * x + 6$$

Method of analysis

- What are the ***basic operations*** here?
  - multiplication, addition, and subtraction
    - We'll only consider the number of multiplications, since the number of additions and subtractions are the same in each solution
- We'll examine the *general form* of a polynomial of degree **n**, and express our result *in terms of **n***
- We'll look at the ***worst case*** (maximum number of multiplications) to get an ***upper bound*** on the work

Method of analysis

General form of a polynomial of degree **n** is

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x^1 + a_0$$

where a_n is non-zero for all **n >= 0**

(this is the worst case)

Analysis of Brute Force Method

$P(x) = a_n \underline{* x * x * \ldots * x * x} +$	**n** multiplications
$a_{n-1} \underline{* x * x * \ldots * x * x} +$	**n-1** multiplications
$a_{n-2} \underline{* x * x * \ldots * x * x} +$	**n-2** multiplications
$\ldots +$	…
$a_2 \underline{* x * x} +$	**2** multiplications
$a_1 \underline{* x} +$	**1** multiplication
a_0	

Number of multiplications needed in the ***worst case*** is

$$T(n) = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$$
$$= n\,(n + 1) / 2 \quad \textit{(see below)}$$
$$= n^2 / 2 + n / 2$$

Sum of first n natural numbers:

Write the n terms of the sum in forward and reverse orders:

$$T(n) = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$
$$T(n) = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$$

Add the corresponding terms:

$$2*T(n) = (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) + (n+1)$$
$$= n\,(n+1)$$

Therefore, $T(n) = n\,(n+1) / 2$

Analysis of Horner's Method

$P(x) = (\ldots (((a_n * x +$	1 multiplication
$a_{n-1}) * x +$	1 multiplication
$a_{n-2}) * x +$	1 multiplication
$\ldots +$	
$a_2) * x +$	1 multiplication
$a_1) * x +$	1 multiplication
a_0	

n times

Analysis of Horner's Method

Number of multiplications needed in the ***worst case*** is *:*

$T(n) = n$

Big-O Notation

- Analysis of Brute Force and Horner’s methods came up with ***exact formulae*** for the maximum number of multiplications
- In general, though, we want ***upper bounds*** rather than exact formulae: we will use the Big-O notation introduced earlier …

Big-O : Formal Definition

- ***Time complexity*** ***T(n)*** of an algorithm is ***O(f(n))***

  (we say "***of the order f(n)*** ")

  if for some positive constant **C**
  and for all but finitely many values of **n**
  (**i.e.** as **n** gets large)

  T(n) <= C * f(n)

- What does this mean? this gives an ***upper bound*** on the number of operations, ***for sufficiently large n***

Big-O Analysis

- We want our **f(n)** to be an easily recognized ***elementary function*** that describes the performance of the algorithm

Big-O Analysis

Example: Polynomial Evaluation

- What is **f(n)** for Horner's method?
  - **T(n) = n**, so choose **f(n) = n**
  - So, we say that the number of multiplications in Horner's method is **O(n)** ("*of the order of n*") and that ***the time complexity of Horner's method is O(n)***

