Reducing the Computation of Convex Hulls to Linear Programming

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Western Science

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Plan

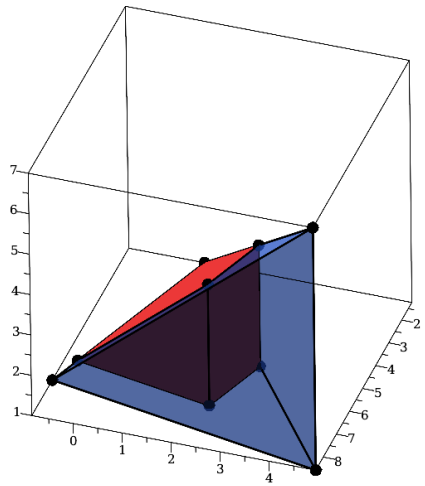
Overview

The wrapping procedure

Computing convex hulls

Plan

Motivating example



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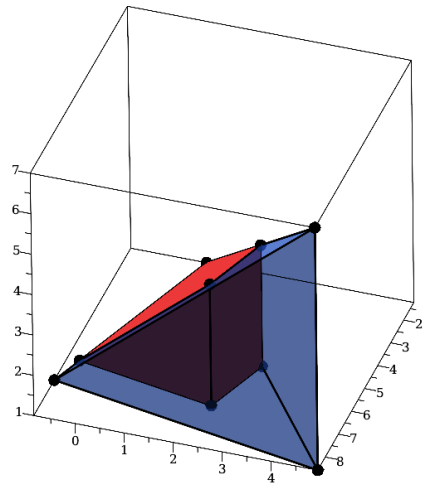

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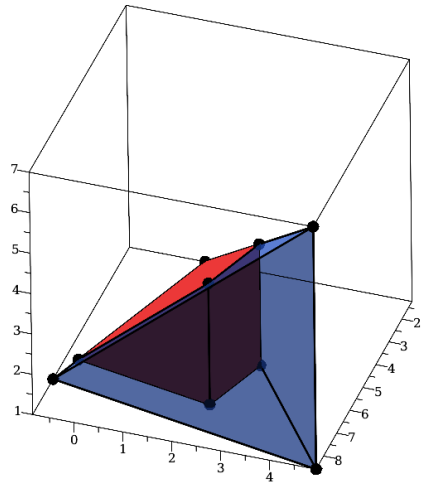

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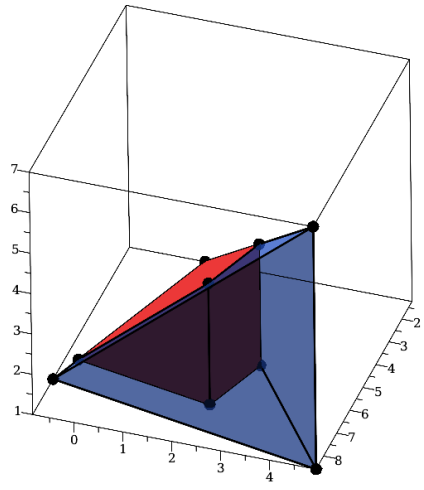

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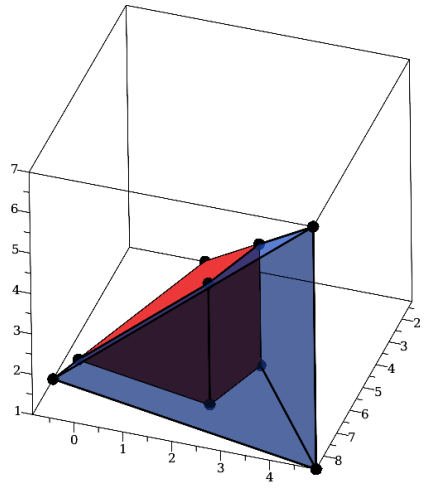

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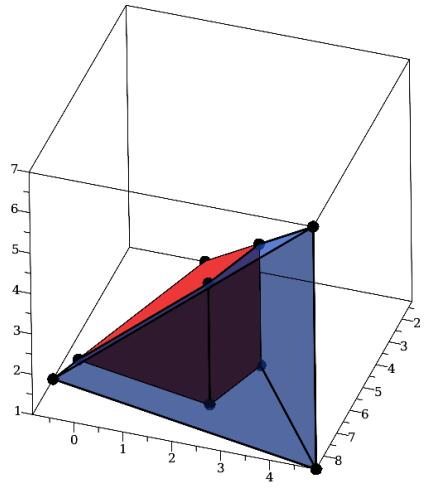

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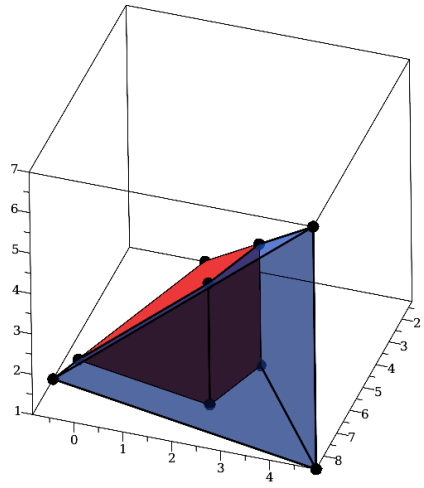

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- Our ultimate goal is to decide whether or not the union of finitely many $\mathbb{Z}$-polyhedral sets is a $\mathbb{Z}$-polyhedra set, extending Verdoolaege's 2D constructions in [21].

