On The Parallelization of Triangular Decompositions

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Outline

Decomposing a Non-Linear System

Many ways to “solve” a system

$$\begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{cases} \xrightarrow{\text{Gröbner Basis}} \begin{cases} x + y + z^2 = 1 \\ (y + z - 1)(y - z) = 0 \\ z^2 \left(z^2 + 2y - 1\right) = 0 \\ z^2 \left(z^2 + 2z - 1\right)(z - 1)^2 = 0 \end{cases}$$

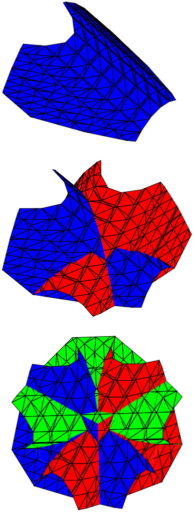
$\Downarrow$ Triangular Decomposition

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y = 0 \\ z - 1 = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}, \quad \begin{cases} x - 1 = 0 \\ y = 0 \\ z = 0 \end{cases}$$

Both solutions are equivalent (via a union).

→ by using triangular decomposition, **multiple components** are found, suggesting possible **component-level parallelism**

Incremental Decomposition of a Non-Linear System



$$F = \begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{cases}$$

$$\varnothing$$

$$F[1] \quad \downarrow$$

$$\left\{x^2 + y + z = 1\right\}$$

$$F[2] \quad \downarrow$$

$$\left\{ \begin{array}{r} x + y^2 + z = 1 \\ y^4 + (2z-2)y^2 + y + (z^2 - z) = 0 \end{array} \right\}$$

$$F[3] \qquad \swarrow \qquad \swarrow \qquad \searrow \qquad \searrow$$

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y = 0 \\ z - 1 = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}, \quad \begin{cases} x - 1 = 0 \\ y = 0 \\ z = 0 \end{cases}$$

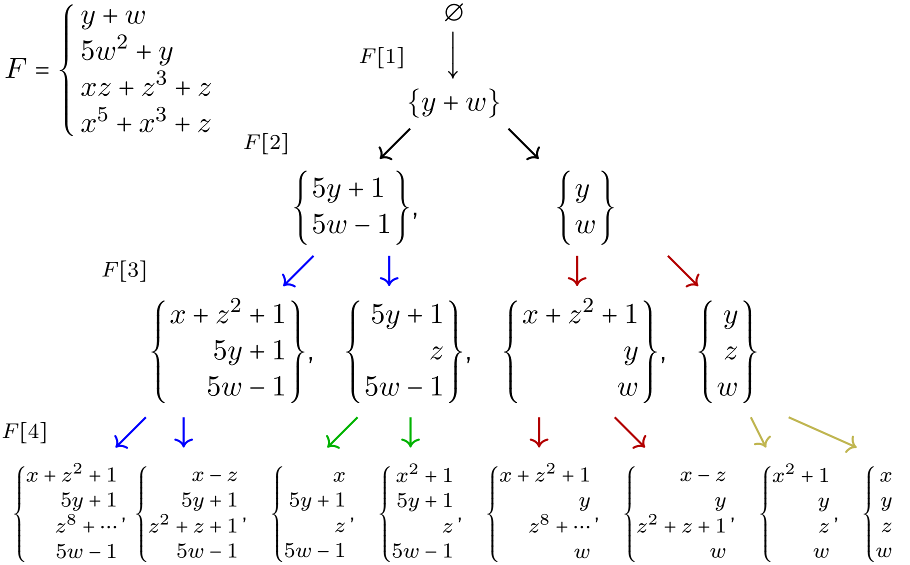
Our Goal: take advantage of different components to gain better performance in high-level decomposition algorithms via **parallelism**

