Parallel Programming and Triangular Decompositions

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Outline

Solving a Linear System of Equations

$$\begin{cases} x + 3y - 2z = 6 \\ 3x + 5y + 6z = 7 \\ 2x + 4y + 3z = 8 \end{cases}$$

Step 1: triangularization

(a) by *elimination of variables*:

$$\begin{cases} x + 3y - 2z = 6 \\ 3x + 5y + 6z = 7 \\ 2x + 4y + 3z = 8 \end{cases} \xrightarrow[\text{substitue } x]{\text{solve for } x} \begin{cases} x = 5 - 3y + 2z \\ -4y + 12z = -8 \\ -2y + 7z = -2 \end{cases} \xrightarrow[\text{substitue } y]{\text{solve for } y} \begin{cases} x = 5 + 2z - 3y \\ y = 2 + 3z \\ z = 2 \end{cases}$$

(b) by *Gaussian elimination*:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Step 2: back-substitution to find particular values for x, y, z

Solving a Non-Linear System of Equations

Via Gröbner Basis we can "solve" a non-linear system

$$\begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{cases} \implies \begin{cases} x + y + z^2 = 1 \\ (y + z - 1)(y - z) = 0 \\ z^2\left(z^2 + 2y - 1\right) = 0 \\ z^2\left(z^2 + 2z - 1\right)(z - 1)^2 = 0 \end{cases}$$

"Solving" a system is not just about finding particular values, rather:

> *"find a description of the solutions from which we can easily extract relevant data."*

Why?

→ A **positive-dimensional system** has *infinitely many solutions*

→ *Underdetermined* linear systems, and most non-linear systems

Decomposing a Non-Linear System

Many ways to "solve" a system

$$\begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{cases} \xrightarrow{\text{Gröbner Basis}} \begin{cases} x + y + z^2 = 1 \\ (y + z - 1)(y - z) = 0 \\ z^2\left(z^2 + 2y - 1\right) = 0 \\ z^2\left(z^2 + 2z - 1\right)(z - 1)^2 = 0 \end{cases}$$

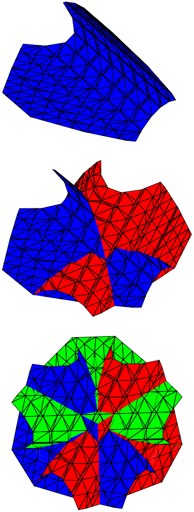
$\Downarrow$ **Triangular Decomposition**

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y = 0 \\ z - 1 = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}, \quad \begin{cases} x - 1 = 0 \\ y = 0 \\ z = 0 \end{cases}$$

Both solutions are equivalent (via a union).

→ by using triangular decomposition, **multiple components** are found, suggesting possible **component-level parallelism**

Incremental Decomposition via Intersection



$$F = \begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{cases}$$

$$\varnothing$$

$$F[1] \quad \downarrow$$

$$\left\{ x^2 + y + z = 1 \right\}$$

$$F[2] \quad \downarrow$$

$$\left\{ \begin{array}{r} x + y^2 + z = 1 \\ y^4 + (2z - 2)y^2 + y + (z^2 - z) = 0 \end{array} \right\}$$

$$F[3] \quad \swarrow \quad \swarrow \quad \searrow \quad \searrow$$

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y = 0 \\ z - 1 = 0 \end{cases}, \quad \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}, \quad \begin{cases} x - 1 = 0 \\ y = 0 \\ z = 0 \end{cases}$$

Our Goal: take advantage of different, independent components to gain performance via concurrency and **thread-level parallelism**

Motivations and Challenges

Component-level parallelism

↳ when a splitting is found during an *intermediate step*, subsequent operations can be performed on each component concurrently

Solving systems by intersection exhibits **irregular parallelism**: parallelism is **problem-dependent** and not algorithmic

↳ Finding splittings in the geometry is as difficult as solving the system

↳ Some systems never split

↳ Some split only at the final step, resulting in no concurrency

↳ Some split irregularly into one big component and many small ones

A **dynamic, adaptable** solution is needed to find, and exploit possible parallelism, without adding excessive overhead in cases where there is none.