Computing convex hulls (I/II)

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- But if k input polyhedral sets $P_1, \ldots, P_k$ of $\mathbb{R}^d$ are given in H-representation rather than V-representation, reducing to the latter makes no sense since $|V| \in \mathcal{O}(m^{\lfloor d/2 \rfloor})$ where $m := |H|$, see [18].

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  - running in polynomial time in the size of the input and output.
- They rely on two core ideas:
  - the wrapping procedure of Chand and Kapur [7], and
  - the reverse search enumeration developed by Avis and Fukuda [2–4].

Our contribution and work in progress

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In what follows:

1. we start by illustrating a core procedure: the wrapping,
2. we explain the top-level procedure for convex hull,
3. we explain the wrapping procedure,
4. all our algorithms require LP in exact arithmetic (GMP),
5. experimentation is work in progress.

Plan

Recap on linear programming

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- As a result, if h is the bit size of Π, one can decide
  - whether P is empty (using $c = 0$) in time $\mathsf{LP}(d, h)$, and
  - in time $2\mathsf{LP}(d, h)$ whether $c^t x = 0$ supports P (using $x \mapsto c^t x$ and then $x \mapsto -c^t x$, as objective functions).

Valid facets and wrapping

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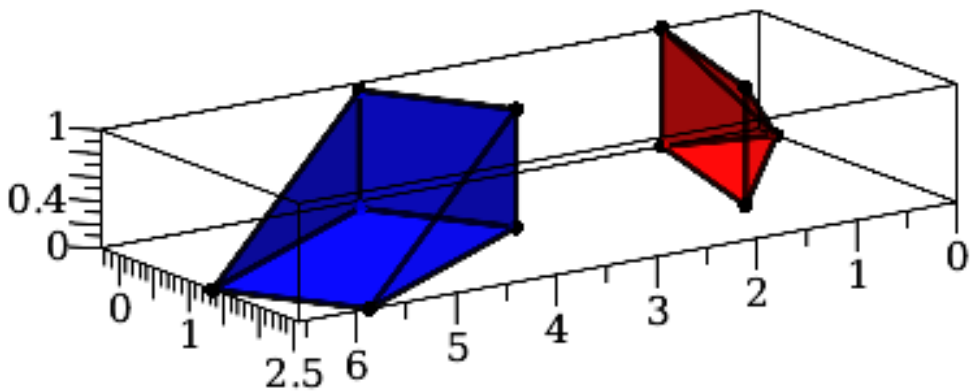
- How to wrap? The rotation angle can be obtained by solving an LP problem.

The wrapping procedure: example (I/V)



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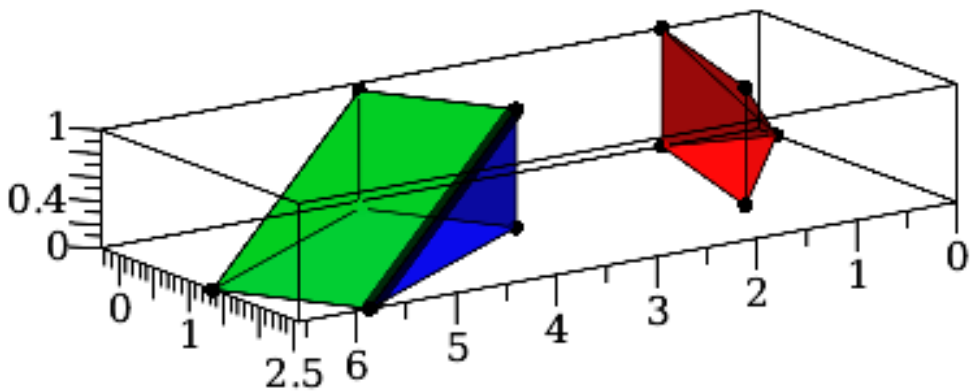

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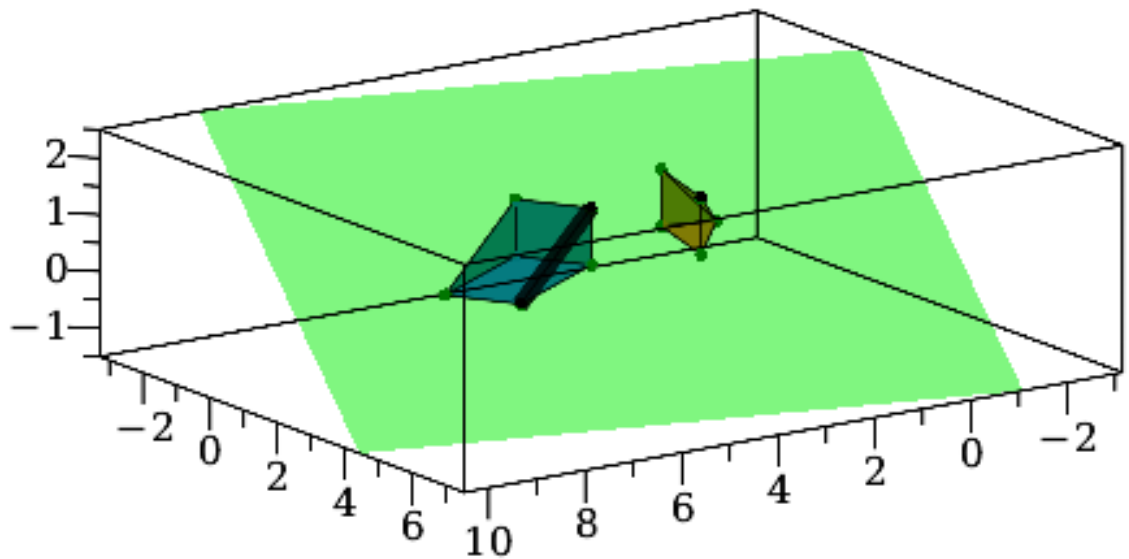

