Quasi-Polynomials, Z-Polyhedra and Parametric Integer Linear Programming

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July 21, 2026

Acknowledgements

- This talk is based on research projects in which **many former and current ORCCA students** have played an essential role. By alphabetic order: Xiaohui Chen, Rui-Juan Jing, Yuzhuo Lei, Christopher Maligec, Chirantan Mukherjee, Delaram Talaashrafi, Linxiao Wang, Ning Xie, Rong Xiao and Haoze Yuan.

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- Most of the algorithms presented in this software demo are **available in** MAPLE**'s** `PolyhedralSets` library.

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  - **compiler optimization** in particular, the analysis, transformation, and scheduling of nested loops in computer programs.

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- All libraries and commands are MAPLE code, except `FME_SatMat` which is is written in C/C++ in support of efficiency critical routines.

Plan

1. Overview
2. Integer hulls of polyhedra
3. Integer point counting for parametric polyhedra
4. Parametric Integer Linear Programming
5. Array Bound Checking via PILP
6. Concluding remarks

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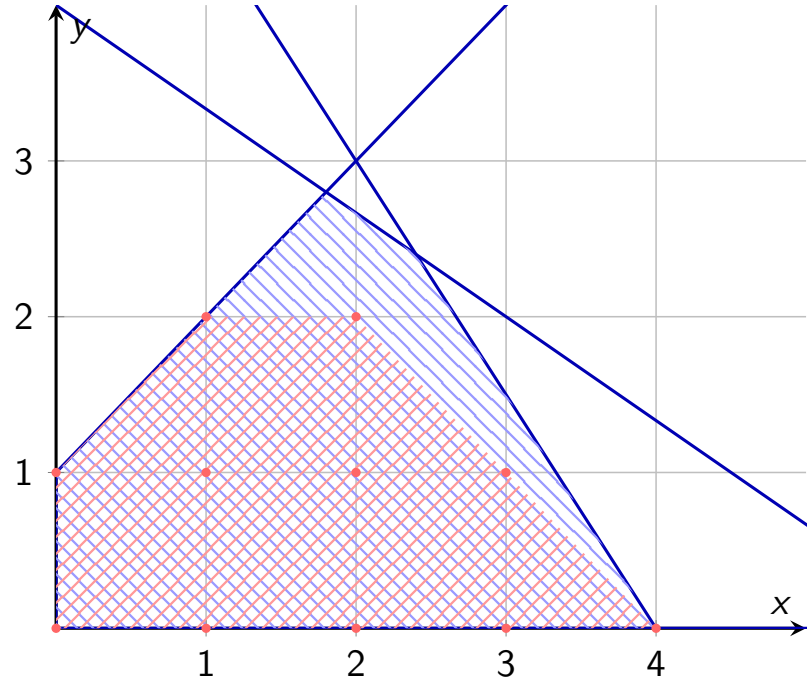
$$\begin{cases} 0 \leq x \\ 0 \leq y \\ 3x + 2y \leq 12 \\ 2x + 3y \leq 12 \\ -x + y \leq 1 \end{cases}$$

$$\begin{cases} 0 \leq x \\ 0 \leq y \\ y \leq 2 \\ x + y \leq 4 \\ -x + y \leq 1 \end{cases}$$



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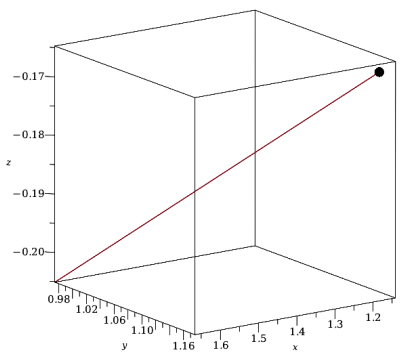
Input system:

$$\begin{cases} 7x + 12y + 31z &= 17, \\ 3x + 5y + 14z &= 7, \\ x + y &\geq 0, \\ y - x &\leq 0. \end{cases}$$



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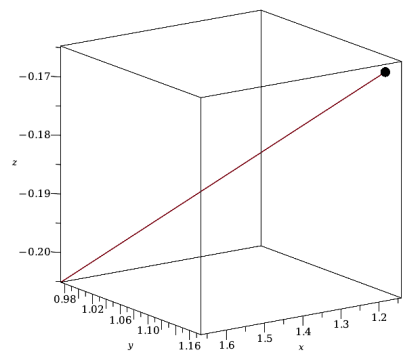

