A New Algorithm for Computing Integer Hulls of 2D Polyhedral Sets

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Ontario Research Centre for Computer Algebra

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One-dimensional array

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for(int i = 0; i < n; i++){
  for(int j = i + 1; j < n; j ++)
    A[i * n + j] = A[(n * j - n + j - i - 1];
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2. Is there data dependence between them?
3. Are there integer solutions to the system of linear inequalities?

$$\begin{cases} 0 \leq i_1 < n \\ i_1 + 1 \leq j_1 < n \\ 0 \leq i_2 < n \\ i_2 + 1 \leq j_2 < n \\ i_1 \times n + j_1 = n \times j_2 - n + j_2 - i_2 - 1 \end{cases}$$

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There is no integer solution, therefore, no dependence.

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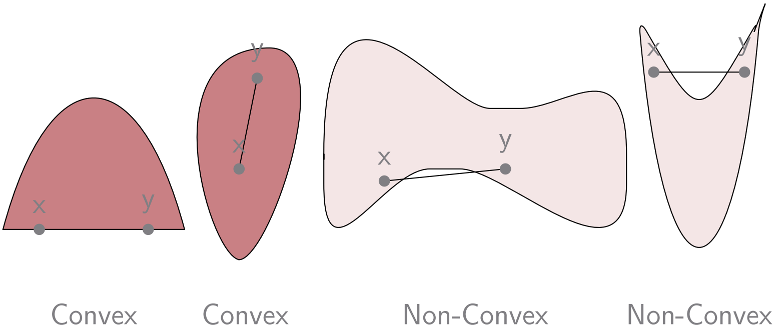
6. This algorithm relies on a recursive construction, effectively reducing the computation of integer hulls in an arbitrary dimensions to that of dimension 2.
7. Hence, we will focus on the 2D case.

Definition

A set S is called convex if the line joining any two points in S is in S, i.e.,

$$\forall x, y \in S, \forall \lambda \in [0,1], \lambda x + (1-\lambda) y \in S.$$



Definition

A convex polyhedral set (or simply a polyhedral set) P is a set $\{\mathrm{x} \in \mathbb{R}^n \mid A\mathrm{x} \leq \mathrm{b}\}$, where $A \in \mathbb{R}^{m \times n}$ is a matrix and $\mathrm{b} \in \mathbb{R}^m$ is a vector.

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A face F of P is a subset $\{\mathrm{x} \in P \mid A_{\mathrm{sub}}\mathrm{x} = \mathrm{b}_{\mathrm{sub}}\}$ for a sub-matrix A_{sub} of A and a sub-vector $\mathrm{b}_{\mathrm{sub}}$ of b.

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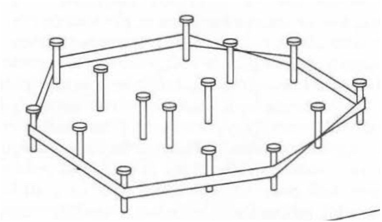
Definition

A vertex cone of P at v is the intersection of the half-spaces defining P and whose boundaries intersect at v.

Definition

The convex hull of a polyhedral set S is the set of all convex combination of S, given by,

$$conv(S) := \{\Sigma_{i=1}^{n} \lambda_i x_i \mid \Sigma_{i=1}^{n} \lambda_i = 1, \lambda_i \in [0, 1]\}.$$



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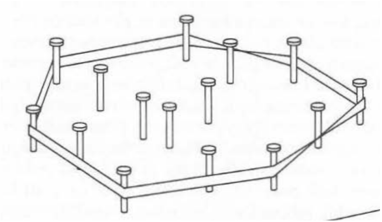

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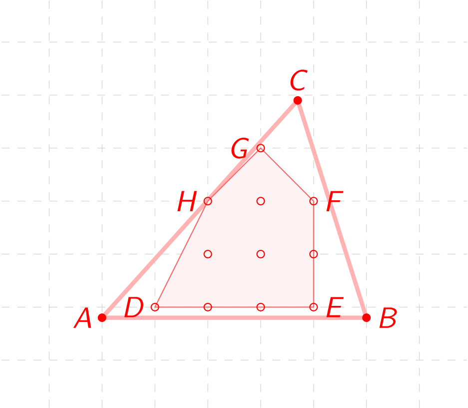
The $conv(S)$ is the smallest convex set containing S, i.e. it is the intersection of all convex sets containing S.