A more interesting example (1/2)

$$F = \begin{cases} y + w \\ 5w^2 + y \\ xz + z^3 + z \\ x^5 + x^3 + z \end{cases}$$

$$\varnothing$$

$F[1]$

$$\{y + w\}$$

$F[2]$

$$\begin{Bmatrix} 5y+1 \\ 5w-1 \end{Bmatrix}, \quad \begin{Bmatrix} y \\ w \end{Bmatrix}$$

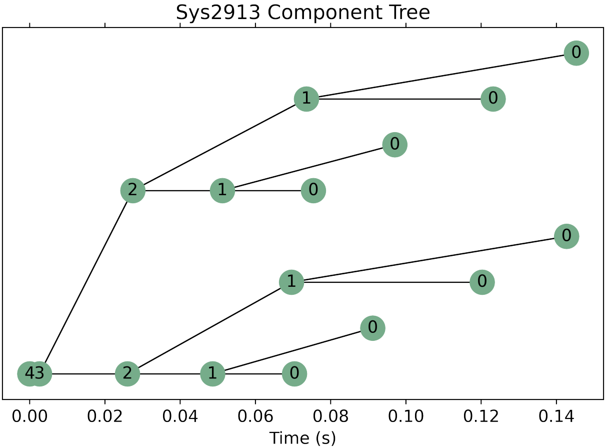
$F[3]$

$$\begin{Bmatrix} x+z^2+1 \\ 5y+1 \\ 5w-1 \end{Bmatrix}, \begin{Bmatrix} 5y+1 \\ z \\ 5w-1 \end{Bmatrix}, \begin{Bmatrix} x+z^2+1 \\ y \\ w \end{Bmatrix}, \begin{Bmatrix} y \\ z \\ w \end{Bmatrix}$$

$F[4]$

$$\begin{Bmatrix} x+z^2+1 \\ 5y+1 \\ z^8+\cdots \\ 5w-1 \end{Bmatrix}, \begin{Bmatrix} x-z \\ 5y+1 \\ z^2+z+1 \\ 5w-1 \end{Bmatrix}, \begin{Bmatrix} x \\ 5y+1 \\ z \\ 5w-1 \end{Bmatrix}, \begin{Bmatrix} x^2+1 \\ 5y+1 \\ z \\ 5w-1 \end{Bmatrix}, \begin{Bmatrix} x+z^2+1 \\ y \\ z^8+\cdots \\ w \end{Bmatrix}, \begin{Bmatrix} x-z \\ y \\ z^2+z+1 \\ w \end{Bmatrix}, \begin{Bmatrix} x^2+1 \\ y \\ z \\ w \end{Bmatrix}, \begin{Bmatrix} x \\ y \\ z \\ w \end{Bmatrix}$$

A more interesting example (2/2)



→ more parallelism exposed as more components found

→ yet, work unbalanced between branches

→ mechanism needed for dynamic parallelism: “workpile” or “task pool”

Previous Works

- Parallelization of high-level algebraic and geometric algorithms was more common roughly 30 years ago
  - ↳ Such as in Gröbner Bases [1, 3, 4] and CAD [11]

- Recent work on parallelism in computer algebra has been on *low-level* routines with *regular parallelism*:
  - ↳ Polynomial arithmetic [5, 8]
  - ↳ Modular methods for GCDs and Factorization [6, 9]

- Recently, high-level algorithms, often with *irregular parallelism* have neither seen much attention nor received thorough parallelization
  - ↳ The normalization algorithm of [2] finds components serially, then processes each component with a simple parallel map
  - ↳ Early work on parallel triangular decomposition was limited by symmetric multi-processing and inter-process communication [10]