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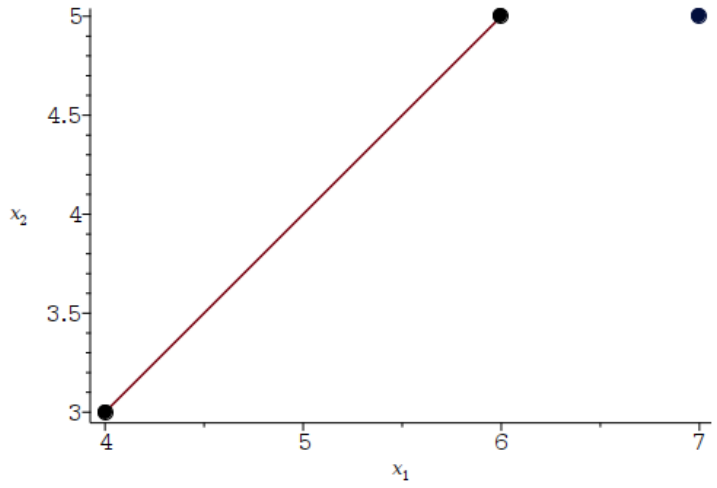

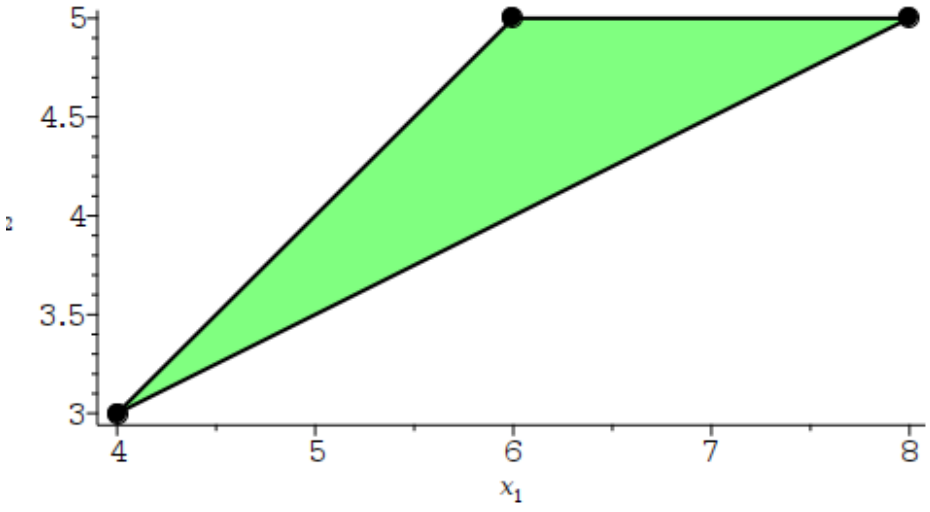

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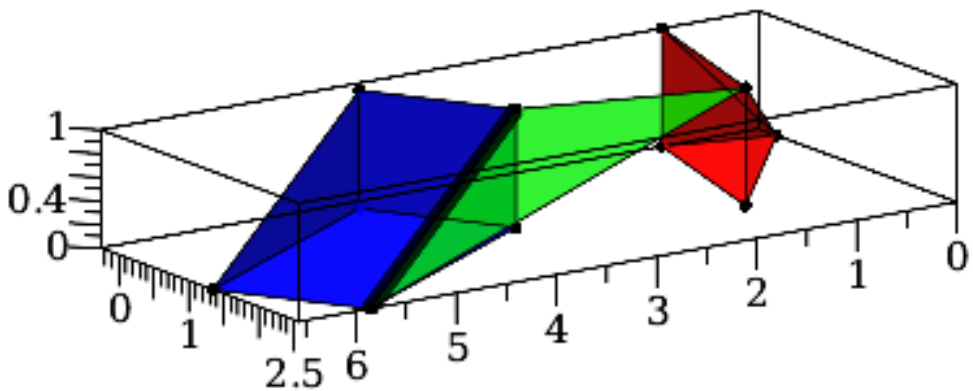

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- Once this rotation angle is known, the corresponding facet of P_0 is deduced by a recursive call to the convex hull procedure.
- This avoids the simplicial decomposition of [14] and relaxes the limitations of [9].