Definition

The integer hull P_I of a convex polyhedral set P is the convex hull of integer points of P.

Example



Background

1. The cutting plane method by Gomory [Go10] to solve integer linear programming (ILP),
   - ILP is solved by introducing constraints at each step until an integer solution is found.
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   - Seghir, Loechner and Meister [SLM12] developed a method for the parametric polytope case.

$\boxed{\text{Integer Hull Algorithm in Maple}}$

Three main steps of the algorithm:

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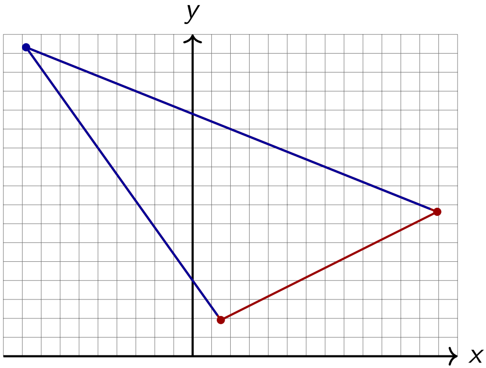
  1. either has integer points as vertices so that $R_I = R$,
  2. or any brute force algorithm can be applied to compute R_I.
- Merging: Once the integer hull of each part of the partition is computed and given by the list of its vertices, an algorithm for computing the convex hull of a set points, such as QuickHull, can be applied to deduce P_I.

Consider the triangle given by $(-\frac{44}{5}, \frac{408}{25}), (\frac{349}{27}, \frac{206}{27})$ and $(\frac{85}{57}, \frac{109}{57})$. We want to find the integer hull.

$$\begin{cases} 2x + 5y \leq & 64 \\ 7x + 5y \geq & 20 \\ 3x - 6y \leq & -7 \end{cases}$$



Normalization

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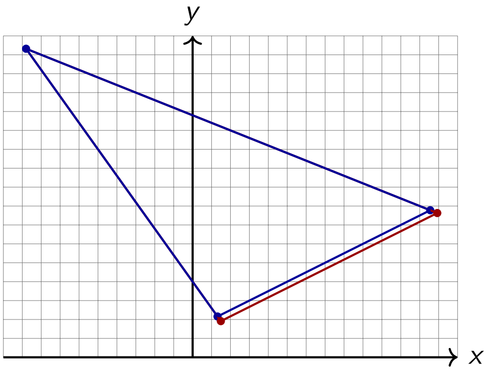
$$\downarrow$$

$$\begin{cases} 2x + 5y \leq & 64 \\ 7x + 5y \geq & 20 \\ 3x - 6y \leq & -9 \end{cases}$$



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$with(PolyhedralSets):$
$inset1 := PolyhedralSet([2\cdot x+5\cdot y \le 64, 7\cdot x+5\cdot y \ge 20, 3\cdot x-6\cdot y \le -9], [x, y]):$
$IntegerHull(inset1);$

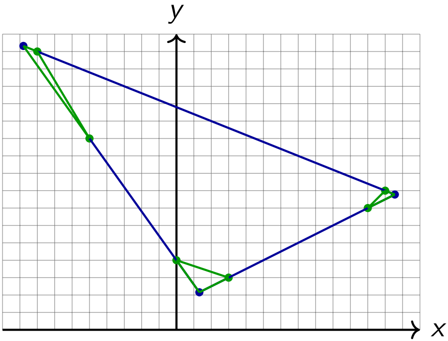
$$[[[12, 8], [-8, 16], [-7, 14], [-5, 11], [0, 4], [1, 3], [3, 3], [11, 7]], [\,]]$$

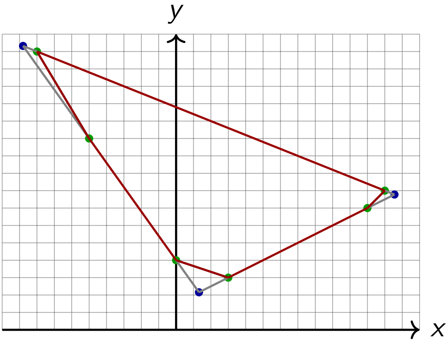
$inset2 := PolyhedralSet([2\cdot x+5\cdot y \le 64, 7\cdot x+5\cdot y \ge 20, 3\cdot x-6\cdot y \le -7], [x, y]):$
$IntegerHull(inset2);$

$$[[[12, 8], [-8, 16], [-7, 14], [-5, 11], [0, 4], [1, 3], [3, 3], [11, 7]], [\,]]$$

