Power Series Arithmetic with the BPAS Library

Basic Polynomial Algebra Subprograms

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Past, Present, Future

The **Basic Polynomial Algebra Subprograms (BPAS)** library [3] provides support for high-performance polynomial algebra.

→ At CASC 2018 we presented sparse polynomial arithmetic [4, 6]

→ These polynomials were employed in a polynomial system solving framework based on regular chains [5]

In this talk we present our high-performance implementation of **multivariate power series** written in C.

We are motivated by: (see [1])

→ Computation of limits of multivariate rational functions

→ New applications of Hensel lifting: Extended Hensel Construction, Jung-Abhyankar Theorem

→ Computation of topological closures, resolution of singularities

Outline

Goals and Previous Work

Our goal is a high-performance power series implementation

- → a lazy implementation in a compiled language (for performance)
- → ability to exploit opportunities for concurrent programming

Lazy evaluation is not new:

- → univariate power series in Scratchpad II using Lisp [7]
- → univariate power series and relaxed algorithms [8]
- → polynomial arithmetic [10]

Yet, no general implementation of (compiled) multivariate power series

- → SageMath provides **truncated** multivariate power series
- → multivariate power series in `PowerSeries` library of Maple [2, 9]

What is a power series?

Let $\mathbf{k}$ be a field (often algebraic closed) then $\mathbf{k}[[X_1, \ldots, X_n]]$ is the **ring of formal power series**

→ indeterminates are $X_1, \ldots, X_n$, coefficients in $\mathbf{k}$

Let $f = \sum_e a_e X^e \in \mathbf{k}[[X_1, \ldots, X_n]]$

→ $a_e \in \mathbf{k}$

→ $e = (e_1, \ldots, e_n)$ is a multi-index with n coordinates

→ $|e| = e_1 + \ldots + e_n$

→ **homogeneous part**: $f_{(d)} = \sum_{|e|=d} a_e X^e$

→ **polynomial part**: $f^{(d)} = \sum_{k \le d} f_{(k)}$

Example: $f = 1 + X_1 + X_1 X_2 + X_2^2 + X_1 X_2^2 + X_1^3 + \cdots$

$$f_{(2)} = X_1 X_2 + X_2^2 \qquad\qquad f^{(2)} = 1 + X_1 + X_1 X_2 + X_2^2$$

We say f is *known to precision 3*

Outline

Design Motivations

1. Only compute terms explicitly needed: requested by user, needed for subsequent operations

2. Ability to **resume** and increase precision of an existing power series

This suggests the need for:

→ **power series ancestry**, a history of operands and operators leading to a particular "child" power series

→ **generator functions**, a function to produce new terms of a power series on demand

Ancestry Example

$$
\begin{tikzcd}[column sep=small]
\begin{array}{l} f = \\ 1 + x + yz + \ldots \end{array} \arrow[dr] & & \begin{array}{l} g = \\ 1 + z + y + \ldots \end{array} \arrow[dl] & & \begin{array}{l} a = \\ 1 + y + x^2 + \ldots \end{array} \arrow[dr] & & \begin{array}{l} b = \\ 1 + yz + xz + \ldots \end{array} \arrow[dl] \\
& \times \arrow[d] & & & & \times \arrow[d] & \\
& \begin{array}{l} h = \\ 1 + z + y + x + yz + xz + xy + \ldots \end{array} \arrow[drr] & & & & \begin{array}{l} c = \\ 1 + y + yz + xz + x^2 + \ldots \end{array} \arrow[dll] & \\
& & & + \arrow[d] & & & \\
& & & \begin{array}{l} h + c = \\ 2 + z + 2y + x + 2yz + 2xz + xy + x^2 + \ldots \end{array} & & &
\end{tikzcd}
$$

Generator Functions

Generators could be **co-routines** or **iterators**, continually **yield**-ing terms of a power series in increasing order.