Equivalent Z-Polyhedron:

$$\begin{cases} x &= -13z - 1, \\ y &= 5z + 2, \\ z &\leq -1. \end{cases} \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -13 \\ 5 \\ 1 \end{pmatrix}\mathbf{t} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \in \text{Lattice}\left(\begin{pmatrix} -13 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\right)$$

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3. The algorithm IntegerPointDecomposition [7] decomposes any $\mathbb{Z}$-polyhedron into **normalized** $\mathbb{Z}$-polyhedra.

Plan

Generating functions of polyedra

- Consider a polyhedral set $P \subseteq \mathbb{Q}^d$ and map each integer point $\mathbf{e} = (e_1, \ldots, e_d)$ of P to the monomial $\mathbf{x}^{\mathbf{e}} = {x_1}^{e_1} \cdots {x_d}^{e_d}$

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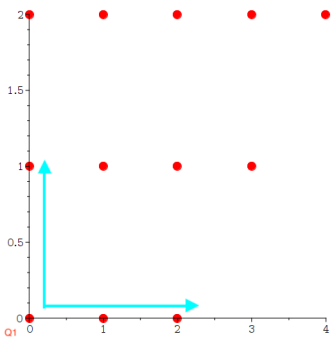

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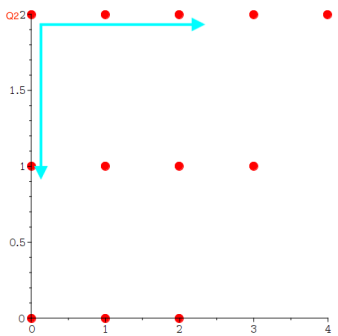

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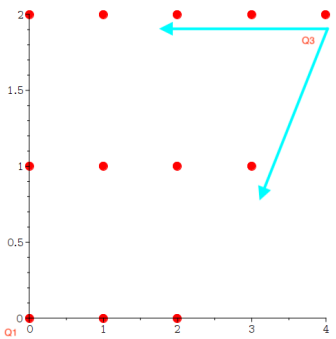

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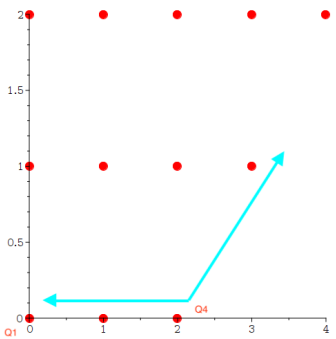

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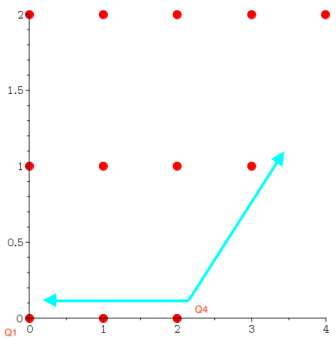

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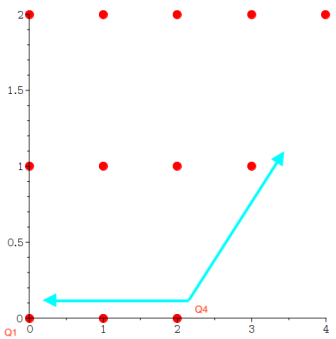

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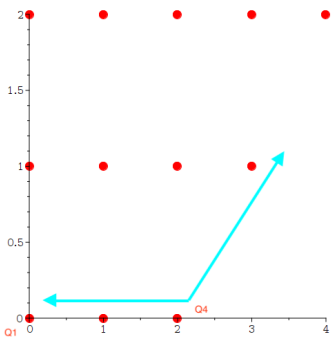

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What is Integer Linear Programming?

$$\max(\textit{or}\ \min)\ \mathbf{c}^\top \mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}$$

where $\mathbf{x} \in \mathbb{Z}^d$, $A \in \mathbb{R}^{m \times d}$ is a matrix and $\mathbf{b} \in \mathbb{R}^m$ is a vector.