Partitioning

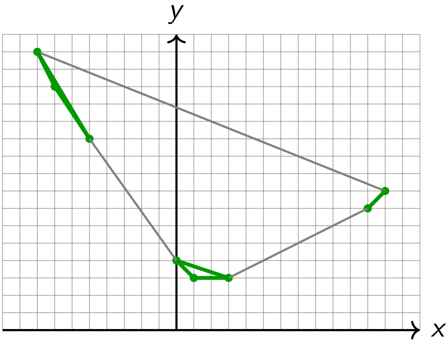
Find the closest integer points to each vertex on its adjacent facets. Apply brute force method to compute the integer hull of each corner.





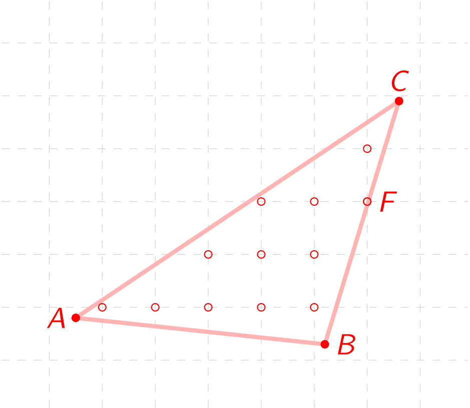
Merging

Compute the convex hull of all the integer hulls.



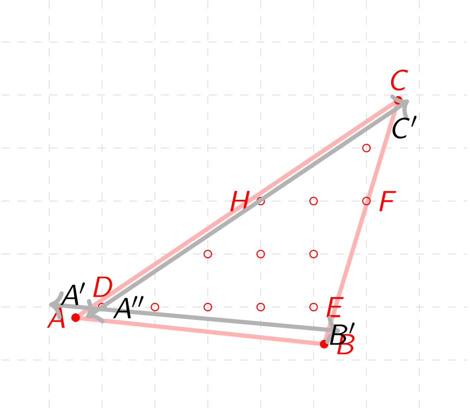
Example

Consider the $\triangle ABC$, where F is an integer point. We want to find the integer hull of $\triangle ABC$.



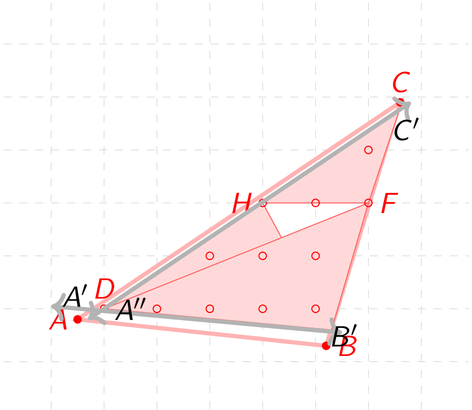
Example

To do so, we translate AB, AC till it touches D and H respectively. BC already has F. The new hyperplanes are $A'B'$ and $A''C'$ respectively.



Example

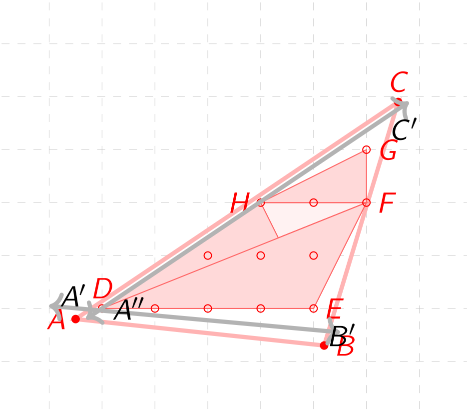
We apply brute force algorithm to compute integer hull of the corner.



The computational cost of brute force algorithm is inherently high, as such it is favorable to apply such a method only if the area is significantly small.

We apply `QuickHull` to all the integer hulls. The integer hull of $\triangle ABC$ is the pentagon $DEFGH$.