→ Power series operations/arithmetic also necessitates dynamic combinations of generator functions

→ Easy in a scripting language, Harder in a compiled language

In a more "closed-form" solution, our generators:

→ **generate a homogeneous part** of a power series, for a particular (total) degree, where

→ the **degree is a parameter** of the function

Top-level `homogeneous_part` and `polynomial_part` functions call the generators generically, as needed for particular degrees

Encoding a Power Series

Our power series struct:

- → **dense array** of homogeneous polynomials

$f_{(0)}$	$f_{(1)}$	$f_{(2)}$	$f_{(3)}$	$f_{(4)}$	$f_{(5)}$	$\cdots$

- → `int`'s for current allocation, precision
- → a function pointer to a **generator**
- → the arguments to pass to the generator function.

The struct emulates a **function closure** for the generator:

- → captures and stores all necessary variables by reference (pointer) to pass as arguments to the generator
- → uses `void*` parameters for generality
- → The ancestry is implied by storing power series pointers as parameters
  - ↳ use **reference counting** on the power series

The PowerSeries struct

```c
typedef Poly_ptr (*homog_part_gen)(int);
typedef Poly_ptr (*homog_part_gen_unary)(int, void*);
typedef Poly_ptr (*homog_part_gen_binary)(int, void*, void*);
typedef Poly_ptr (*homog_part_gen_tert)(int,void*,void*,void*);

typedef union HomogPartGenerator {
    homog_part_gen nullaryGen;
    homog_part_gen_unary unaryGen;
    homog_part_gen_binary binaryGen;
    homog_part_gen_tert tertiaryGen;
} HomogPartGenerator_u;

typedef struct PowerSeries {
    int deg, alloc;
    Poly_ptr* homog_polys;

    HomogPartGenerator_u gen;
    int genOrder;
    void *genParam1, *genParam2, *genParam3;

    int refCount;
} PowerSeries_t;
```

Power Series Arithmetic: Multiplication

1. A top-level lazy function sets up a `PowerSeries` with generator and generator parameters and immediately returns
2. A **void generator** (wrapper function) is called generically with the `void*` params, casting them to the correct type, and then calls...
3. The **true generator** creates and returns $f_{(d)}$ for input d :

```
Poly_ptr homogPart_prod(int d, PowerSeries_t* f, PowerSeries_t* g){
    Poly_ptr sum = zeroPolynomial();
    for (int i = 0; i <= d; i++) {
        Poly_ptr p = multPolys(homogPart(d-i,f), homogPart(i,g));
        sum = addPolynomials(sum, p);
    }
    return sum;
}
```

→ "Top-level" `homogPart` immediately returns already computed terms, or calls the generator through the function pointer as needed

→ Other supported operations: addition, subtraction, negation, inversion

The Ancestry and Generators

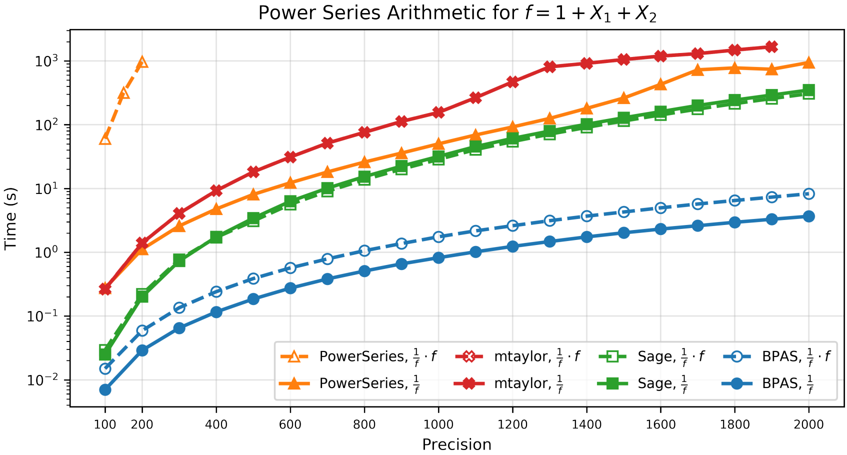
The power series ancestry is implied by a **generator's parameters**

- → For a power series `f`, `f.genParam1`, `f.genParam2`, ... are its **parents**
- → Relationship is one-sided; parents don't know about their children

For a generator to make use of its parents, they must be kept "alive"

- → **reference counting**
- → a parent's reference count is incremented when a child is created
- → "destroying" only decrements reference count...
- → when count ≤ 0, then data is actually `free`'d
- → when a child is `free`'d, its parents get "destroyed"

