Employing C++ Templates in the Design of a Computer Algebra Library

Basic Polynomial Algebra Subprograms

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The BPAS Library

Basic Polynomial Algebra Subprograms (BPAS) [2] provides **high-performance polynomial algebra**.

→ High performance: core implementation in C considers data locality, cache complexity, and parallelism for modern multi-core architectures

→ Easy to use: **"Dynamic" Object-Oriented interface** in C++ is a light-weight wrapper of the underlying, optimized C code.

Notable highly-optimized operations include:

→ FFTs, parallel integer polynomial multiplication, modular polynomial arithmetic, parallel Taylor shift, real root isolation (ICMS 2014 [5])

→ Big prime field FFTs, arithmetic in $\mathbb{Z}/p\mathbb{Z}$ for large characteristic [6]

→ Sparse polynomial arithmetic, pseudo-division, normal form [4]

→ Polynomial system solving: parallel triangular decomposition via regular chains [3]

Outline

Motivation: Usability

BPAS is concerned with **accessibility**, **interoperability**, and **usability**.

→ Open-source and written in C/C++ provides the former two.

To achieve usability, we consider best practices for its interface.

1. Natural: a symmetric encoding of the algebraic hierarchy

   field ⊂ Euclidean domain ⊂ GCD domain ⊂ integral domain ⊂ ring

2. Easy to use: an object-oriented design with well-defined interfaces. A so-called **algebraic class hierarchy**: rings are classes and elements of a ring are objects

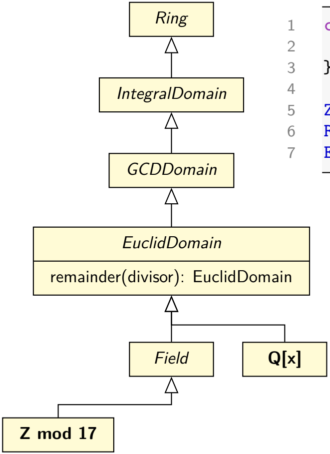
3. Encapsulation: hide complexity of low-level code; class interfaces

4. Extensible: adaptable to new (user-created) types, type composition

5. Type safe: compile-time type safety and mathematical type safety

Motivation: Type Safety

A naive implementation of the algebraic hierarchy as a class hierarchy creates mathematically unsafe operations via polymorphism.




```
class EuclidDomain {
    EuclidDomain remainder(EuclidDomain& divisor);
}

Zmod17 a;
RationalPoly b;
EuclidDomain r = a.remainder(b);
```

→ $\mathbb{Z}/17\mathbb{Z}$ and $\mathbb{Q}[x]$ are Euclidean domains

→ the code is valid via polymorphism

→ could compile, but then issues at runtime.

Existing Solutions

In other compiled libraries, mathematical type safety is only a runtime property maintained through runtime value checks.

→ In Singular's libpolys [7], all algebraic types are a single class. Instance variables (Booleans, enums) store properties of rings

→ In CoCoA [1] rings and elements of a ring are separate classes. Elements hold references to their "owning" ring which are compared at runtime and errors thrown if not identical.

→ In LinBox [8] rings and elements are again distinct, with references to abstract ring elements being downcasted for operations.

Our Goal: provide both compile-time mathematical type safety and a natural, extensible object-oriented hierarchy for the algebraic hierarchy

Outline

Templates in C++

A **class template** in C++ is a parameterized class definition

→ *template instantiation*: providing a particular value for a template parameter

→ **compile-time** code generation and *overload resolution* occurs

→ synonymous definitions for function templates

Template metaprogramming uses templates to control and modify a program's code or compilation

→ facilitates compile-time code generation

→ facilitates **compile-time code evaluation**

Compile-Time Introspection

Templates allow for *compile-time introspection*

→ compile-time evaluation of code to determine properties of a type

```
struct Foo {
    typedef int X;
};

template<typename T> char test(typename T::X const*);

template<typename T> int test(...);

#define type_has_X(T) (sizeof(test<T>(NULL)) == 1);

std::cout << "Foo has X: " << type_has_X(Foo);
std::cout << "int has X: " << type_has_X(int);
```

→ if `T` has type `X`, then compile-time overload resolution chooses first definition with return type `char` (size == 1)

→ else, second definition is chosen with return type `int` (size >= 2)

Outline

Algebraic Class Hierarchy

The algebraic hierarchy as a class hierarchy with mathematical type safety

Solution: an **abstract class template hierarchy**.

- → abstract classes: well-defined interfaces, default behaviour
- → inheritance incrementally extends/builds interface
- → template parameter modifies interface to restrict method parameters

```cpp
template <class Derived>
class Ring {...};

template <class Derived>
class IntegralDomain : Ring<Derived> {...};

template <class Derived>
class GCDDomain : IntegralDomain<Derived> {...};

template <class Derived>
class EuclidDomain : GCDDomain<Derived> {
    Derived remainder(Derived& divisor);
}
```

Algebraic Class Hierarchy: Static Polymorphism

Static polymorphism via *Curiously Recurring Template Pattern*: concrete class is used as template parameter of super class.