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$$\max_{x,y \in \mathbb{Z}} \ y$$
$$-x + y \le 1$$
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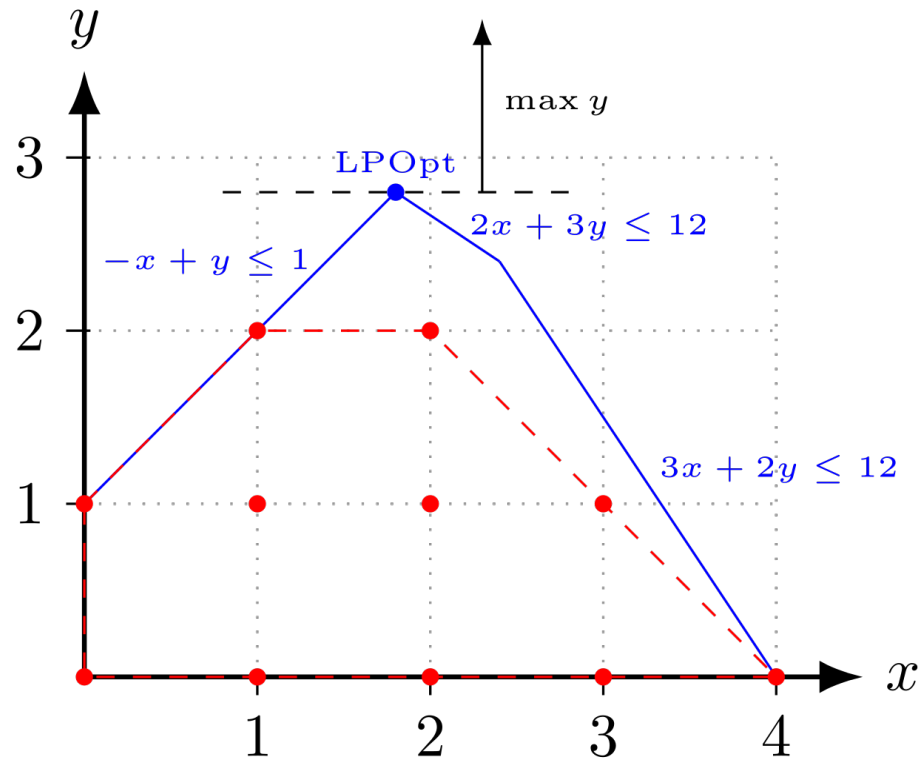

Figure: https://commons.wikimedia.org/w/index.php?curid=79967308

Parametric Integer Linear Programming (PILP)

Given a *parametric* $\mathbb{Z}$*-polyhedron*

$$\mathcal{P}(\mathbf{y}) = \{\mathbf{x} \in \mathbb{Z}^m \mid M\mathbf{x} \leq N\mathbf{y} + \mathbf{p}, \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \in L\}$$

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Goal:

We want to compute two functions:

$$\begin{aligned} \text{Max}&: \mathbb{Z}^n \to \mathbb{Z} \\ &\mathbf{y} \mapsto \max\{g(\mathbf{x},\mathbf{y}) \mid \mathbf{x} \in \mathcal{P}(\mathbf{y})\} \end{aligned}$$

and

$$\begin{aligned} X_{\max}&: \mathbb{Z}^n \to 2^{\mathbb{Z}^n} \\ &\mathbf{y} \mapsto \{\mathbf{x} \mid g(\mathbf{x},\mathbf{y}) = \text{Max}(\mathbf{y}),\ \mathbf{x} \in \mathcal{P}(\mathbf{y})\} \end{aligned}$$

Reduction to *IntegerPointDecomposition* (*IPD*)

Key idea: introduce an auxiliary variable z and consider

$$F(z, \mathbf{y}) \equiv \exists\, \mathbf{x} \in \mathbb{Z}^m : z = g(\mathbf{x}, \mathbf{y}) \wedge \mathbf{x} \in \mathcal{P}(\mathbf{y})$$

Then $\mathrm{Max}(\mathbf{y}) = \max\{z \in \mathbb{Z} \mid F(z, \mathbf{y})\}$.