Big-O Analysis

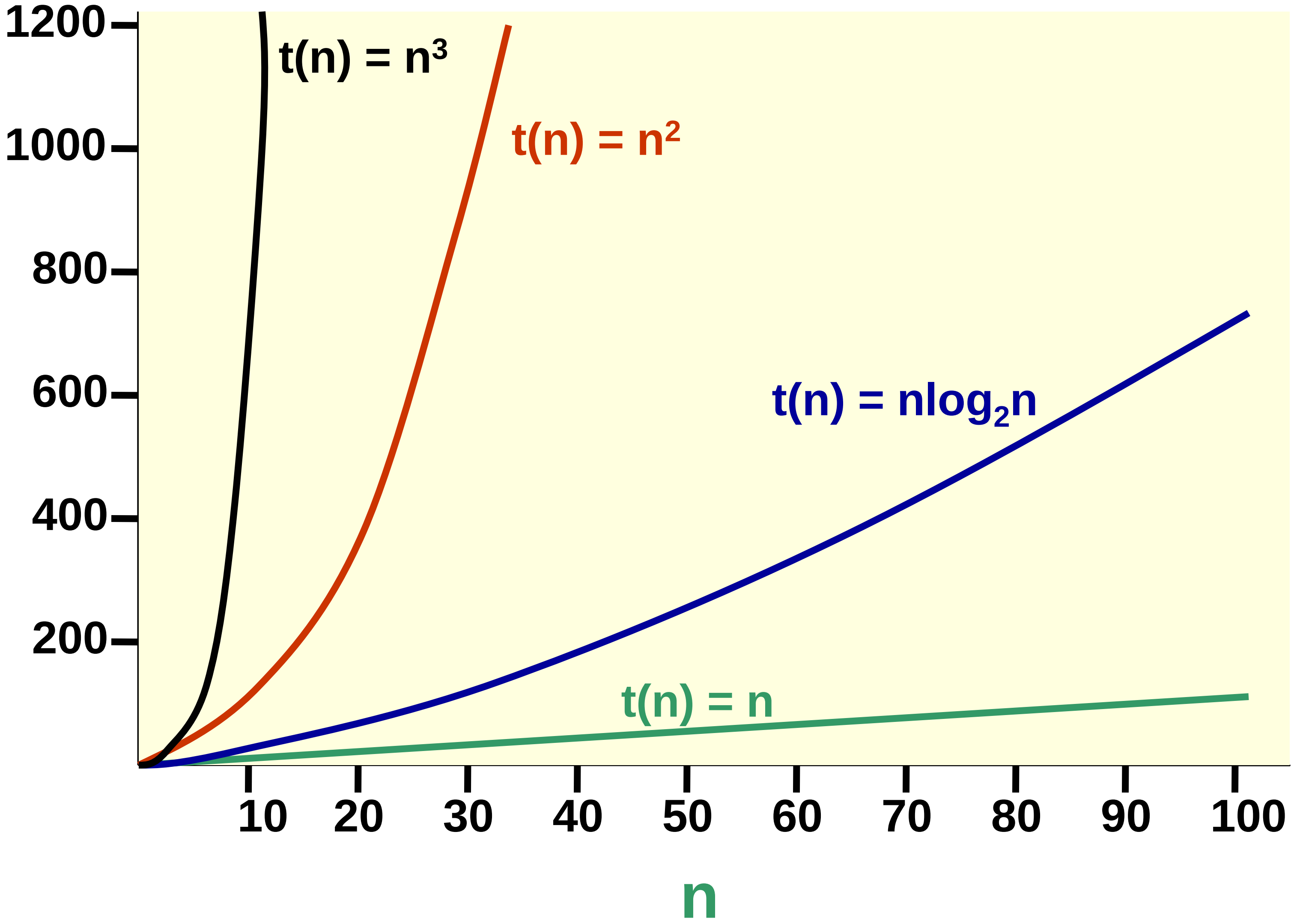
Example: Polynomial Evaluation

- What is **f(n)** for the Brute Force method?
  - Choose the highest order (***dominant***) term of
    $T(n) = n^2/2 + n/2$
    with a coefficient of **1**, so that
    $f(n) = n^2$

Discussion

- Why did we use the dominant term?
  - It determines the basic ***shape*** of the function
- Why did we use a coefficient of 1?
  - For large $\mathbf{n}$, $\mathbf{n^2/2}$ is essentially $\mathbf{n^2}$

Recall: Shape of Some Typical Functions



Big-O Analysis

Example: Polynomial Evaluation

- Is **$f(n) = n^2$** a good choice? i.e. for large **n**, how does **T(n)** compare to **f(n)**?
  - $T(n)/ f(n)$ approaches ½ for large n
  - So, $T(n)$ is approximately $n^2/2$ for large n
- $n^2/2 \le T(n) \le n^2$, so $f(n)$ is a close upper bound

Big-O Analysis

Example: Polynomial Evaluation

- So, we say that the number of multiplications in the Brute Force method is ***O(n²)*** (“*of the order of n²*”) and that ***the time complexity of the Brute Force method is O(n²)***
  - Think of this as “***proportional to n²***”