→ function resolution occurs at compile-time

→ method declaration restricts params to be compile-time compatible

```cpp
template <class Derived>
class EuclidDomain : GCDDomain<Derived> {
    Derived remainder(const Derived& divisor);
};

class Integer : EuclidDomain<Integer> {...}; //CRTP
//Integer remainder(const Integer& divisor);

class RationalPoly : EuclidDomain<RatonalPoly> {...}; //CRTP
//RationalPoly remainder(const RationalPoly& divisor);

Integer x;  RationalPoly p;

//compiler error: EuclidDomain<RationalPoly>::remainder
//                takes RationalPoly as parameter
RationalPoly r = p.remainder(x);
```

Algebraic Class Hierarchy: Adding Flexibility

Disjoint class hierarchies is too restrictive. Allow *implicit conversion* by defining constructors, e.g. for natural ring embeddings $\mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{Q}[x]$

```
class Integer : EuclidDomain<Integer> {};

class RationalPoly : EuclidDomain<RatonalPoly> {
    RationalPoly(Integer x) {...};
};

Integer x;
RationalPoly p;

//no error: implicit conversion, Integer to RationalPoly
RationalPoly r = p.remainder(x);
```

→ Explicitly defining constructors gives permission for compatibility between types at compile-time

→ *cf.* working in a restrictive manner: allow everything at compile-time and catch incompatibility at runtime

Outline

Algebraic Class Hierarchy with Polynomials

Extend abstract class template hierarchy to include polynomials

→ parameterize polynomial abstract classes by coefficient ring

```
template <class Derived>
class Ring {...};

template <class CoefRing, class Derived>
class Poly : Ring<Derived> {...};

class RationalPoly : Poly<RationalNumber, RationalPoly> {...};
```

Problem: What if `CoefRing` is not actually a ring?

→ e.g. `Poly<std::string>` or `Poly::<Apple>`

Problem: polynomial rings form different algebraic types depending on the ground ring

→ e.g. $\mathbb{Q}[x]$ is a Euclidean domain, $\mathbb{Z}[x]$ is an integral domain

Constraining the Ground Ring

At compile-time ensure that a polynomial's coefficient ring is an actual ring with template metaprogramming.

`Derived_from<T, Base>`: statically determines if `T` is a subclass of `Base`, creating a compiler-error if not

→ inheriting from `Derived_from` forces evaluation at compile-time during template instantiation

→ Coefficient ring must be a subclass of `Ring`

→ `Poly` can assume `CoefRing` has a certain interface at minimum

```
template <class T, class Base>
class Derived_from {...};

template <class CoefRing, class Derived>
class Poly : Ring<Derived>,
             Derived_from<CoefRing, Ring<CoefRing>> {...};
```

Adapting to Different Coefficient Rings (1/2)

Determine type of coefficient ring using *compile-time introspection*

- → **Conditional inheritance** then determines correct algebraic type and interface for polynomials over that ring
- → "Dynamic" type creation via introspection, template instantiation

`is_base_of<T, Base>::value`

- → compile-time Boolean value determines if `T` is a subclass of `Base`

`conditional<Bool, T1, T2>::value`

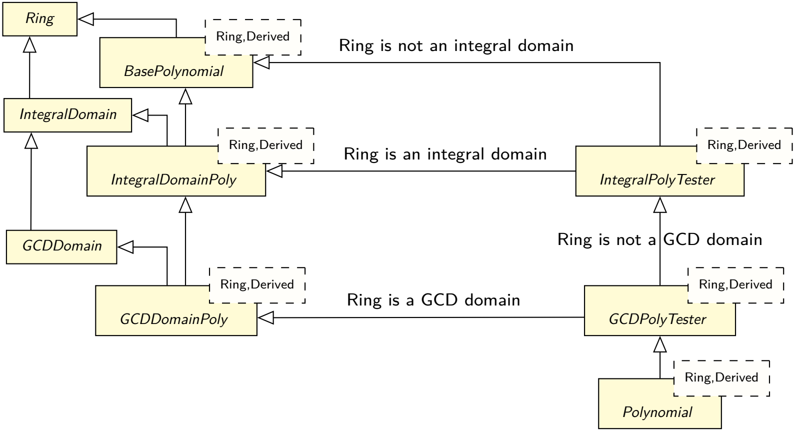
- → A compile-time tertiary conditional operator for choosing types
- → `Bool ? T1 : T2`

```
template <class CRing, class Derived>
class Poly : conditional< is_base_of<CRing, Field<CRing>>::value,
                          EuclidDomain<Derived>,
                          Ring<Derived>
                        >::value {...};
```

Adapting to Different Coefficient Rings (2/2)

A chain of `conditional`'s create a case-discussion at compile-time

- → *Tester* hierarchy separates introspection from actual interface
- → Concrete classes inherit from *Polynomial* to automatically determine their type and interface



Conclusions and Future Work

Algebraic Class Hierarchy as an abstract class template hierarchy

→ Direct object-oriented encoding of algebraic types for strict interfaces

→ Compile-time type safety, implicit conversion allows compatibility

→ Properties of rings (classes) can be exploited with introspection

→ Adaptive polynomial class hierarchy through conditional inheritance

↳ More generally, conditionally exposes/adds methods to an interface

In future:

→ The hierarchy will be extended to include power series, polynomials over prime fields

→ A Python interface will be added on top of the C++ interface

Thank You!

I look forward to your questions:

→ during the live Q/A session, or

→ via email, Alex Brandt <abrandt5@uwo.ca>

References

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[3] M. Asadi, A. Brandt, R. H. C. Moir, M. Moreno Maza, and Y. X. "On the Parallelization of Triangular Decomposition of Polynomial Systems". In: *ISSAC 2020, Proceedings*. (to appear). 2020.

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