Example

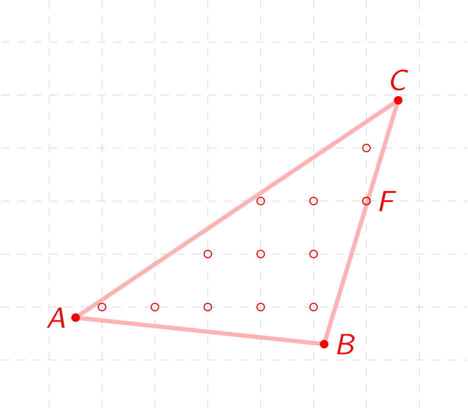


The compiler spends more time on the brute force algorithm than the "real" algorithm.

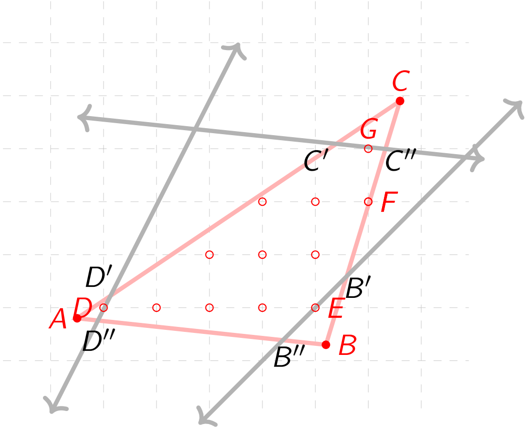
$\boxed{\text{New method}}$

Example

Consider the same $\triangle ABC$, where F is an integer point. We want to find the integer hull of $\triangle ABC$.



To do so, we translate AB, BC and AC to G, D and E respectively. The new hyperplanes are $C'C''D''D'$, and $B''B'$ respectively.



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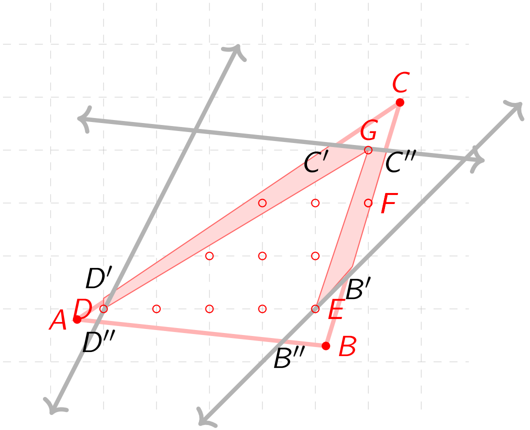

The area on which brute force algorithm is applied is significantly small.

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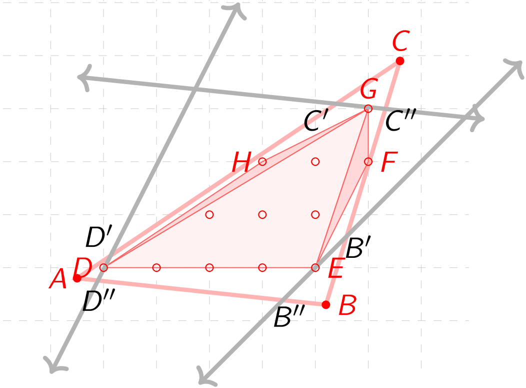

Preliminary Benchmark

The Maple version used is the 2024 release of Maple. All the benchmarks are done on an Intel Core i7-7700T with Clockspeed: 2.9 GHz and Turbo Speed: 3.8 GHz. It has 4 cores and 8 threads.

Preliminary Comparison Test			
Vertices	Volume	New Algo.	Existing Algo.
10	29.9	663.00ms	823.00ms
13	35.08	906.00ms	1.06ms
13	40.56	996.00ms	1.10s
12	40.63	898.00ms	928.00ms
1000	69829.26	855ms	952ms
15	263124.06	1.65s	1.73s
1000	6.54×10^6	1.98s	2.44s
1500	3.13×10^9	74.11s	89.72s

1. In this implementation we have only reduced the area on which the brute force algorithm is applied during partitioning to search for integer vertices in the corner.

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5. Brute force algorithm gets worse with increase in dimension. We expect to see imporvements in complexity for the general integer hull algorithm.

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