Plan

Specifications of our convex hull procedure

We make three assumptions:

1. P_0 is full-dimensional,
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The homogenization cone of P_0

- polyhedral sets P_i in H-rep: $A_i\vec{x_i} \leq \vec{b_i}$, $1 \leq i \leq k$.
- Assume these H-reps are minimal [15]
- C_i homogen. cone of P_i: $A_i\vec{x_i} \leq \vec{b_i}\lambda_i, \lambda_i \geq 0$, $1 \leq i \leq k$.
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Therefore, C_0 is given by the vectors $(\vec{x}, \lambda)$ satisfying the following system of Minkowski sums of C_i for $i = 1, \ldots, k$:

$$C_0 = \begin{cases} A_i^{\leq}\vec{x_i} \leq \vec{b_i}^{\leq}\lambda_i, & 1 \leq i \leq k \\ A_i^{=}\vec{x_i} = \vec{b_i}^{=}\lambda_i, & 1 \leq i \leq k \\ 0 \leq \lambda_i, & 1 \leq i \leq k \\ (\vec{x}, \lambda) = \sum_{i=1}^{k}(\vec{x_i}, \lambda_i). \end{cases} \tag{1}$$

where $A_i^{\leq}, \vec{b_i}^{\leq}$ (resp. $A_i^{=}, \vec{b_i}^{=}$) come from the inequalities (resp. equations) of P_i's H-rep.

When valid facets give rise to facets of P_0

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Lemma (2)

Fix $1 \leq i \leq k$. *Assume* P_i *is full-dimensional. Let* $1 \leq c \leq m_i$. *If the half-space* $A_{i,c}x \leq b_{i,c}$ *is valid for* $P_1, \ldots, P_k$, *then the facet of* P_i *corresponding to* $A_{i,c}x \leq b_{i,c}$ *is contained in a facet of* P_0.

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Lemma (3)

Assume, P_i *is of co-dimension* 1 *and let* $A_{i,1}x = b_{i,1}$ *be the equation of* $A_i\vec{x_i} \leq \vec{b_i}$. *If either* $A_{i,1}x \leq b_{i,1}$ *or* $-A_{i,1}x \leq -b_{i,1}$ *is valid for* $P_1, \ldots, P_k$, *then* P_i *is contained in a facet of* P_0.

Determining the initial ridges

- Assume P_i is full-dimensional
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Lemma (4)

Let ε be a new variable. Consider the linear program Π whose feasible region is given by

$$\begin{cases} A_{i,j}x &= b_{i,j} & (H_{i,j}) \\ A_{i,\ell}x &= b_{i,\ell} & (H_{i,\ell}) \\ A'_i x &\leq b'_i - (\varepsilon, \ldots, \varepsilon)^t, \end{cases}$$

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with ε as objective function to be maximized.
Then, $H_{i,j} \cap H_{i,\ell} \cap P_i$ is a ridge of P_i if and only if Π's feasible region is not empty and the maximum value of Π's objective function is strictly positive.

High-level of view of our convex hull algorithm

- The algorithm manages two data-structures:
  1. a set $\mathcal{F}$, under construction, of half-spaces S valid for $P_1, \ldots, P_k$ so that a facet of P_0 has S as supporting hyperplane; $\mathcal{F}$ is initialized with Lemmas (2) and (3),

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  4. using T and F'' we update $\mathcal{F}$ and $\mathcal{R}$ so as to maintain their invariants.

How the wrapping procedure yields a new facet p_0

- Recall that applying the wrapping to a ridge R,
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Lemma (5)

We have
$H \cap \mathsf{ConvexHull}(P_1, \ldots, P_k) = \mathsf{ConvexHull}(P_1', \ldots, P_k')$.

Therefore, this recursive call to the convex hull algorithm returns the desired facet F''.