Big-O Analysis: Summary

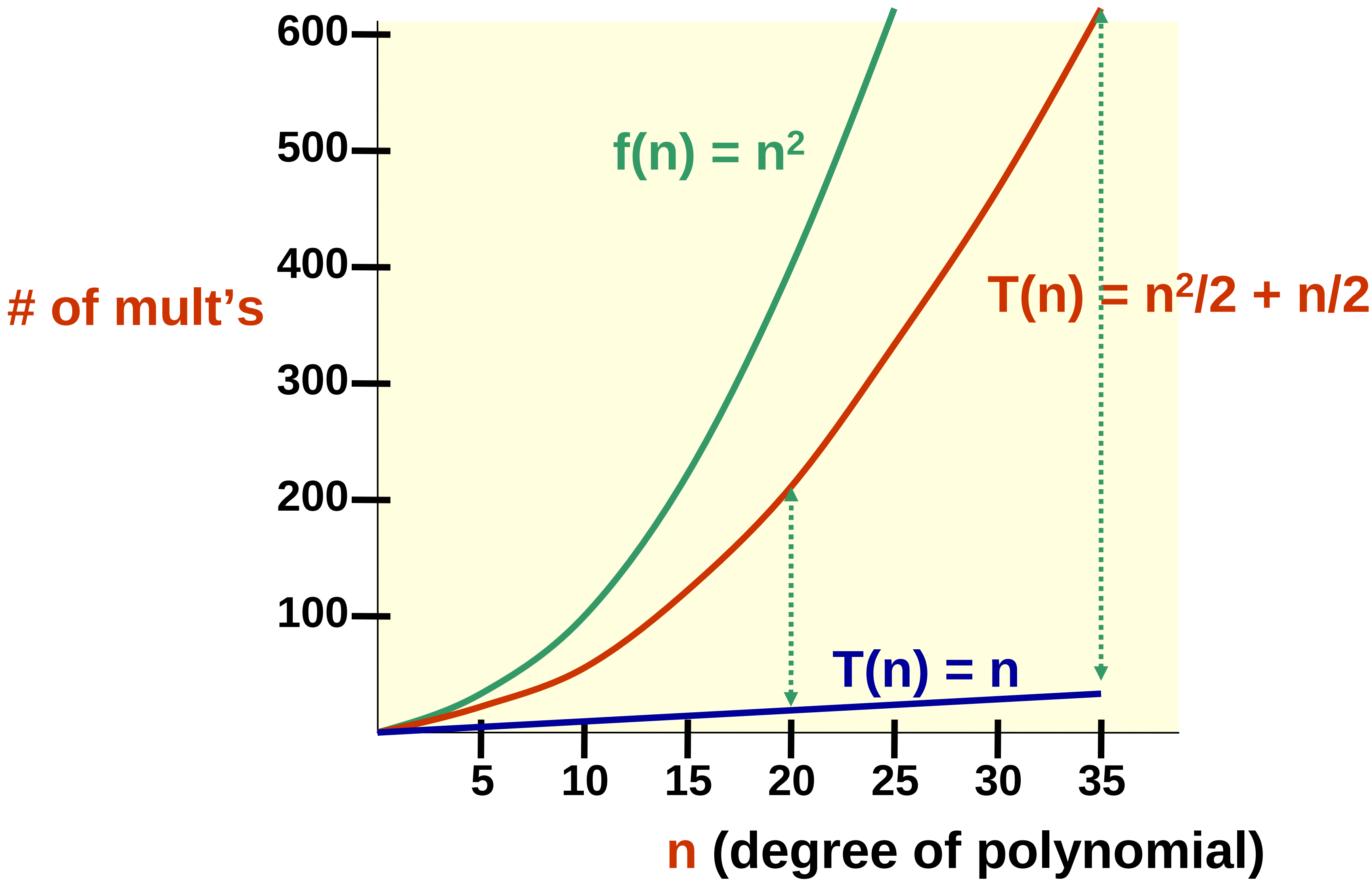
- We want **f(n)** to be an easily recognized elementary function
- We want a ***tight upper bound*** for our choice of **f(n)**
  - $\mathbf{n^3}$ is also an upper bound for the Brute Force method, but not a good one!
    - Why not? Look at how T(n) compares to f(n) = $\mathbf{n^3}$
      - T(n) / n^3 approaches 0 for large n, i.e. T(n) is miniscule when compared to n^3 for large n

Big-0 Example: Polynomial Evaluation Comparison

n and T(n) =n (Horner)	$T(n)= n^2/2 + n/2$ (Brute Force)	$f(n) = n^2$ (upper bound for Brute Force)
5	15	25
10	55	100
20	210	400
100	5050	10000
1000	500500	1000000

n is the degree of the polynomial.
Recall that we are comparing the number of multiplications.