Experimentation: Integer coefs, 2 vars

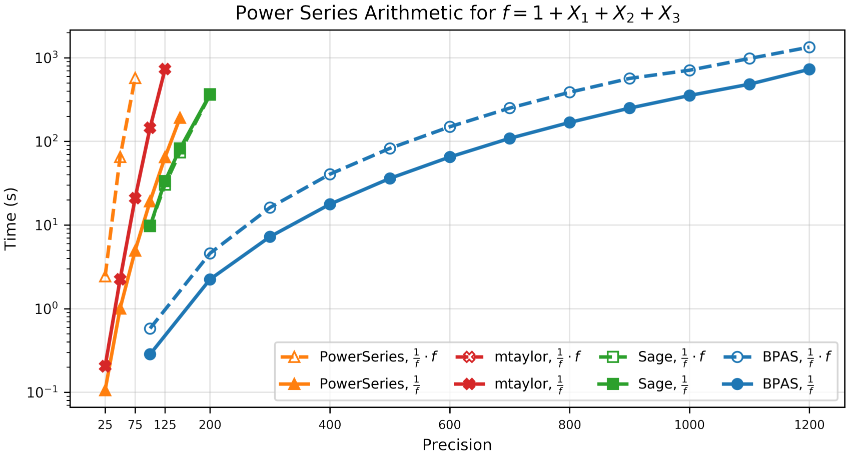


→ `PowerSeries` library in MAPLE [2, 9]

→ `mtaylor` in MAPLE 2020

→ Truncated multivariate power series in SAGE

Experimentation: Integer coefs, 3 vars

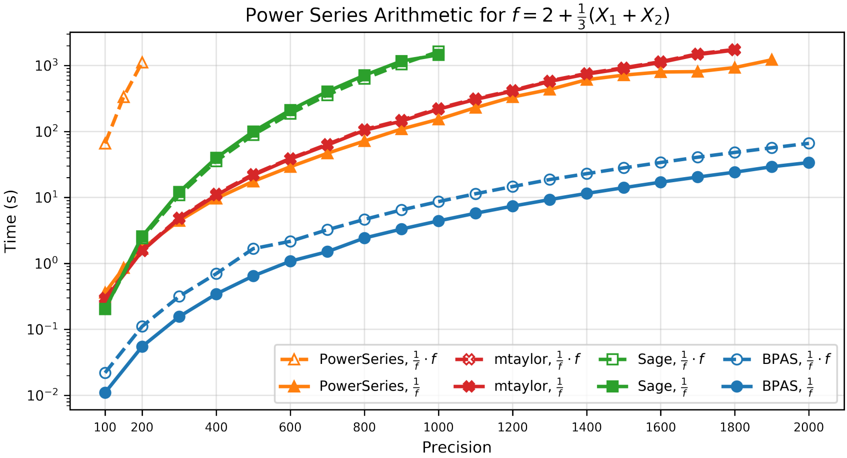


→ `PowerSeries` library in MAPLE [2, 9]

→ `mtaylor` in MAPLE 2020

→ Truncated multivariate power series in SAGE

Experimentation: Rat. Num coefs, 2 vars



- → `PowerSeries` library in MAPLE [2, 9]
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Outline

Weierstrass Preparation Theorem

Let $\mathbb{A} = \mathbf{k}[[X_1, \ldots, X_n]]$ and $\mathcal{M} = \langle X_1, \ldots, X_n \rangle$ be its maximal ideal

Let $f = \sum_i a_i Y^i \in \mathbb{A}[[Y]]$, $d \geq 0$ be the smallest integer s.t. $a_d \notin \mathcal{M}$

→ at the origin $(X_1 = \cdots = X_n = 0)$, $f \neq 0$ and $a_d \neq 0$

WPT yields a **polynomial approximation of a power series** around the origin through factorization

Weierstrass Preparation Theorem: there exists unique $\alpha,\ p$ s.t.

(i) $\alpha \in \mathbb{A}[[Y]]$ is invertible,

(ii) $p = Y^d + \sum_{i=0}^{d-1} b_i Y^i \in \mathbb{A}[Y]$ with $b_0, \ldots, b_{d-1} \in \mathcal{M}$

(iii) $f = \alpha p$

↳ $p \in \mathbb{A}[Y]$ is a monic Univariate Polynomial over Power Series, *UPoPS*

↳ if f is a UPoPS then so is α

Computability of WPT: A Lemma

Let $f, g, h \in \mathbb{A}$ with $f = gh$. With f and h known, compute g.