Wrapping without Projection

- The procedure input consists of
  - H_1: $a_1^t x - b_2 \leq 0$ a half space containing $P_1, \ldots, P_k$,
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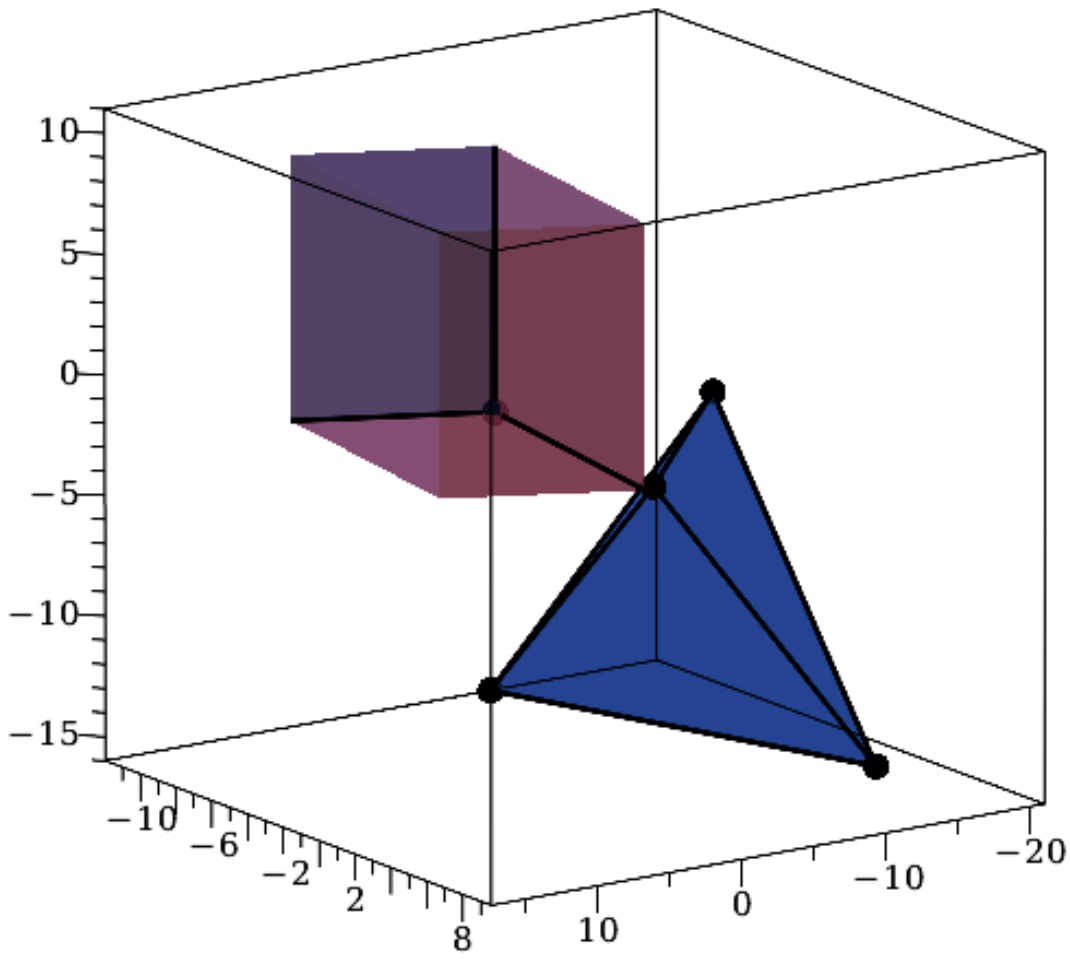
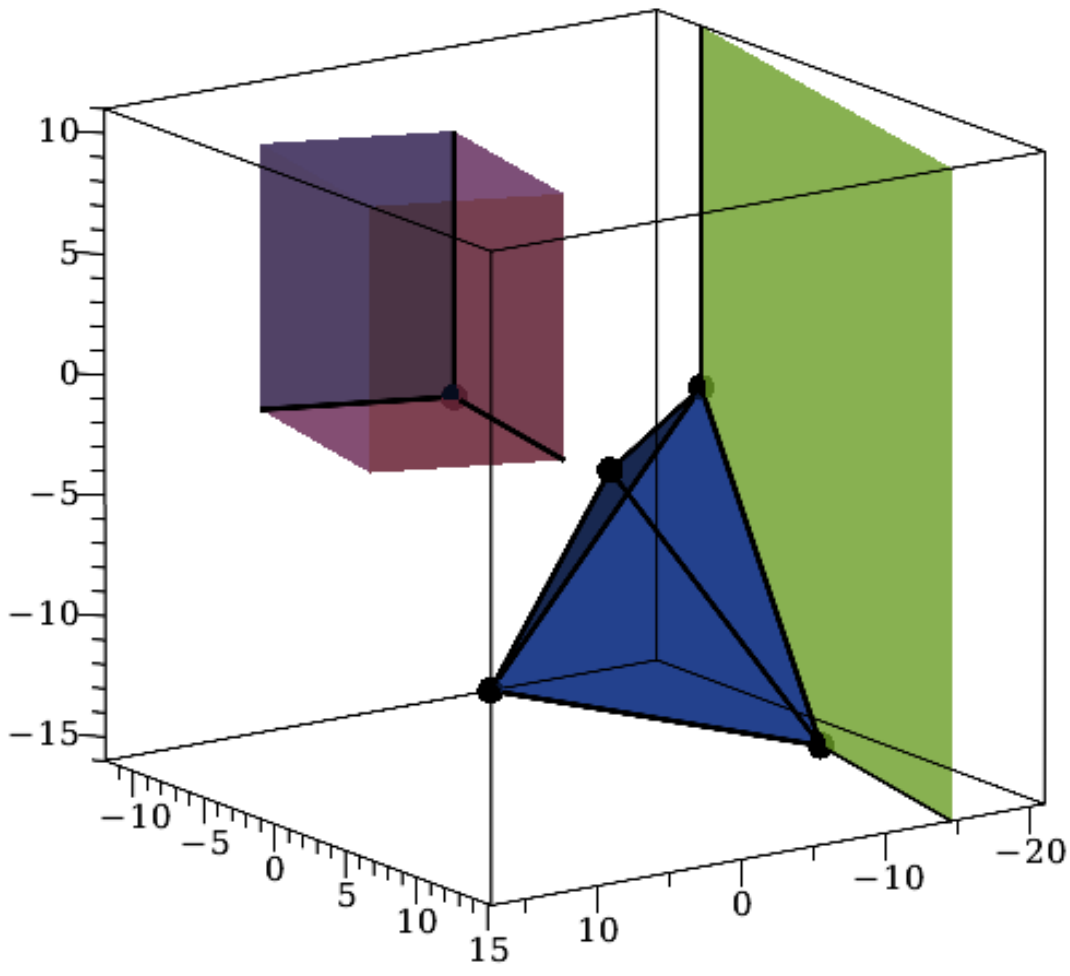
- Hence, we search for H_3 in the form $H_3 = H_1 + kH_2$, with k real.
- Applying the Charnes-Cooper transformation, the desired k is obtained as

$$\begin{aligned} &\min -(a_1^t x - b_1 \lambda) \\ &\text{Subject to: Conditions 1} \\ &a_2^t x - b_2 \lambda = 1. \end{aligned} \tag{2}$$