Big-0 Example: Polynomial Evaluation



Time Complexity and Input

- Run time can depend on the ***size of the input*** only (***e.g.*** sorting 5 items vs. 1000 items)
- Run time can also depend on the ***particular input*** (***e.g.*** suppose the input is already sorted)
- This leads to several kinds of time complexity analysis:
  - ***Worst case*** analysis
  - ***Average case*** analysis
  - ***Best case*** analysis

Worst, Average, Best Case

- ***Worst case analysis***: considers the ***maximum*** of the time over all inputs of size **n**
  - Used to find an upper bound on algorithm performance for large problems (large **n**)
- ***Average case analysis***: considers the ***average*** of the time over all inputs of size **n**
  - Determines the average (or expected) performance
- ***Best case analysis***: considers the ***minimum*** of the time over all inputs of size **n**

Discussion

- What are some difficulties with average case analysis?
  - Hard to determine
  - Depends on distribution of inputs
    (they might not be evenly distributed)
- So, we usually use ***worst case*** analysis
  (why not best case analysis?)

Example: Linear Search

- ***The problem***: search an array **a** of size **n** to determine whether the array contains the value **key**
  - Return ***array index*** if found, ***-1*** if not found

```
Set k to 0.
While (k < n-1) and (a[k] is not key)
    Add 1 to k.
If a[k] == key
    Return k.
Else
    Return –1.
```

- Total amount of work done:
  - ***Before loop***: a constant amount of work
  - ***Each time through loop***: 2 comparisons and a constant amount of work (the **and** operation and addition)
  - ***After loop***: a constant amount of work
  - ***So, we consider the number of comparisons only***
- ***Worst case***: need to examine all **n** array locations
- So, **T(n) = 2*n**, and time complexity is **O(n)**

- Simpler (less formal) analysis:
  - Note that work done before and after the loop is independent of **n**, and work done during a single execution of loop is independent of **n**
  - In worst case, loop will be executed **n** times, so amount of work done is proportional to **n**, and algorithm is **O(n)**

- ***Average*** case for a ***successful*** search:
  - Number of comparisons necessary to find the key? **1** or **2** or **3** or **4** … or **n**
  - Assume that each possibility is equally likely
  - Average number of comparisons :
    (1+2+3+ … +n)/n = (n*(n+1)/2)/n
    = (n+1)/2
  - Average case time complexity is therefore **O(n)**

Example: Binary Search

- ***General case***: search a ***sorted*** array **a** of size **n** looking for the value **key**

- ***Divide and conquer*** approach:
  - Compute the middle index **mid** of the array
  - If **key** is found at **mid**, we're done
  - Otherwise repeat the approach on the half of the array that might still contain **key**
  - etc…

Binary Search Algorithm

```
Set first to 0.

Set last to n-1.

Do {

   Set mid to (first + last) / 2.

   If key < a[mid], Set last to mid – 1.

   Else Set first to mid + 1.

} While (a[mid] is not key) and (first <= last).

If a[mid] == key Return mid.

Else Return –1.
```

- Amount of work done before and after the loop is a constant, and is independent of **n**
- Amount of work done during a single execution of the loop is constant
- Time complexity will therefore be proportional to number of times the loop is executed, so that is what we will analyze

Worst case: **key** is not found in the array

- Each time through the loop, at least half of the remaining locations are rejected:
  - After ***first*** time through, **<= n/2** remain
  - After ***second*** time through, **<= n/4** remain
  - After ***third*** time through, **<= n/8** remain
  - After ***k^{th}*** time through, **$\leq n/2^k$** remain

- Suppose in the ***worst case*** that the maximum number of times through the loop is **k**; we must express **k** in terms of **n**