Assume $f_{(0)} = 0$ and $h_{(0)} \neq 0$, then:

$$f_{(0)} = g_{(0)} h_{(0)} = 0 \implies g_{(0)} = 0$$

By induction, $g_{(r)}$ is uniquely determined by $f_{(1)}, \ldots, f_{(r)}, h_{(0)}, \ldots, h_{(r-1)}$

$$\begin{aligned}
f_{(1)} + f_{(2)} + \cdots + f_{(r)} &= (g_{(1)} + g_{(2)} + \cdots + g_{(r)})(h_{(0)} + h_{(1)} + \cdots + h_{(r)}) \\
f_{(1)} &= g_{(1)} h_{(0)} \\
f_{(2)} &= g_{(2)} h_{(0)} + g_{(1)} h_{(1)} \\
&\vdots \\
f_{(r)} &= g_{(r)} h_{(0)} + g_{(r-1)} h_{(1)} + \cdots + g_{(1)} h_{(r-1)}
\end{aligned}$$

We can compute $g_{(r)}$ for $r = 1, 2, \ldots$ **using only polynomial arithmetic**:

$$\frac{1}{h_{(0)}} \left(f_{(r)} - g_{(r-1)} h_{(1)} - \cdots - g_{(1)} h_{(r-1)} \right) = g_{(r)}$$

Lazy Weierstrass Preparation

Let $f = \sum_{\ell}^{d+m} a_\ell Y^\ell$, $p = Y^d + \sum_j^{d-1} b_j Y^j$, $\alpha = \sum_i^m c_i Y^i$ be UPoPS.

↳ a_ℓ, b_j, c_i are power series

↳ $b_j \in \mathcal{M}$ for $j = 0, \ldots, d-1$

$$f = \alpha p \implies \begin{array}{rcl} a_0 & = & b_0 c_0 \\ a_1 & = & b_0 c_1 + b_1 c_0 \\ & \vdots & \\ a_{d-1} & = & b_0 c_{d-1} + b_1 c_{d-2} + \cdots + b_{d-2} c_1 + b_{d-1} c_0 \\ a_d & = & b_0 c_d + b_1 c_{d-1} + \cdots + b_{d-1} c_1 + c_0 \\ & \vdots & \\ a_{d+m-1} & = & b_{d-1} c_m + c_{m-1} \\ a_{d+m} & = & c_m \end{array}$$

We update p and α by solving these equations modulo $\mathcal{M}^r$, $r = 1, 2, \ldots$

↳ "ping-pong" updates: p to mod $\mathcal{M}^2$, α to mod $\mathcal{M}^2$, p to mod $\mathcal{M}^3 \ldots$

(1) $b_j \equiv 0 \mod \mathcal{M}$, $j = 0, \ldots, d-1$ (2) $c_i \equiv a_i \mod \mathcal{M}$ for $i = 0, \ldots, m$

Lazy Weierstrass Phase 1: Updating p

Let $f = \sum_{\ell}^{d+m} a_\ell Y^\ell$, $p = Y^d + \sum_j^{d-1} b_j Y^j$, $\alpha = \sum_i^m c_i Y^i$ be UPoPS.

↳ a_ℓ, b_j, c_i are power series ↳ $b_j \in \mathcal{M}$ for $j = 0, \ldots, d-1$

$$
\begin{aligned}
a_0 &= b_0 c_0 \\
a_1 - b_0 c_1 &= b_1 c_0 \\
a_2 - b_0 c_2 - b_1 c_1 &= b_2 c_0 \\
&\vdots \\
a_{d-1} - b_0 c_{d-1} - b_1 c_{d-2} + \cdots - b_{d-2} c_1 &= b_{d-1} c_0
\end{aligned}
$$

$b_j \equiv 0 \mod \mathcal{M}$, $j = 0, \ldots, d-1$. Then, for $\mathcal{M}^r$, $r > 1$:

- → let $F_j = a_j - \sum_{k=0}^{j-1} b_k c_{i-k}$
- → the previous lemma applies to each equation $F_j = b_j c_0$ to update each b_j **in succession**, from $j = 0$ to $d-1$
- → Each F_j automatically updated through updated b_k and lazy power series arithmetic

Lazy Weierstrass Phase 2: Updating α

Let $f = \sum_{\ell}^{d+m} a_\ell Y^\ell$, $p = Y^d + \sum_{j}^{d-1} b_j Y^j$, $\alpha = \sum_{i}^{m} c_i Y^i$ be UPoPS.