Main Results

Basic Polynomial Algebra Subprograms

- An implementation of triangular decomposition fully in C/C++
- Parallelization dynamically finds and exploits as much parallelism as possible throughout the triangular decomposition algorithm
- Implementation framework for parallelization based on task pools, generating functions, pipelines, fork-join
- An extensive evaluation of our implementation against over 3000 real-world polynomial systems

Outline

Polynomial Notations

- Let $\mathbf{k}$ be a perfect field, such as $\mathbb{Q}$ (and its extensions) or $\mathbb{C}$
- Let $\mathbf{k}[\underline{X}]$ be the set of multivariate polynomials (a *polynomial ring*) with n ordered variables, $\underline{X} = X_1 < \cdots < X_n$.
- For $p \in \mathbf{k}[\underline{X}]$:
  - ↳ the main variable of p is the maximum variable with positive degree
  - ↳ the initial of p is the leading coeff. of p with respect to its main variable
  - ↳ the tail of p is the terms leftover after setting its initial to 0

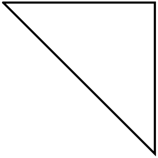
$$(2y + ba)x^2 + (by)x + a^2 \quad \in \; \mathbb{Q}[b < a < y < x]$$

- Any set of polynomials $F \subset \mathbf{k}[\underline{X}]$ can form a **system of equations** by setting $f = 0$ for each $f \in F$.
- The **algebraic variety** of F is the geometric representation of the solution set of F
  - ↳ $V(F) = \{(a_1, \ldots, a_n) \in \mathbf{k}^n \mid f(a_1, \ldots, a_n) = 0, \forall f \in F\}$

Triangular Sets and Regular Chains

A **triangular set** $T \subset \mathbf{k}[\underline{X}]$ is a collection of polynomials with pairwise different main variables.

$$T = \left\{ \begin{array}{l} T_v = h\,v^d + \mathrm{tail}(T_v) \\ T_v^- = \left\{ \quad \right\} \end{array} \right\}$$



$\subset \mathbf{k}[\underline{X}]$

Example:

$$T = \left\{ \begin{array}{r} (2y + ba)x - by + a^2 \\ 2y^2 - by - a^2 \\ a + b \end{array} \right\}$$

$\subset \mathbb{Q}[b < a < y < x]$

A **regular chain** is a triangular set if:

(i) T_v^- is a regular chain, and

(ii) initial of T_v (h) is **regular** with respect to T_v^-

In triangular decomposition, **every component is a regular chain**

Regularity

$$F_1 = \begin{cases} yx - 1 = 0 \\ y = 0 \\ z - 1 = 0 \end{cases}$$

- → This set is inconsistent; there are no solutions
- → Back-substituting $y = 0$, $yx - 1 = 0$ yields $-1 = 0$

$$F_2 = \begin{cases} (y+1)x^2 - x = 0 \\ y^2 - 1 = 0 \\ z - 1 = 0 \end{cases}$$

- → y has two solutions: $y^2 - 1 = (y+1)(y-1)$
- → For $y = -1$, x has 1 solution
- → For $y = 1$, x has 2 solutions

A polynomial is **regular** (w.r.t. a particular regular chain) if it is neither:

(i) zero (e.g. y in F_1), nor

(ii) a *zero-divisor* (e.g. $(y+1)$ in F_2)

The foundation of splitting: regularity testing

To intersect a polynomial with an existing regular chain, it must have a regular initial, regularizing finds splittings via a **case discussion**

→ either the initial is regular, or it is not regular

$$f = (y+1)x^2 - x$$

$$T = \begin{cases} y^2 - 1 = 0 \\ z - 1 = 0 \end{cases}$$

$y + 1 = 0$:

$$T_1 = \begin{cases} y + 1 = 0 \\ z - 1 = 0 \end{cases} \xrightarrow{f = x} T_1 = \begin{cases} x = 0 \\ y + 1 = 0 \\ z - 1 = 0 \end{cases}$$