- Exit the do..while loop when the number of remaining possible locations is less than 1 (that is, when **first > last**): this means that $\mathbf{n/2^k < 1}$

- Also, $\mathbf{n/2^{k-1} >= 1}$; otherwise, looping would have stopped after **k-1** iterations

- Combining the two inequalities, we get

  $$n/2^k < 1 <= n/2^{k-1}$$

- Invert and multiply through by **n** to get

  $$2^k > n >= 2^{k-1}$$

- Next, take base-2 logarithms to get

  $$\mathbf{k > \log_2(n) >= k-1}$$

  which is equivalent to

  $$\mathbf{\log_2(n) < k <= \log_2(n) + 1}$$

- So, binary search algorithm is **$O(\log_2(n))$** in terms of the number of array locations examined

Big-O Analysis in General

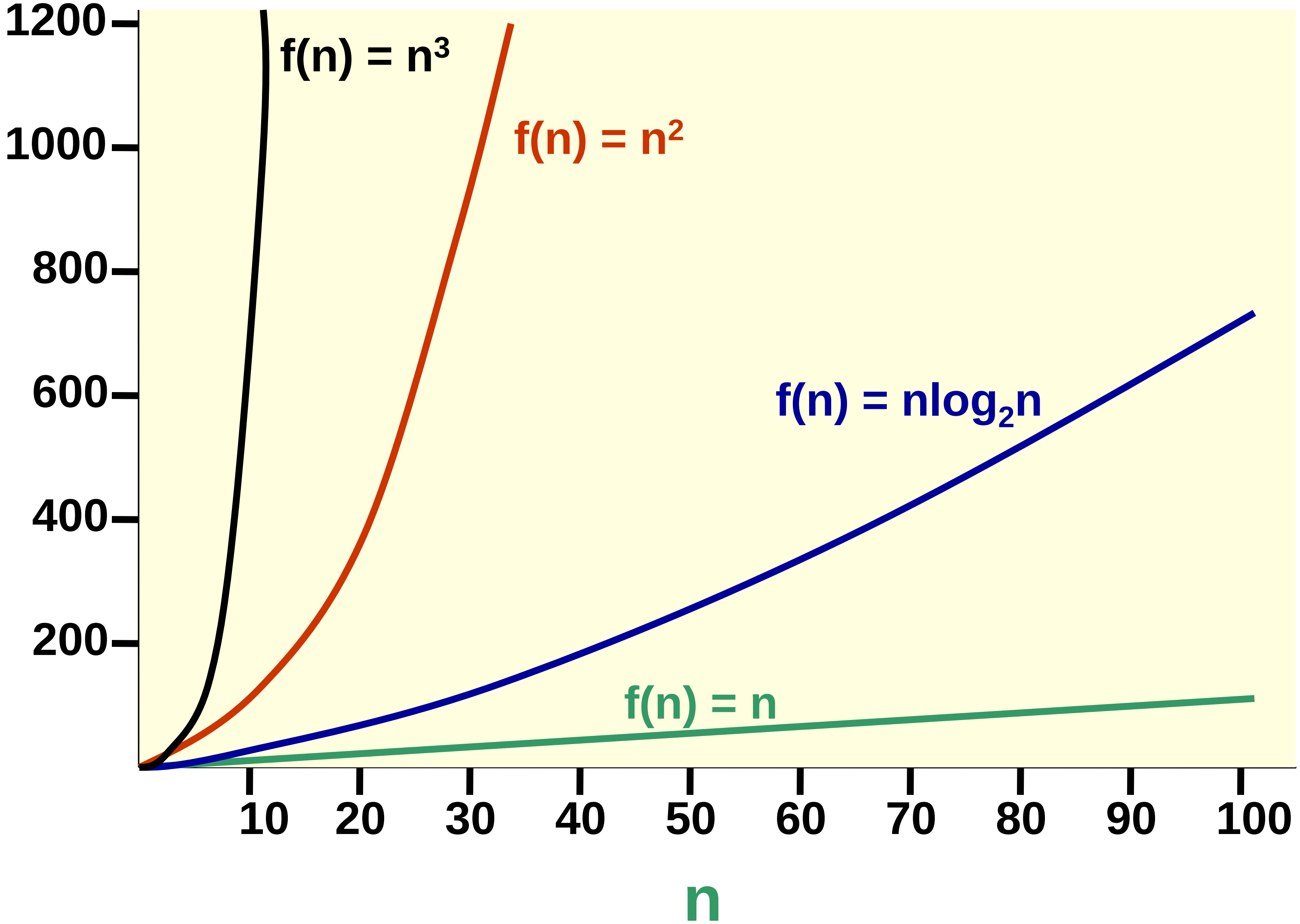
- To determine the time complexity of an algorithm:
  - Look at the loop structure
  - Identify the basic operation(s)
  - Express the number of operations as $f_1(n) + f_2(n) + \ldots$
  - Identify the ***dominant term*** f_i
  - Then the time complexity is $O(f_i)$