Motivations and Challenges

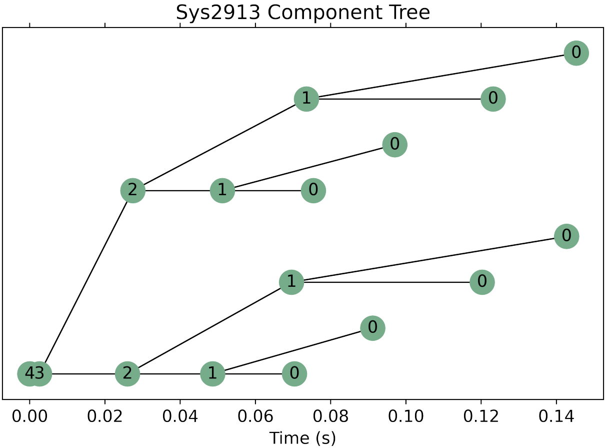
- Many challenges exist in parallelizing triangular decompositions:
  - ↳ Some systems never split
  - ↳ Some split only at the final step, leaving very little concurrency
  - ↳ Some split into one "main" component and several degenerative cases
- Potential **parallelism is problem-dependent** and not algorithmic; it exhibits **irregular parallelism**
- Where a splitting is found in an **intermediate step**, subsequent steps can operate concurrently on each independent component
  - ↳ Finding splittings in the geometry is as difficult as solving the system
- A solution must exploit all possible parallelism, without adding too much overhead in the cases where there is none

A more interesting example (1/2)

$$F = \begin{cases} y + w \\ 5w^2 + y \\ xz + z^3 + z \\ x^5 + x^3 + z \end{cases}$$



A more interesting example (2/2)



→ more parallelism exposed as more components found

→ yet, work unbalanced between branches

→ mechanism needed for dynamic parallelism: “workpile” or “task pool”

Previous Works

- Parallelization of high-level algebraic and geometric algorithms was more common roughly 30 years ago
  - ↳ Such as in Gröbner Bases [1, 3, 4] and CAD [10]
- Recent work on parallelism has been on *low-level* routines with *regular parallelism*:
  - ↳ Polynomial arithmetic [5, 7]
  - ↳ Modular methods for GCDs and Factorization [6, 8]
- Recently, high-level algorithms, often with *irregular parallelism* have neither seen much attention nor received thorough parallelization
  - ↳ The normalization algorithm of [2] finds components serially, then processes each component with a simple parallel map
  - ↳ Early work on parallel triangular decomposition was limited by symmetric multi-processing and inter-process communication [9]

Main Results

Basic Polynomial Algebra Subprograms

- An implementation of triangular decomposition fully in C/C++
- Parallelization effectively exploits as much parallelism as possible throughout the triangular decomposition algorithm
- Implementation framework for parallelization based on task pools, generating functions, pipelines, fork-join
- An extensive evaluation of our implementation against over 3000 real-world polynomial systems

Outline

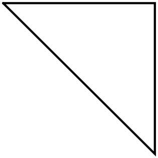
Regular Chains, Notations

Let $\mathbf{k}$ be a perfect field, and $\mathbf{k}[\underline{X}]$ have ordered vars. $\underline{X} = X_1 < \cdots < X_n$

A triangular set T is a regular chain if h is regular modulo $\mathrm{sat}(T_v^-)$ and T_v^- is a regular chain

$T = \Big\{\ T_v = h\,v^d + \mathrm{tail}(T_v)$

$T_v^- = \Big\{$



$\Big\}\ \Big\}$

$\subset \mathbf{k}[\underline{X}]$

Example:

$$T = \left\{ \begin{array}{r} (2y + ba)x - by + a^2 \\ 2y^2 - by - a^2 \\ a + b \end{array} \right\} \subset \mathbb{Q}[b < a < y < x]$$

Saturated ideal of a regular chain:

- → $\mathrm{sat}(T) = (\mathrm{sat}(T_v^-) + \langle T_v \rangle) : \langle h \rangle^\infty$
- → $\mathrm{sat}(\varnothing) = \langle 0 \rangle$

Quasi-component of a regular chain:

- → $W(T) := V(T) \setminus V(h_T)$, $h_T := \prod_{p \in T} h_p$
- → $\overline{W(T)} = V(\mathrm{sat}(T))$

Triangular Decomposition Algorithms

A **triangular decomposition** of an input system $F \subseteq \mathbf{k}[\underline{X}]$ is a set of regular chains $T_1, \ldots, T_e$ such that:

(a) $V(F) = \bigcup_{i=1}^{e} \overline{W(T_i)}$, in the sense of Kalkbrener, or

(b) $V(F) = \bigcup_{i=1}^{e} W(T_i)$, in the sense of Wu and Lazard

Triangular decomposition by incremental **intersection** has key subroutines:

Intersect. Given $p \in \mathbf{k}[\underline{X}]$, $T \subset \mathbf{k}[\underline{X}]$, compute $T_1, \ldots, T_e$ such that:
$V(p) \cap W(T) \ \subseteq \ \bigcup_{i=1}^{e} W(T_i) \ \subseteq \ V(p) \cap \overline{W(T)}$