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Key idea: introduce an auxiliary variable z and consider

$$F(z, \mathbf{y}) \equiv \exists\, \mathbf{x} \in \mathbb{Z}^m : z = g(\mathbf{x}, \mathbf{y}) \wedge \mathbf{x} \in \mathcal{P}(\mathbf{y})$$

Then $\mathrm{Max}(\mathbf{y}) = \max\{z \in \mathbb{Z} \mid F(z, \mathbf{y})\}$.

Algorithm outline:

1. Apply *IPD* to $\begin{pmatrix} \mathbf{x} \\ z \\ \mathbf{y} \end{pmatrix} \in \bigvee_{i=1}^{e} \mathbb{Z}\mathrm{Polyhedron}(P_i, L_i)$ with variable ordering $x_1 > \cdots > x_m > z > y_1 > \cdots > y_n$
2. Obtain normalized $\mathbb{Z}\mathrm{Polyhedron}(P_i, L_i)$ such that each P_i has *at most one upper bound for* z
3. For each $\mathbb{Z}\mathrm{Polyhedron}(P_i, L_i)$, extract $\mathrm{Max}(\mathbf{y})$ as a piecewise quasi-polynomial

Reducing to One Upper Bound for z

After IPD, each normalized $\mathbb{Z}$Polyhedron(P_i, L_i) may have *several* upper bounds for z: $b_1 z \leq A_1(\mathbf{y})$ and $b_2 z \leq A_2(\mathbf{y})$.

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The idea: Let $l = \text{lcm}(b_1, b_2)$, $c_1 = l/b_1$, $c_2 = l/b_2$. The two upper bounds together say:

$$lz \le c_1 A_1(\mathbf{y}) \quad \text{and} \quad lz \le c_2 A_2(\mathbf{y})$$

This is equivalent to a *case split* on $\mathbf{y}$:

- **Region Δ_{12}:** $c_1 A_1(\mathbf{y}) \le c_2 A_2(\mathbf{y})$ where only the bound $b_1 z \le A_1(\mathbf{y})$ is active
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The process repeats until all pieces have at most one upper bound for z.

Extracting Max from Each Piece

Once every $\mathbb{Z}\text{Polyhedron}(P_i, L_i)$ has at most one upper bound for z, reading off $\text{Max}(\mathbf{y})$ is straightforward.

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The idea: Because the $\mathbb{Z}$Polyhedron is *normalized*, the lattice L_i expresses z as a linear combination of a free integer parameter t and the $\mathbf{y}$-variables:

$$z = q_i \cdot t + f_i(\mathbf{y}), \quad t \in \mathbb{Z}, \quad q_i \in \mathbb{Q}$$

To maximize z, we simply maximize t, which is bounded above by the single upper bound $b_i z \leq A_i(\mathbf{y})$:

$$t \leq \left\lfloor \frac{A_i(\mathbf{y}) - b_i f_i(\mathbf{y})}{b_i q_i} \right\rfloor$$

Plugging the maximum t back in gives

$$\text{Max}(\mathbf{y}) = q_i \left\lfloor \frac{A_i(\mathbf{y}) - b_i f_i(\mathbf{y})}{b_i q_i} \right\rfloor + f_i(\mathbf{y})$$

a *piecewise quasi-polynomial* in $\mathbf{y}$, valid over the region of P_i.

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Edge cases: If no upper bound exists on z, then $\text{Max}(\mathbf{y}) = +\infty$. If $q_i = 0$, then z is constant and $\text{Max}(\mathbf{y}) = f_i(\mathbf{y})$ directly.

Two Strategies for Finding the Optimizer

Given Max $= z^*$, they differ in *how* the optimizer $(x_1^*, \ldots, x_m^*)$ is recovered.

Sample Point Method

1. Set $z = z^*$
2. **Pick a concrete integer point** from the resulting feasible slice.
3. Return that point as $(x_1^*, \ldots, x_m^*)$.

Key idea: existence of a solution is guaranteed by normalization; any witness suffices.
Drawback: may require solving an auxiliary ILP to find the witness, which can be expensive on hard instances.

Locus Method

1. Set $z = z^*$
2. **Back-substitute** through the polyhedron structure to express each x_i^* as a closed-form function of the parameters $p_0, p_1, \ldots$
3. Return the full parametric optimizer locus.

Key idea: normalization of IPD output enables direct symbolic back-substitution without search.
Advantage: avoids auxiliary ILP; slightly faster in practice.