↳ a_ℓ, b_j, c_i are power series

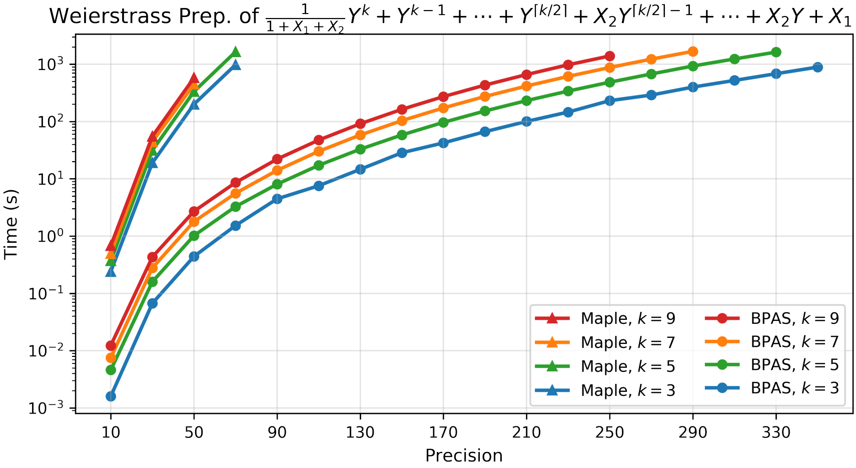
↳ $b_j \in \mathcal{M}$ for $j = 0, \ldots, d-1$

$$\begin{array}{rcl} c_m & = & a_{d+m} \\ c_{m-1} & = & a_{d+m-1} - b_{d-1}c_m \\ c_{m-2} & = & a_{d+m-2} - b_{d-2}c_m - b_{d-1}c_{m-1} \\ & \vdots & \\ c_0 & = & a_d - b_0 c_d - b_1 c_{d-1} - \cdots - b_{d-1}c_1 \end{array}$$

$c_i \equiv a_i \mod \mathcal{M}$ for $i = 0, \ldots, m$. Then, for $\mathcal{M}^r$, $r > 1$:

- → In Phase 1, b_j, $j = 0, \ldots, d-1$ updated to modulo $\mathcal{M}^r$
- → c_i then automatically updated (by lazy arithmetic) to modulo $\mathcal{M}^r$
- → Note: updating each c_i is independent since $b_j \in \mathcal{M}$.
  e.g. b_j known $\mathcal{M}^r$, c_i known $\mathcal{M}^{r-1} \implies b_j c_i$ known modulo $\mathcal{M}^r$

Experimentation



Outline

Hensel's Lemma

Let $f = \sum_i^k a_i Y^i \in \mathbf{k}[[X_1, \ldots, X_n]][Y]$ be a monic polynomial of degree k.

Let $\bar{f} = f(0, \ldots, 0, Y) \in \mathbf{k}[Y]$. Assuming $\mathbf{k}$ is algebraically closed, $\bar{f}$ factorizes into linear factors $\bar{f} = (Y - c_1)^{k_1} \cdots (Y - c_r)^{k_r}$.

Hensel's Lemma: there exists monic $f_1, \ldots, f_r \in \mathbf{k}[[X_1, \ldots, X_n]][Y]$ s.t.