Regularize: Given $p \in \mathbf{k}[\underline{X}]$, $T \subset \mathbf{k}[\underline{X}]$, compute $T_1, \ldots, T_e$ such that:

i. $W(T) \ \subseteq \ \bigcup_{i=1}^{e} W(T_i) \ \subseteq \ \overline{W(T)}$, and

ii. $p \in \mathrm{sat}(T_i)$ or p is regular modulo $\mathrm{sat}(T_i)$, for $i = 1, \ldots, e$

RegularGCD: Given $p \in \mathbf{k}[\underline{X}]$ with main variable v, $T = \{T_v\} \cup T_v^-$, find pairs (g_i, T_i) such that:

i. $W(T_v^-) \ \subseteq \ \bigcup_{i=1}^{e} W(T_i) \ \subseteq \ \overline{W(T_v^-)}$, and

ii. g_i is a *regular gcd* of p, T_v w.r.t. T_i

Finding Splittings: GCDs and Regularize

Let $p \in \mathbf{k}[\underline{X}] \setminus \mathbf{k}$ with main variable v. Let $T = T_v^- \cup T_v$. All are square free.

A **regular GCD** g of p and T_v w.r.t. $\mathrm{sat}(T_v^-)$ has:

1. h_g is regular modulo $\mathrm{sat}(T_v^-)$
2. $g \in \langle p, T_v \rangle$ (every solution of p and T_v solves g as well)
3. if $\deg(g, v) > 0$, then g pseudo-divides p and T_v.

Let $q = \mathrm{pquo}(T_v, g)$. In Regularize, g says where p vanishes or is regular:

$$W(T) \subseteq W(T_v^- \cup g) \cup W(T_v^- \cup q) \cup (V(h_g) \cap W(T)) \subseteq \overline{W(T)}$$

In Intersect, splittings are found via recursive calls:

$$V(p) \cap W(T) \subseteq W(T_v^- \cup g) \cup (V(p) \cap (V(h_g) \cap W(T))) \subseteq V(p) \cap \overline{W(T)}$$

Parallel Programming Patterns

Parallel Map, Workpile

- → Map a function to each item in a collection, executing each function call simultaneously. Requires *lockstep threads*.
- → Workpile generalizes map to a "pile" of tasks and a set of workers. Allows intermediate tasks to add more tasks, enables **load-balancing**

Asynchronous Generators, Pipeline

- → A generator function (a.k.a iterator, coroutine) which produces data to be consumed in parallel; special-case of *producer-consumer problem*
- → Async generators calling other async generators create a **pipeline**

Divide-and-Conquer, Fork-Join Parallelism

- → Divide a problem, solve recursively, then combine sub-solutions.
- → When >1 recursive call *fork* computations, perform each recursive call concurrently, then *join* before combining sub-solutions.

Outline

Triangularize: incremental triangular decomposition

Algorithm 1 Triangularize(F)

Input: a finite set $F \subseteq \mathbf{k}[\underline{X}]$

Output: regular chains $T_1, \ldots, T_e \subseteq \mathbf{k}[\underline{X}]$ such that $V(F) = W(T_1) \cup \cdots \cup W(T_e)$

```
𝒯 := {∅}
for p ∈ F do
    𝒯′ := {}
    for T ∈ 𝒯 do                         ▷ map Intersect over the current components
        𝒯′ := 𝒯′ ∪ Intersect(p, T)
    𝒯 := 𝒯′
return RemoveRedundantComponents(𝒯)
```

- **Coarse-grained parallelism**: each Intersect represents substantial work
- At each "level" there $|\mathcal{T}|$ components with which to intersect, yielding $|\mathcal{T}| - 1$ additional threads
- Performs a *breadth-first search*, with synchronization at each level

Triangularize: a task-based approach

Algorithm 2 TriangularizeByTasks(F)

Input: a finite set $F \subseteq \mathbf{k}[\underline{X}]$

Output: regular chains $T_1, \ldots, T_e \subseteq \mathbf{k}[\underline{X}]$ such that $V(F) = W(T_1) \cup \cdots \cup W(T_e)$

```
Tasks ← { (F, ∅) };  𝒯 ← {}
while |Tasks| > 0 do
    (P, T) ← pop a task from Tasks
    Choose a polynomial p ∈ P; P′ ← P ∖ {p}
    for T′ in Intersect(p, T) do
        if |P′| = 0 then 𝒯 ← 𝒯 ∪ {T′}
        else Tasks ← Tasks ∪ {(P′, T′)}
return RemoveRedundantComponents(𝒯)
```

