Integer Hulls, $\mathbb{Z}$-Polyhedra and Presburger Arithmetic in Action

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Acknowledgements

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- Most of the algorithms presented in this software demo are **available in MAPLE's `PolyhedralSets`** library.

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- One important objective for us is to support **Presburger arithmetic**, that is, quantifier elimination over the integers, see our ISSAC 2025 paper.

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- All libraries and commands are MAPLE code, except `FME_SatMat` which is is written in C/C++ in support of efficiency critical routines.

Plan

1. Overview
2. Integer hulls of polyhedra
3. Integer point counting for parametric polyhedra
4. Quantifier elimination over the integers
5. Concluding remarks

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Integer hulls and lattices

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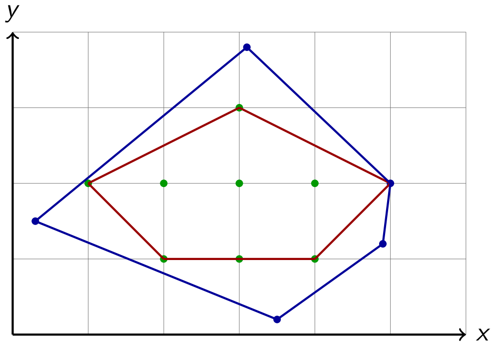

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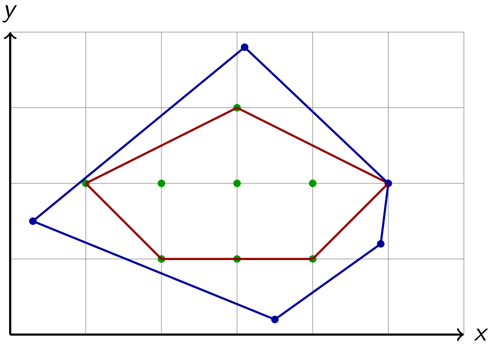

2. A subset $L \subseteq \mathbb{Z}^d$ is called an **integer lattice** (or simply a lattice) if
$$L = \{\mathbf{x} \in \mathbb{Z}^d \mid (\exists \mathbf{t} \in \mathbb{Z}^c)\ \mathbf{x} = A\mathbf{t} + \mathbf{b}\}$$
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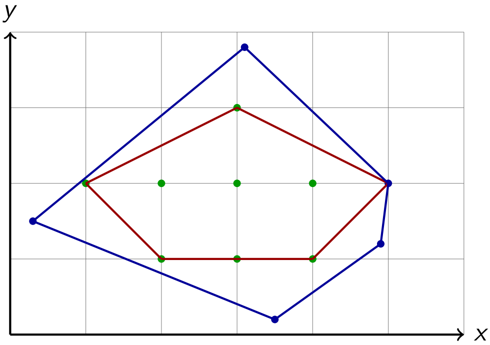

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3. It is convenient to see this lattice as the solution set of the systems of congruence relations $\mathbf{x} \equiv \mathbf{b} \mod A$.

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3. The algorithm IntegerPointDecomposition [3] decomposes any $\mathbb{Z}$-polyhedron into **normalized** $\mathbb{Z}$-polyhedra.

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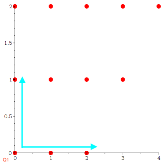
- Still for $d = 2$, $G(P, \mathbf{x})$ is computed as the sum of the generating functions of its vertex cones, thanks to Brion's theorem (1988) [1].



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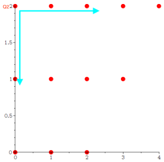

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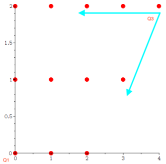

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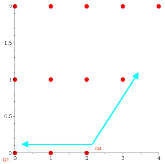

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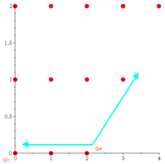

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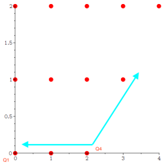

$$\begin{array}{rcl} G(P, \mathbf{x}) & = & G(Q_1, \mathbf{x}) + G(Q_2, \mathbf{x}) + G(Q_3, \mathbf{x}) + G(Q_4, \mathbf{x}) \\ & = & \frac{1}{1-x} \frac{1}{1-y} + \frac{1}{1-x} \frac{y^2}{1-y^{-1}} + \frac{x^4 y^2}{(1-x^{-1})(1-x^{-1}y^{-1})} + \frac{x^2 y^0}{(1-xy)(1-x^{-1})} \end{array}$$

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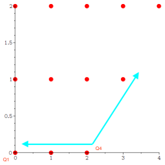

$$\begin{aligned}
G(P, \mathbf{x}) &= G(Q_1, \mathbf{x}) + G(Q_2, \mathbf{x}) + G(Q_3, \mathbf{x}) + G(Q_4, \mathbf{x}) \\
&= \frac{1}{1-x}\frac{1}{1-y} + \frac{1}{1-x}\frac{y^2}{1-y^{-1}} + \frac{x^4y^2}{(1-x^{-1})(1-x^{-1}y^{-1})} + \frac{x^2y^0}{(1-xy)(1-x^{-1})} \\
&= y^2 + xy^2 + x^2y^2 + x^3y^2 + x^4y^2 + y + xy + x^2y + \\
&\quad + x^3y + 1 + x + x^2.
\end{aligned}$$

Plan

Presburger arithmetic

The language of **Presburger arithmetic** is:

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3. $\phi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a quantifier-free formula, where each **atom** (= formula free of quantifiers and connectives) is either
   a. a non-strict inequality $\ell(x_1, \ldots, x_m, y_1, \ldots, y_n) \leq 0$,
   b. or a divisibility relation $k \mid \ell(x_1, \ldots, x_m, y_1, \ldots, y_n)$,

   where:

   a. $k \in \mathbb{Z}_{>0}$ is a constant, and
   b. $\ell(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a **linear integer** polynomial, thus with total degree at most 1.

Quantifier elimination

Theorem 1

Presburger arithmetic admits quantifier elimination.

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Remark 1

Recall

$$F = Q_1 x_1 \cdots Q_m x_m \ \phi(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

Our goal is to determine the set $D(y_1, \ldots, y_n) \subseteq \mathbb{Z}^n$ *of* **ALL** *integer tuples of* $(y_1, \ldots, y_n)$ *for which the formula* $F(x_1, \ldots, x_m, y_1, \ldots, y_n)$ *is true.*

Concluding remarks

Summary and notes

1. We have presented software libraries for computing with integer hulls and $\mathbb{Z}$-polyhedra.
2. They support integer point counting of parametric polyhedra (= computing Ehrhart polynomials) as well as Presburger arithmetic (= quantifier elimination over the integers).
3. The algorithm IntegerPointDecomposition plays an essential role.
4. Most of these functionalities are available in Maple 2025, except QE.

Work in progress

1. In the presence of free variables, QE tend to split computations more than necessary and we are developing algorithms dealing with this issue.
2. We are extending our implementation of Presburger arithmetic to support certain class of non-linear expressions that are of practical interest in compiler theory [2].

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