The Three PILP Implementations

For each parametric $\mathbb{Z}$-polyhedron $\mathcal{P}(\mathbf{y})$, we solve both $\min_{\mathbf{x}\in\mathcal{P}(\mathbf{y})} g(\mathbf{x})$ and $\max_{\mathbf{x}\in\mathcal{P}(\mathbf{y})} g(\mathbf{x})$, where $g(\mathbf{x}) = x_1 + \cdots + x_m$.

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3. **IntegerPointDecomposition [6]**
   - Run *IPD* once with $z = g(\mathbf{x})$ introduced
   - *Both* min and max extracted from the same $\mathbb{Z}$-polyhedron decomposition
   - Implemented in `Maple` *QuantifierEliminationOverZ* [4, 5]

Test	Feautrier (C)	Barvinok (C)	Sample Point (Maple)	Locus (Maple)
boulet	0.032	0.021	2.514	2.493
bouleti	0.039	error	13.069	12.795
cg1	0.024	0.027	1.205	1.167
cnt_sum2	1.114	not supported	>30 mins	>30 mins
difficult	0.079	0.555	1136.988	1133.288
esced	0.051	0.026	394.609	390.996
ex	0.027	0.030	0.900	0.900
ex2	0.027	0.030	0.907	0.900
fimmel	0.038	0.038	0.644	0.640
jcomplex	583.280	not supported	>30 mins	>30 mins
max	0.026	0.030	1.437	1.397
negative	0.021	0.028	0.521	0.521
phideo	0.764	>30 mins	>30 mins	>30 mins
seghir–e1	0.068	error	1278.216	1265.387
seghir–e3	0.031	0.067	73.585	73.353
seghir–e4	0.045	11.675	>30 mins	>30 mins
seghir–e5	0.147	2.533	>30 mins	>30 mins
seghir–e6	0.045	0.366	447.975	445.734
seghir–e7	0.029	0.111	55.668	55.795
seghir–e8	0.031	0.075	925.595	903.961
seghir–e9	0.332	>30 mins	>30 mins	>30 mins
small	0.018	0.026	1.209 (ILP)	1.200 (ILP)
sor1d	0.024	0.022	0.455	0.453
square	0.027	0.029	0.916	0.958
sven	0.024	0.019	0.228	0.278
tobi	0.026	0.025	45.504	45.355

Array Bound Checking: Setup & F_2 Analysis

Input Setup

Triply-Nested Loop

for $(i_1 = 0;\ i_1 \le R_1;\ i_1{+}{+})$
  for $(i_2 = 0;\ i_2 \le R_2 - i_1;\ i_2{+}{+})$
    for $(i_3 = 0;\ i_3 \le R_3 - i_1;\ i_3{+}{+})$

$\tilde{A}[i_1 + i_2 + 1][i_1 + 2i_2 + 1][2i_1 + i_3] \leftarrow \ldots$

- **Domain ($\mathcal{D}$):** $\triangle$-prism
- $(i_1, i_2, i_3) \in \mathbb{Z}^3$
- $0 \le i_1 \le R_1$
- $0 \le i_2 \le R_2 - i_1$
- $0 \le i_3 \le R_3 - i_1$
- **Reference Functions:**

$$f_2(i_1, i_2, i_3) = i_1 + 2i_2 + 1$$
$$f_3(i_1, i_2, i_3) = 2i_1 + i_3$$

Formulation & Solving

Safety Condition via PILP

$$\underbrace{\max_{(i_1,i_2,i_3)\in\mathcal{D}} f_2}_{F_2} \le M_2 \wedge \underbrace{\max_{(i_1,i_2,i_3)\in\mathcal{D}} f_3}_{F_3} \le M_3$$

Exact Piecewise Maxima (F_2)

$$F_2 = \begin{cases} 2R_2 + 1 & \text{if } R_2 \ge 1 \\ 1 & \text{if } R_1 = 0, R_2 = 0 \end{cases}$$

- Max achieved at $i_1 = 0$ (due to $i_2 \le R_2 - i_1$).
- **Naive cuboid:** $R_1 + 2R_2 + 1$ (over-conservative).