- *Tasks* is essentially an underlying data structure for a **task scheduler**
- Use a **thread pool** of workers, each executing the body of the while loop
- Tasks create more tasks, workers pop Tasks until none remain.
- Adaptive to load-balancing, no inter-task synchronization

Outline

Intersect as a Generator

Algorithm 3 **Intersect**(p, T)

Input: $p \in \mathbf{k}[\underline{X}] \setminus \mathbf{k}$, $v := \mathrm{mvar}(p)$, a regular chain T s.t. $T = T_v^- \cup T_v$

Output: regular chains $T_1, \ldots, T_e$ satisfying specs.

for $(g_i, T_i) \in$ **RegularGCD**(p, T_v, v, T_v^-) **do**
    if $\dim(T_i) \neq \dim(T_v^-)$ **then**
        for $T_{i,j} \in$ **Intersect**(p, T_i) **do**
            yield $T_{i,j}$
    else
        if $g_i \notin \mathbf{k}$ and $\deg(g_i, v) > 0$ **then**
            yield $T_i \cup \{g_i\}$
        for $T_{i,j} \in$ **Intersect**$(\mathrm{lc}(g_i, v), T_i)$ **do**
            for $T' \in$ **Intersect**$(p, T_{i,j})$ **do**
                yield T'

→ **yield** "produces" a single data item, and then continues computation

→ each **for** loop consumes a data one at a time from the generator

Generators are both Producers and Consumers

Algorithm 3 **Intersect**(p, T)

for $(g_i, T_i) \in$ **RegularGCD**(p, T_v, T_v^-) **do**
  if $\dim(T_i) \neq \dim(T_v^-)$ **then**
    for $T_{i,j} \in$ **Intersect**(p, T_i) **do**
      yield $T_{i,j}$
  else
    if $g_i \notin \mathbf{k}$ and $\deg(g_i, v) > 0$ **then**
      yield $T_i \cup \{g_i\}$
    for $T_{i,j} \in$ **Intersect**$(\mathrm{lc}(g_i, v), T_i)$ **do**
      for $T' \in$ **Intersect**$(p, T_{i,j})$ **do**
        yield T'

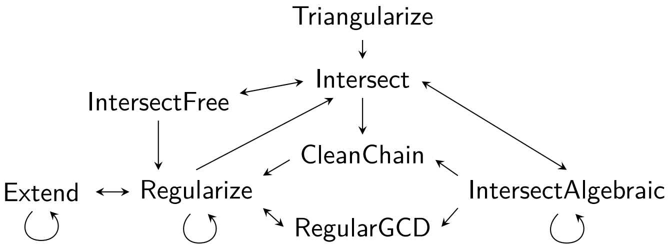
Algorithm 4 **Regularize**(p, T)

for $(g_i, T_i) \in$ **RegularGCD**(p, T_v, T_v^-) **do**
  ▷ assume $\dim(T_i) = \dim(T_v^-)$
  if $0 < \deg(g_i, v) < \deg(T_v, v)$ **then**
    yield $T_i \cup g_i$
    yield $T_i \cup \mathrm{pquo}(T_v, g_i)$
    for $T_{i,j} \in$ **Intersect**$(\mathrm{lc}(g_i, v), T_i)$ **do**
      for $T' \in$ **Regularize**$(p, T_{i,j})$ **do**
        yield T'
  else
    yield T_i

→ Establishing mutually recursive functions as generators allows data to **stream** between subroutines; subroutines are effectively *non-blocking*

→ function call stack of generators creates a *dynamic parallel pipeline*.