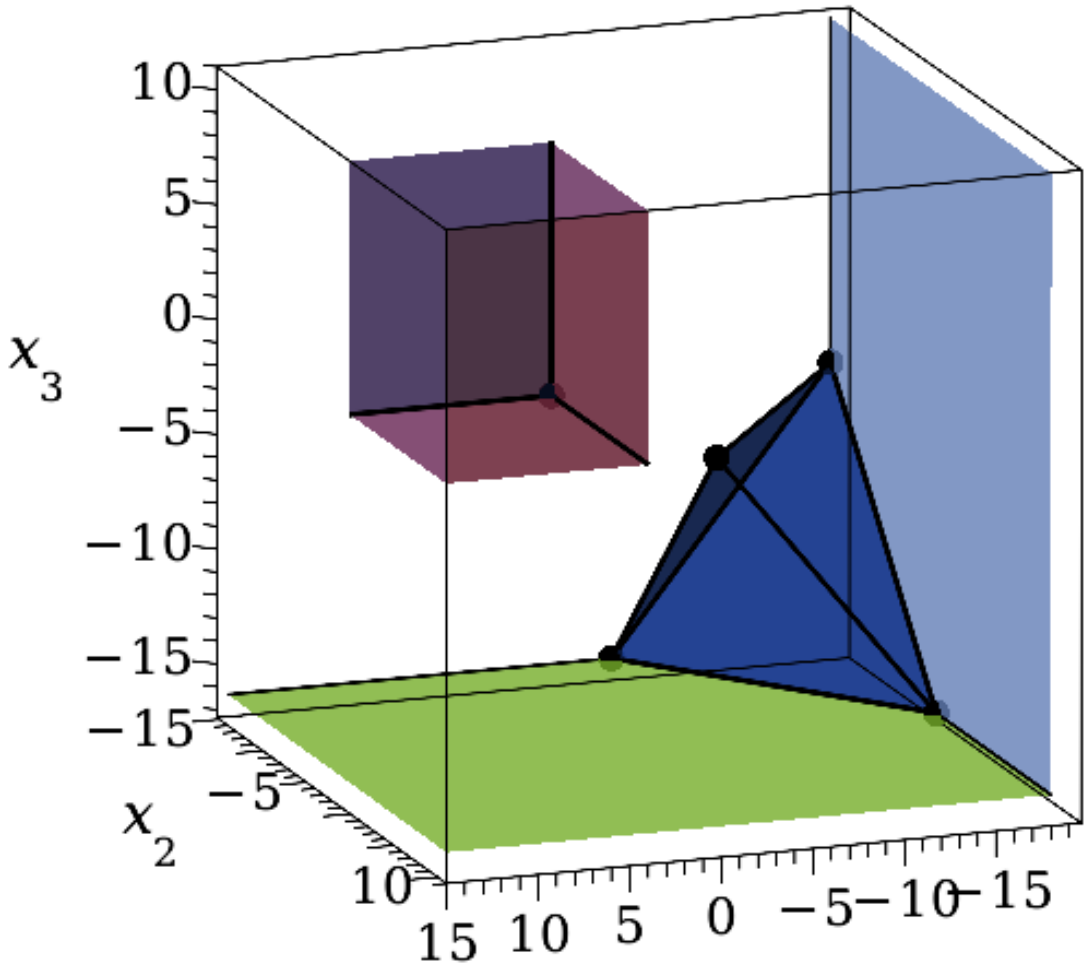
The extended convex hull (I/V)