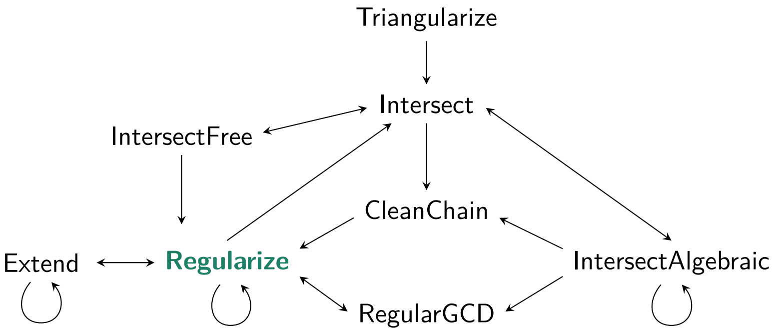
$y + 1 \neq 0$:

$$T_2 = \begin{cases} y - 1 = 0 \\ z - 1 = 0 \end{cases} \xrightarrow{f = 2x^2 - x} T_2 = \begin{cases} 2x^2 - x = 0 \\ y - 1 = 0 \\ z - 1 = 0 \end{cases}$$

All roads lead to Regularize

The Triangularize algorithm iteratively calls intersect, then a network of mutually recursive functions do the heavy-lifting.

↳ In all cases, polynomials are forced to be regular and splittings are (possibly) found via **Regularize**



Outline

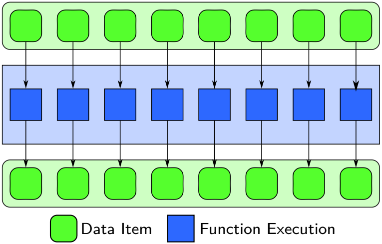
Parallel Map and Workpile

Map is the possibly the most well-known parallel programming pattern

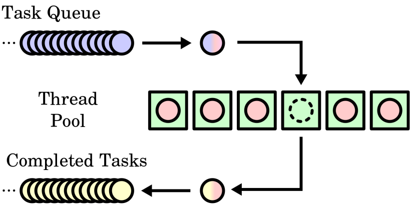
↳ execute a function on each item in a collection concurrently

↳ with multiple Maps, tasks must execute in *lockstep*

Map Pattern [7]



Thread Pool (Wikipedia)



Workpile generalizes Map to a *queue of a tasks*, allowing tasks to add more tasks, thus enabling *load-balancing* as tasks start asynchronously

↳ one possible implementation of workpile is a **thread pool**

Triangularize: incremental triangular decomposition

Algorithm 1 Triangularize(F)

Input: a finite set $F \subseteq \mathbf{k}[\underline{X}]$

Output: regular chains $T_1, \dots, T_e \subseteq \mathbf{k}[\underline{X}]$ encoding the solutions of $V(F)$

```
𝒯 := {∅}
for p ∈ F do
    𝒯′ := {}
    for T ∈ 𝒯 Map                          ▷ map Intersect over the current components
        𝒯′ := 𝒯′ ∪ Intersect(p, T)
    𝒯 := 𝒯′
return RemoveRedundantComponents(𝒯)
```

- **Coarse-grained parallelism**: each Intersect represents substantial work
- At each "level" there are $|\mathcal{T}|$ components with which to intersect, yielding $|\mathcal{T}|$ concurrent calls to intersect
- Performs a *breadth-first search*, with intersects occurring in lockstep

Triangularize: a task-based approach

Algorithm 2 TriangularizeByTasks(F)

Input: a finite set $F \subseteq \mathbf{k}[\underline{X}]$

Output: regular chains $T_1, \ldots, T_e \subseteq \mathbf{k}[\underline{X}]$ encoding the solutions of $V(F)$