Subroutine Pipeline



- → All subroutines as generators allows pipeline to evolve dynamically with the call stack.
- → call stack forms a **tree** if several generators invoked by one consumer
- → Pipeline creates **fine-grained parallelism** since work diminishes with each recursive call
- → A thread pool is used and shared among all generators; generators run synchronously if pool is empty

Outline

Divide-and-Conquer and Fork-Join

Remove redundancies from a list of regular chains with DnC:

- → Recursively and concurrently obtain two irredundant lists, then merge.
- → **Cilk** is used to **fork**/**spawn** and **join**/**sync**

Algorithm 5 RemoveRedundantComponents($\mathcal{T}$)

Input: a finite set $\mathcal{T} = \{T_1, \ldots, T_e\}$ of regular chains
Output: an irredudant set $\mathcal{T}'$ with the same algebraic set as $\mathcal{T}$

```
if e = 1 then return 𝒯
ℓ ← ⌈e/2⌉; 𝒯_{≤ℓ} ← {T_1, ..., T_ℓ}; 𝒯_{>ℓ} ← {T_{ℓ+1}, ..., T_e}
𝒯_1 := spawn RemoveRedundantComponents(𝒯_{≤ℓ})
𝒯_2 := RemoveRedundantComponents(𝒯_{>ℓ})
sync
𝒯_1' := ∅;  𝒯_2' := ∅
for T_1 ∈ 𝒯_1 do
    if ∀T_2 in 𝒯_2 IsNotIncluded (T_1, T_2) then 𝒯_1' := 𝒯_1' ∪ {T_1}
for T_2 ∈ 𝒯_2 do
    if ∀T_1 in 𝒯_1' IsNotIncluded (T_2, T_1) then 𝒯_2' := 𝒯_2' ∪ {T_2}
return 𝒯_1' ∪ 𝒯_2'
```

Outline

Experimentation Setup

Thanks to Maplesoft, we have a collection of over 3000 real-world systems from: actual user data, the literature, bug reports.

Of these $>$3000 systems, 828 require greater than 0.1s to solve

→ Non-trivial systems to warrant the overheads of parallelism

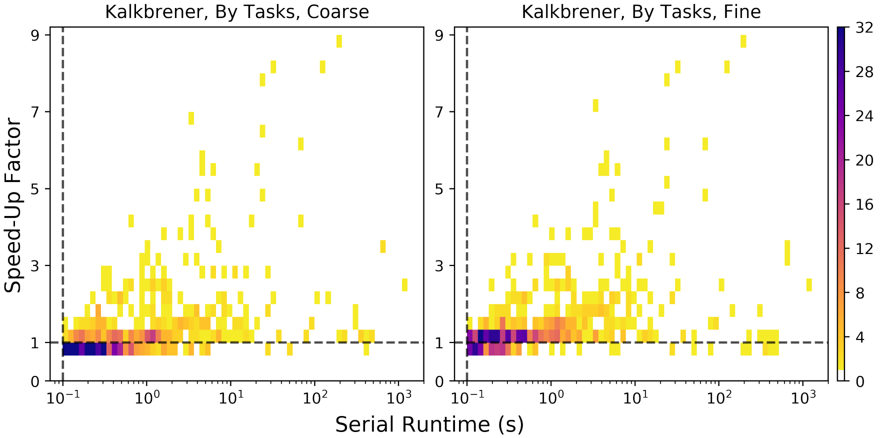
203 of these 828 systems (25%) **do not split** at all

→ No speed-up expected; *some slow-down* is expected in these cases

→ however, we include them to ensure that *slow-down is minimal*

These experiments are run on a node with 2x6-core Intel Xeon X560 processors (24 physical threads with hyperthreading)

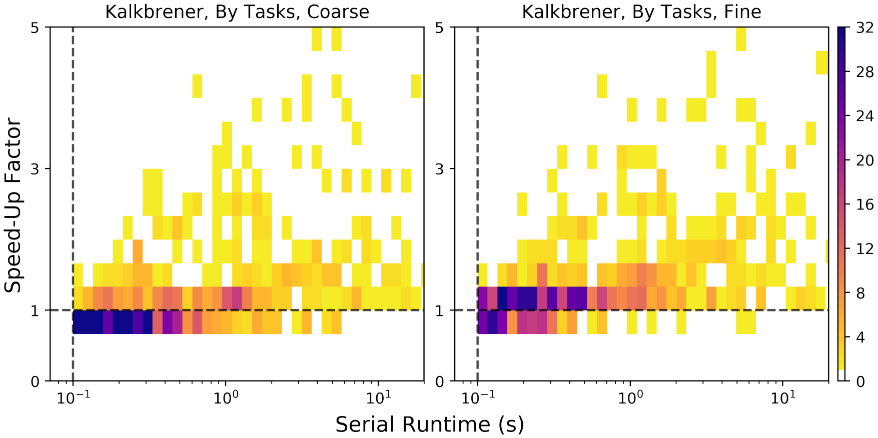
Speed-ups on Non-Trivial Systems (1/2)