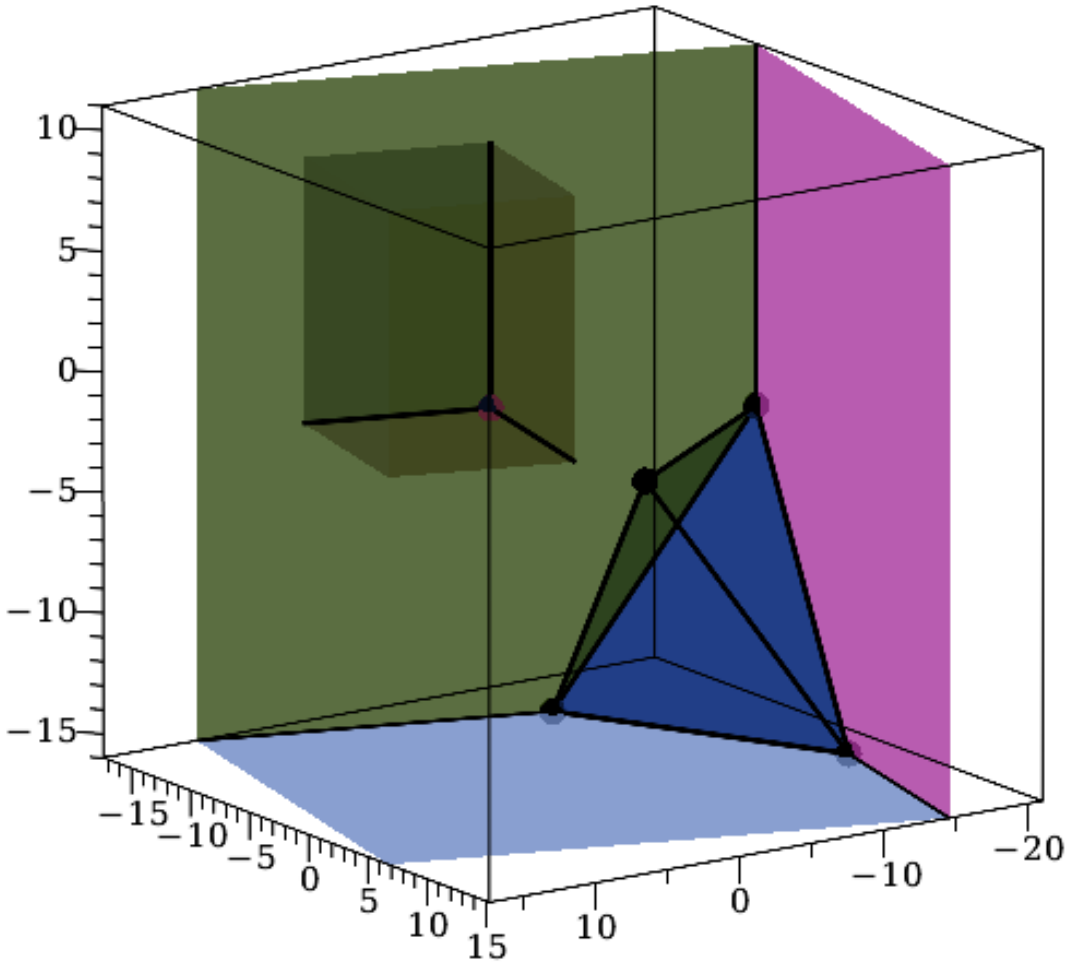
The extended convex hull (II/V)



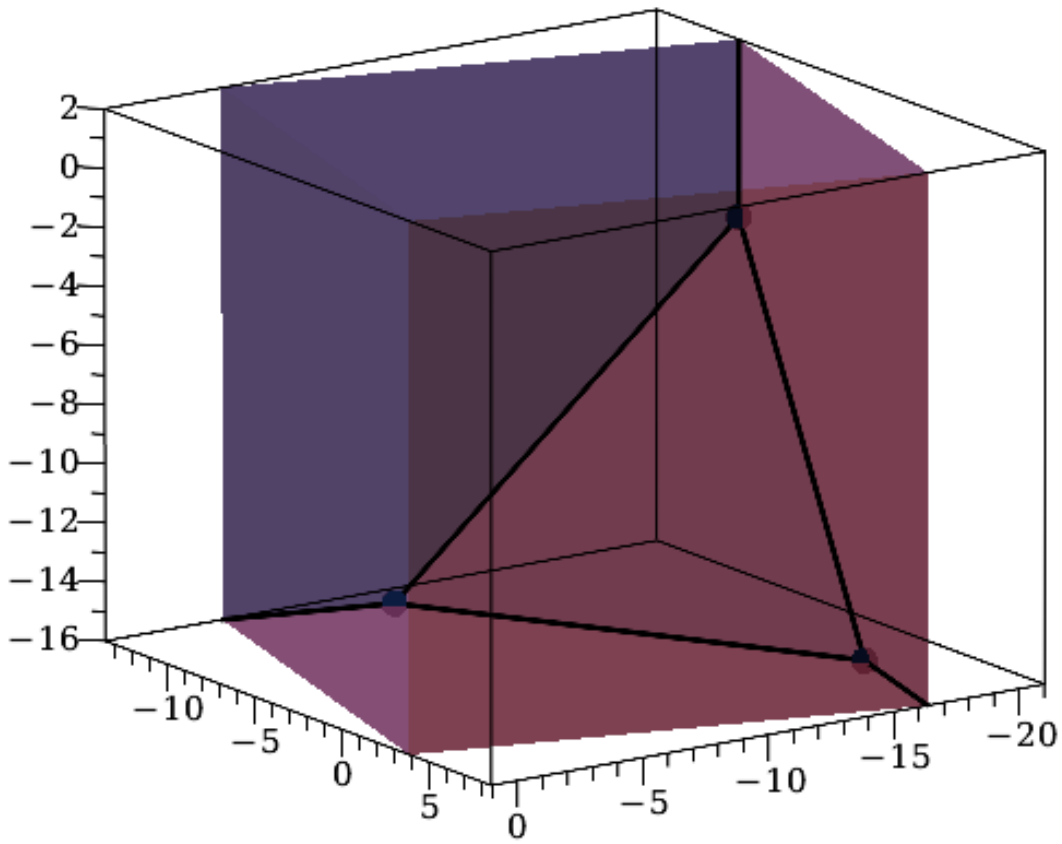
The extended convex hull (III/V)



The extended convex hull (IV/V)



The extended convex hull (V/V)



References I

[1] J. Antony, M. K. Mukundan, M. Thomas, and R. Muthuganapathy. "Convex Hull Computation in a Grid Space: A GPU Accelerated Parallel Filtering Approach". In: *Pacific Graphics Conference Papers and Posters*. Ed. by R. Chen, T. Ritschel, and E. Whiting. The Eurographics Association, 2024. ISBN: 978-3-03868-250-9.