1: *Tasks* $\leftarrow \{ (F, \varnothing) \}$; $\mathcal{T} \leftarrow \{\}$
2: **while** $|$*Tasks*$| > 0$ **do**
3: $(P, T) \leftarrow$ pop a task from *Tasks*
4: Choose a polynomial $p \in P$; $P' \leftarrow P \setminus \{p\}$
5: **for** T' in **Intersect**(p, T) **do**
6: **if** $|P'| = 0$ **then** $\mathcal{T} \leftarrow \mathcal{T} \cup \{T'\}$
7: **else** *Tasks* $\leftarrow$ *Tasks* $\cup \{(P', T')\}$
8: **return** RemoveRedundantComponents($\mathcal{T}$)

- *Tasks* is really a task scheduler augmented with a thread pool
- Tasks create more tasks, workers pop Tasks until none remain.
- Adaptive to load-balancing, no inter-task synchronization

Outline

Generators and Pipelines

Generators

→ A generator function (i.e. iterator) yields data items one a time, allowing the function's control flow to resume on its next execution.

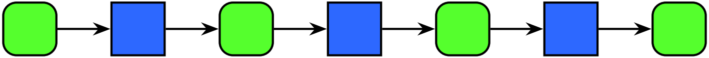
Asynchronous Generators; Producer-Consumer

→ *async generators* can concurrently produce items while the generator's caller is consuming items; creating a producer-consumer pair

Pipeline

→ By connecting many producer-consumer pairs we create a *pipeline*

→ Pipelines need not be linear, they can be *directed acyclic graphs*



Regularize as an Asynchronous Generator

Algorithm 3 **Regularize**(p, T)

Input: $p \in \mathbf{k}[\underline{X}] \setminus \mathbf{k}$, $v := \mathrm{mvar}(p)$, a regular chain $T = T_v^- \cup T_v$

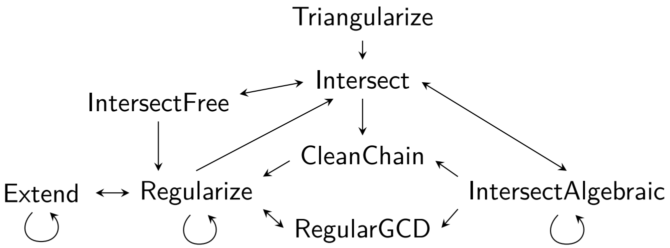
Output: regular chains $T_1, \ldots, T_e$ satisfying specs.

```
for (g_i, T_i) ∈ RegularGCD(p, T_v, T_v^-) do
    if 0 < deg(g_i, v) < deg(T_v, v) then
        yield T_i ∪ g_i
        yield T_i ∪ pquo(T_v, g_i)
        for T_{i,j} ∈ Intersect(lc(g_i, v), T_i) do
            for T' ∈ Regularize(p, T_{i,j}) do
                yield T'
    else
        yield T_i
```

→ **yield** "produces" a single data item, and then continues computation

→ each **for** loop consumes a data one at a time from the generator

Subroutine Pipeline



- → Making all subroutines generators allows a pipeline to evolve dynamically with the call stack.
- → call stack forms a **tree** if several generators invoked by one consumer
- → Asynchronous Generators, Pipelines create **fine-grained parallelism** since work diminishes with each recursive call, pipeline depth
- → In our implementation, a thread pool is used and shared among all generators; generators run synchronously if pool is empty

Outline

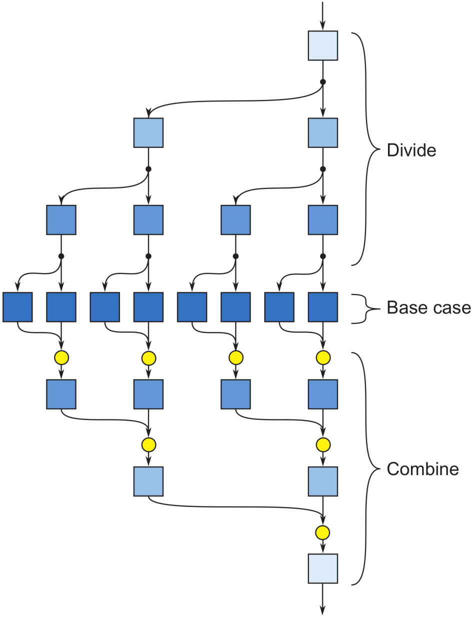
Divide-and-Conquer and Fork-Join

→ Divide a problem into sub-problems, solving each recursively

→ Combine sub-solutions to produce a full solution

→ **Fork**: execute multiple recursive calls in parallel (divide)