- ***Examples*** of ***dominant terms***:
  - $\mathbf{n}$ dominates $\mathbf{\log_2(n)}$
  - $\mathbf{n \log_2(n)}$ dominates $\mathbf{n}$
  - $\mathbf{n^2}$ dominates $\mathbf{n \log_2(n)}$
  - $\mathbf{n^m}$ dominates $\mathbf{n^k}$ when $\mathbf{m > k}$
  - $\mathbf{a^n}$ dominates $\mathbf{n^m}$ for any $\mathbf{a > 1}$ and $\mathbf{m >= 0}$
- That is,

$$\log_2(n) < n < n \log_2(n) < n^2 < \ldots < n^m < a^n$$

for $\mathbf{a > 1}$ and $\mathbf{m > 2}$

Recall: Shape of Some Typical Functions



Examples of Big-O Analysis

- ***Independent nested loops*:**

```
int x = 0;
for (int i = 1; i <= n/2; i++){
  for (int j = 1; j <= n*n; j++){
        x = x + i + j;
  }
}
```

  - Number of iterations of inner loop is **independent** of the number of iterations of the outer loop (***i.e.*** the value of **i**)
  - How many times through outer loop?
  - How many times through inner loop?
  - Time complexity of algorithm?

- ***Dependent nested loops*:**

```
int x = 0;
for (int i = 1; i <= n; i++){
  for (int j = 1; j <= 3*i; j++){
        x = x + j;
  }
}
```

  - Number of iterations of inner loop ***depends on*** the value of **i** in the outer loop
  - On **k**th iteration of outer loop, how many times through inner loop?
  - Total number of iterations of inner loop = sum for **k** running from **1** to **n**
  - Time complexity of algorithm?

Usefulness of Big-O

- We can ***compare algorithms*** for efficiency, for example:
  - ***Linear search*** vs ***binary search***
  - Different sort algorithms
  - Iterative vs recursive solutions (recall Fibonacci sequence!)

- We can ***estimate actual run times*** if we know the time complexity of the algorithm(s) we are analyzing

Estimating Run Times

- Assuming a million operations per second on a computer, here are some typical complexity functions and their associated runtimes:

f(n)	$n = 10^3$	$n = 10^5$	$n = 10^6$
$\log_2(n)$	10^{-5} sec.	$1.7*10^{-5}$ sec.	$2*10^{-5}$ sec.
n	10^{-3} sec.	0.1 sec.	1 sec.
$n \log_2(n)$	0.01 sec.	1.7 sec.	20 sec.
n^2	1 sec.	3 hours	12 days
n^3	17 mins.	32 years	317 centuries
2^n	10^{285} cent.	10^{10000} years	10^{100000} years

Discussion

- Suppose we want to perform a sort that is **$O(n^2)$**. What happens if the number of items to be sorted is 100000?
- Compare this to a sort that is **$O(n \log_2(n))$** . Now what can we expect?
- Is an **$O(n^3)$** algorithm practical for large **n**?
- What about an **$O(2^n)$** algorithm, even for small **n**? e.g. for a Pentium, runtimes are:

n = 30	n = 40	n = 50	n = 60
11 sec.	**3 hours**	**130 days**	**365 years**

Intractable Problems

- A problem is said to be ***intractable*** if solving it by computer is impractical
- Algorithms with time complexity **$O(2^n)$** take too long to solve even for moderate values of **n**
  - What are some examples we have seen?