[2] D. Avis and K. Fukuda. "A basis enumeration algorithm for linear systems with geometric applications". In: *Applied Mathematics Letters* 4.5 (1991), pp. 39–42.

[3] D. Avis and K. Fukuda. "A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra". In: *Proceedings of the seventh annual symposium on Computational geometry*. 1991, pp. 98–104.

References II

[4] D. Avis and K. Fukuda. "Reverse search for enumeration". In: *Discrete applied mathematics* 65.1-3 (1996), pp. 21–46.

[5] E. Bernardi, S. Masi, P. Xu, and P. Bonnifait. "High Integrity Lane-level Occupancy Estimation of Road Obstacles Through LiDAR and HD Map Data Fusion". In: *IEEE Intelligent Vehicles Symposium*. IEEE, 2020, pp. 1873–1878.

[6] T. M. Chan. "Optimal output-sensitive convex hull algorithms in two and three dimensions". In: *Discrete & computational geometry* 16.4 (1996), pp. 361–368.

[7] D. R. Chand and S. S. Kapur. "An algorithm for convex polytopes". In: *Journal of the ACM (JACM)* 17.1 (1970), pp. 78–86.

References III

[8] N. Ding. "An Efficient Convex Hull-Based Vehicle Pose Estimation Method for 3D LiDAR". In: *CoRR* abs/2302.01034 (2023). arXiv: 2302.01034.

[9] K. Fukuda, T. M. Liebling, and C. Lütolf. "Extended convex hull". In: *Computational Geometry* 20.1-2 (2001), pp. 13–23.

[10] M. Gouda, Z. Pawliuk, J. Mirza, and K. El-Basyouny. "Using convex hulls with octree/voxel representations of point clouds to assess road and roadside geometric design for automated vehicles". In: *Automation in Construction* 154 (2023), p. 104967. ISSN: 0926-5805.

[11] M. Greer. *Memory Optimization for Convex Hull Support Point Queries*. 2025. arXiv: 2509.03753 [cs.GR]. URL: https://arxiv.org/abs/2509.03753.

References IV

[12] Z. Guo, C. Liu, X. Zhang, J. Jiao, X. Ji, and Q. Ye. "Beyond Bounding-Box: Convex-Hull Feature Adaptation for Oriented and Densely Packed Object Detection". In: *IEEE Conference on Computer Vision and Pattern Recognition, CVPR 2021, virtual, June 19-25, 2021*. Computer Vision Foundation / IEEE, 2021, pp. 8792–8801.

[13] M. A. Jayaram and H. Fleyeh. "Convex Hulls in Image Processing: A Scoping Review". In: *American Journal of Intelligent Systems* 6.2 (2016), pp. 48–58.

[14] R. Jing, Y. Lei, M. Moreno Maza, and C. Mukherjee. "On the Computation of Extended Convex Hulls". In: *Computer Algebra in Scientific Computing*. Cham: Springer Nature Switzerland, 2027, pp. 189–209. ISBN: 978-3-032-34586-8.

References V

[15] R. Jing, M. Moreno Maza, C. Mukherjee, Y. Xie, and C. Yuan. "Efficient detection of redundancies in systems of linear inequalities". In: *J. Symb. Comput.* 135 (2026), p. 102527.

[16] N. Karmarkar. "A new polynomial-time algorithm for linear programming". In: *Proceedings of the sixteenth annual ACM symposium on Theory of computing*. STOC '84. New York, NY, USA: ACM, 1984, pp. 302–311. ISBN: 0-89791-133-4. DOI: 10.1145/800057.808695. URL: http://doi.acm.org/10.1145/800057.808695.

[17] D. G. Kirkpatrick and R. Seidel. "The ultimate planar convex hull algorithm?" In: *SIAM journal on computing* 15.1 (1986), pp. 287–299.

References VI

[18] P. McMullen. "On the upper-bound conjecture for convex polytopes". In: *Journal of Combinatorial Theory, Series B* 10.3 (1971), pp. 187–200.

[19] S. Meeran and A. Share. "Optimum path planning using convex hull and local search heuristic algorithms". In: *Mechatronics* 7.8 (1997), pp. 737–756. ISSN: 0957-4158.

[20] J. Schulman, Y. Duan, J. Ho, A. X. Lee, I. Awwal, H. Bradlow, J. Pan, S. Patil, K. Goldberg, and P. Abbeel. "Motion planning with sequential convex optimization and convex collision checking". In: *Int. J. Robotics Res.* 33.9 (2014), pp. 1251–1270.

[21] S. Verdoolaege. "Integer set coalescing". In: *International Workshop on Polyhedral Compilation Techniques, Date: 2015/01/19-2015/01/19, Location: Amsterdam, The Netherlands*. 2015.