→ **Join**: merge parallel execution back into serial execution (combine)



Removal of Redundant Components

After a system is solved, and many components found, we can remove components from the solution set that are contained within others

→ Follow a merge-sort approach; **spawn**/fork and **sync**/join

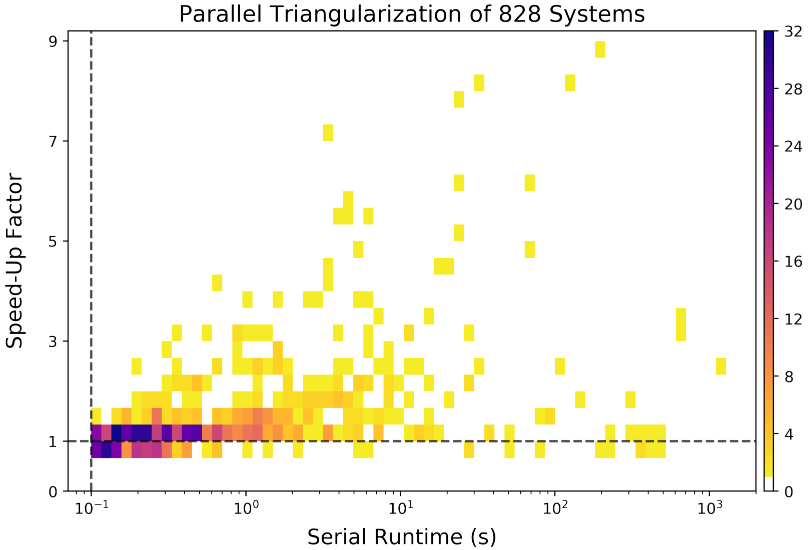
Algorithm 4 RemoveRedundantComponents($\mathcal{T}$)

Input: a finite set $\mathcal{T} = \{T_1, \ldots, T_e\}$ of regular chains
Output: an irredudant set $\mathcal{T}'$ with the same algebraic set as $\mathcal{T}$

```
if e = 1 then return 𝒯
ℓ ← ⌈e/2⌉; 𝒯_{≤ℓ} ← {T_1, ..., T_ℓ}; 𝒯_{>ℓ} ← {T_{ℓ+1}, ..., T_e}
𝒯_1 := spawn RemoveRedundantComponents(𝒯_{≤ℓ})
𝒯_2 := RemoveRedundantComponents(𝒯_{>ℓ})
sync
𝒯_1' := ∅;   𝒯_2' := ∅
for T_1 ∈ 𝒯_1 do
    if ∀T_2 in 𝒯_2 IsNotIncluded (T_1, T_2) then 𝒯_1' := 𝒯_1' ∪ {T_1}
for T_2 ∈ 𝒯_2 do
    if ∀T_1 in 𝒯_1' IsNotIncluded (T_2, T_1) then 𝒯_2' := 𝒯_2' ∪ {T_2}
return 𝒯_1' ∪ 𝒯_2'
```

Outline

Experimentation



Conclusion & Future Work

We have tackled irregular parallelism in a high-level algebraic algorithm

→ our solution dynamically finds and exploits possible parallelism

→ uses dynamic parallel task management, async. generators, and DnC

Further parallelism can be found through:

→ evaluation/interpolation schemes for subresultant chains

→ solving over a prime field produces more splittings; then lift solutions

Our parallel techniques could be employed in further high-level algorithms.

→ e.g. factorization: pipelining between square-free, distinct-degree, and equal-degree factorization

Thank You!

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