F_3 Analysis & Final Safety Checks

Input Setup

Triply-Nested Loop

for $(i_1 = 0;\ i_1 \le R_1;\ i_1{+}{+})$
  for $(i_2 = 0;\ i_2 \le R_2 - i_1;\ i_2{+}{+})$
    for $(i_3 = 0;\ i_3 \le R_3 - i_1;\ i_3{+}{+})$

$\tilde{A}[i_1 + i_2 + 1][i_1 + 2i_2 + 1][2i_1 + i_3] \leftarrow \dots$

- **Domain ($\mathcal{D}$):** $\triangle$-prism
- $(i_1, i_2, i_3) \in \mathbb{Z}^3$
- $0 \le i_1 \le R_1$
- $0 \le i_2 \le R_2 - i_1$
- $0 \le i_3 \le R_3 - i_1$
- **Reference Functions:**

$$f_2(i_1, i_2, i_3) = i_1 + 2i_2 + 1$$
$$f_3(i_1, i_2, i_3) = 2i_1 + i_3$$

Key Trade-off

Increasing i_1 increases the $2i_1$ term but reduces the available range for i_3. Optimal boundary depends on which constraint binds.

Formulation & Solving

Exact Piecewise Maxima (F_3)

$$F_3 = \begin{cases} R_1 + R_3 & \text{if } R_1 \le R_3 \\ 2R_3 & \text{if } R_1 > R_3 \end{cases}$$

Final Runtime Conditions

$$2R_2 + 1 \le M_2$$
and
$$\begin{cases} R_1 + R_3 \le M_3 & \text{if } R_1 \le R_3 \\ 2R_3 \le M_3 & \text{if } R_1 > R_3 \end{cases}$$

Concluding remarks

Summary and notes

1. We have presented software libraries for computing integer hulls and $\mathbb{Z}$-polyhedra.
2. They support integer point counting of parametric polyhedra (= computing Ehrhart polynomials) as well as Presburger arithmetic (= quantifier elimination over the integers).
3. One can replace this QE problems with a number of problem instances from parametric integer linear programming (PILP).
4. The algorithm IntegerPointDecomposition plays an essential role.
5. Most of these functionalities are available in Maple 2025, except QE and PILP.

References

[1] A. Barvinok and K. Woods. "Short Rational Generating Functions for Lattice Point Problems". In: Journal of the American Mathematical Society 16.4 (2003), pp. 957–979.

[2] M. Brion. "Points entiers dans les polyedres convexes". In: Annales scientifiques de l'École normale supérieure. Vol. 21. 4. 1988, pp. 653–663.

[3] P. Feautrier. "Parametric integer programming". In: RAIRO. Recherche opérationnelle 22.3 (1988), pp. 243–268. URL: http://www.numdam.org/item?id=RO_1988__22_3_243_0.

[4] R.-J. Jing, Y. Lei, C. F. S. Maligec, M. M. Maza, and C. Mukherjee. "Integer Hulls, Z-Polyhedra and Presburger Arithmetic in Action". In: ACM Commun. Comput. Algebra 59.3 (Jan. 2026), pp. 41–46. ISSN: 1932-2232. DOI: 10.1145/3787957.3787958. URL: https://doi.org/10.1145/3787957.3787958.

[5] R.-J. Jing, Y. Lei, C. F. S. Maligec, M. Moreno Maza, and C. Mukherjee. “Quantifier Elimination Over the Integers”. In: ISSAC ’25 (2025), pp. 353–362. DOI: 10.1145/3747199.3747580. URL: https://doi.org/10.1145/3747199.3747580.

[6] R.-J. Jing, Y. Lei, M. Moreno Maza, and C. Mukherjee. “Quantifier Elimination Over the Integers”. In: To appear. DOI: 10.2139/ssrn.6178290. URL: https://ssrn.com/abstract=6178290.

[7] R. Jing and M. Moreno Maza. “Computing the Integer Points of a Polyhedron, I: Algorithm”. In: CASC 2017, Proceedings. Vol. 10490. LNCS. Springer, 2017, pp. 225–241.

[8] S. Verdoolaege. “isl: An Integer Set Library for the Polyhedral Model”. In: Lecture Notes in Computer Science 6327 (2010). Ed. by K. Fukuda, J. van der Hoeven, M. Joswig, and N. Takayama, pp. 299–302.