→ "Coarse": task manager only, "Fine": tasks and generators

→ Adding generators **increases parallelism**: *streaming* components allows Triangularize to create a new task as early as possible

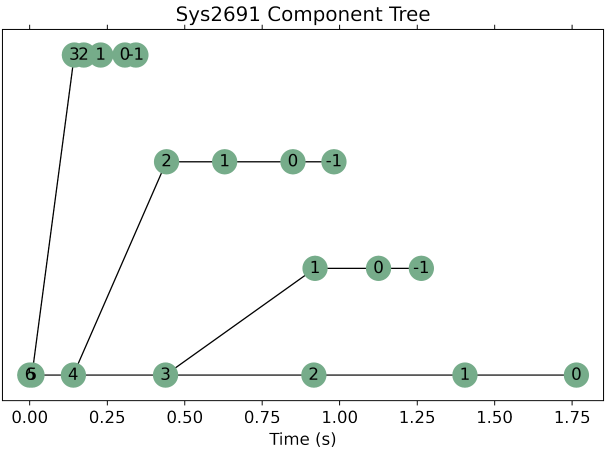
Speed-ups on Non-Trivial Systems (2/2)



→ "Coarse": task manager only, "Fine": tasks and generators

→ Adding generators **increases parallelism**: *streaming* components allows Triangularize to create a new task as early as possible

Inspecting the Geometry: Sys2691



- → Bottom "main" branch is majority of the work.
- → Little overlap with the quickly-solved degenerative branches
- → 2.13× speedup achieved; 88% efficient compared to work/span ratio

Conclusion & Future Work

We have tackled irregular parallelism in a high-level algebraic algorithm

- → our solution dynamically finds and exploits possible parallelism
- → uses dynamic parallel task management, async. generators, and DnC

Further parallelism can be found through:

- → evaluation/interpolation schemes for subresultant chains
- → solving over a prime field produces more splittings; then lift solutions

Our parallel techniques could be employed in further high-level algorithms.

- → e.g. factorization: pipelining between square-free, distinct-degree, and equal-degree factorization

Thank You!

I look forward to your questions:

→ during the live Q/A session,

→ at a Zoom Meeting 18:00-19:00 EEST July 21 2020:
https://westernuniversity.zoom.us/j/93900888047
(Meeting ID: 93900888047)

→ via email:
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References

[1] G. Attardi and C. Traverso. "Strategy-Accurate Parallel Buchberger Algorithms". In: *J. Symbolic Computation* 22 (1996), pp. 1–15.

[2] J. Böhm, W. Decker, S. Laplagne, G. Pfister, A. Steenpaß, and S. Steidel. "Parallel algorithms for normalization". In: *J. Symb. Comput.* 51 (2013), pp. 99–114.

[3] B. Buchberger. "The parallelization of critical-pair/completion procedures on the L-Machine". In: *Proc. of the Jap. Symp. on functional programming*. 1987, pp. 54–61.

[4] J. C. Faugere. "Parallelization of Gröbner Basis". In: *Parallel Symbolic Computation PASCO 1994 Proceedings*. Vol. 5. World Scientific. 1994, p. 124.

[5] M. Gastineau and J. Laskar. "Parallel sparse multivariate polynomial division". In: *Proceedings of PASCO 2015*. 2015, pp. 25–33.

[6] J. Hu and M. B. Monagan. "A Fast Parallel Sparse Polynomial GCD Algorithm". In: *ISSAC 2016, Waterloo, ON, Canada, July 19-22, 2016*. 2016, pp. 271–278.

[7] M. Monagan and R. Pearce. "Parallel sparse polynomial multiplication using heaps". In: *ISSAC*. 2009, pp. 263–270.

[8] M. Monagan and B. Tuncer. "Sparse Multivariate Hensel Lifting: A High-Performance Design and Implementation". In: *ICMS 2018*. 2018, pp. 359–368.

[9] M. Moreno Maza and Y. Xie. "Component-level parallelization of triangular decompositions". In: *PASCO 2007 Proceedings*. ACM. 2007, pp. 69–77.

[10] B. D. Saunders, H. R. Lee, and S. K. Abdali. "A parallel implementation of the cylindrical algebraic decomposition algorithm". In: *ISSAC*. Vol. 89. 1989, pp. 298–307.