(i) $f = f_1 \cdots f_r$,

(ii) $\deg(f_i, Y) = k_i$, for $i = 0, \ldots, r$

(iii) $\bar{f}_i = (Y - c_i)^{k_i}$, for $i = 0, \ldots, r$

A factorization routine:

1. Translate f by c_i, it now has order k_i
2. Weierstrass preparation can then be applied to obtain p with degree k_i and α with degree $k - k_i$
3. After the reverse translation, p is f_i, and α is the "new" f

Factorization via Hensel's Lemma

Algorithm 1 HenselFactorization(f)

Input: $f = \sum_{i=0}^{k} a_i Y^i, a_i \in \mathbf{k}[[X_1, \ldots, X_n]]$.
Output: $f_1, \ldots, f_r$ satisfying Hensel's Lemma

1: $\bar{f} = f(0, \ldots, 0, Y)$
2: $c_1, \ldots, c_r \leftarrow$ obtain roots of $\bar{f}$ in $\mathbf{k}$ ▷ factor $\bar{f}$
3: $f^* = f$
4: **for** $i = 1$ to r **do**
5: $\quad g \leftarrow f^*(Y + c_i)$
6: $\quad p, \alpha \leftarrow$ WeierstrassPreparation(g)
7: $\quad f_i \leftarrow p(Y - c_i)$
8: $\quad f^* \leftarrow \alpha(Y - c_i)$
9: **return** $f_1, \ldots, f_r$

→ The Taylor shifts $f^*(Y + c_i)$, $p(Y - c_i)$, $\alpha(Y - c_i)$ are implemented lazily. Combined with lazy WPT, this entire factorization is lazy.

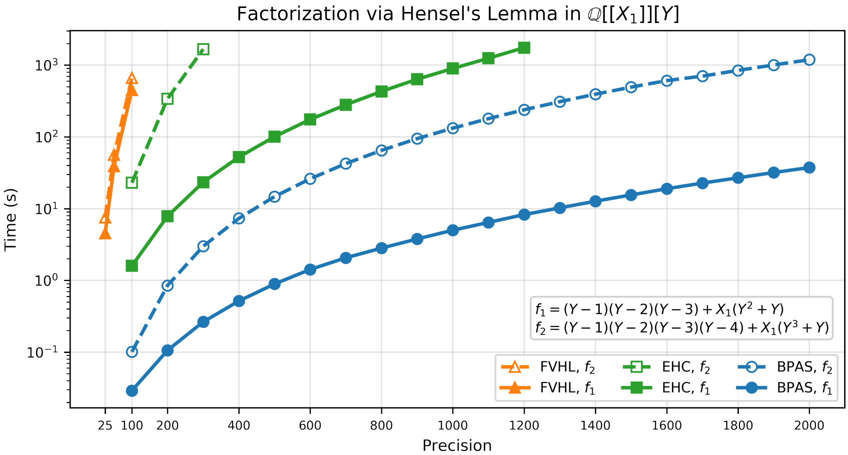
A Factorization Pipeline (w.i.p.)

Viewing factorization via Hensel's lemma as a pipelined computation provides opportunities for parallelism.

- → Updating f_i automatically updates its corresponding α, thus allowing f_{i+1} to be updated
- → Perform each Weierstrass update (and reverse shift) as stages in a parallel pipeline

	Stage 1 (f_1)	Stage 2 (f_2)	Stage 3 (f_3)	Stage 4 (f_4)
Time 1	f_1 to prec. 1			
Time 2	f_1 to prec. 2	f_2 to prec. 1		
Time 3	f_1 to prec. 3	f_2 to prec. 2	f_3 to prec. 1	
Time 4	f_1 to prec. 4	f_2 to prec. 3	f_3 to prec. 2	f_4 to prec. 1
Time 5	f_1 to prec. 5	f_2 to prec. 4	f_3 to prec. 3	f_4 to prec. 2
Time 6	f_1 to prec. 6	f_2 to prec. 5	f_3 to prec. 4	f_4 to prec. 3

Experimentation



From `PowerSeries` library:

→ FVHL: Factorization via Hensel's Lemma

→ EHC: Extended Hensel Construction (Puiseux series)

Conclusions and Future Work

We have implemented high-performing lazy implementations of Power Series and UPoPS, including:

- → Power series arithmetic: $\pm$, $\times$, $\div$
- → Weierstrass preparation
- → Taylor shift by elements of $\mathbf{k}$
- → Factorization via Hensel's lemma

Further performance to be obtained through:

- → Parallelization internal to a Weierstrass Update
- → Pipelined computation in factorization via Hensel's lemma
- → Relaxed algorithms [8]

Thank You!

Questions?

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