Quantifier Elimination Over the Integers

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Plan

1. Overview
2. Basic concepts
3. Quantifier elimination over the integers
4. Integer projection
5. Experimentation
6. Concluding remarks

Quantifier elimination

Input

Consider a formula in prenex normal form,

$$F = Q_1 x_1 \ldots Q_m x_m \; \phi(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

where:

1. $Q_1, \ldots Q_m$ is a sequence of quantifiers (existential $\exists$ or universal $\forall$),
2. $x_1, \ldots, x_m$ are bound variables,
3. $y_1, \ldots, y_n$ are free variables and,
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1. **A sample point:** a tuple of values for $(y_1, \ldots, y_n)$ making F true, if such a tuple exists, `false` otherwise,
2. **An equivalent formula:** describing the set $D(y_1, \ldots, y_n)$ consisting of all tuples of values for $(y_1, \ldots, y_n)$ making F true.

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4. Still for a sequence of existential quantifiers in LIA, the algorithm [10] by Christoph Haase et al. **computes sample points in singly exponential time**.
5. Can we eliminate a sequence of existential quantifiers for LIA in the sense of computing a formula in single exponential time?

Applications

- Optimizing Compilers:
  - Array Dependence Analysis
  - Polyhedral frameworks (GCC's Graphite [22], LLVM's Polly [8]).
- Program Verification:
  - Stanford Pascal Verifier [18]
  - CompCert [16]
- Theorem Proving:
  - SAT/SMT Solvers (Z3 [20], CVC5 [1])
  - Proof Assistants (Coq [4], Isabelle [21], HOL Light [11], Lean [19]).

Software Implementations

- ISL (Integer Set Library) [27]
- TaPAS (Talence Presburger Arithmetic Suite) [15]
- Yices [7]
- Princess (Scala Theorem Prover) [25]
- Our Software, see the ISSAC 2025 software demo.

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5. Experimental results are provided.

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$$P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}$$
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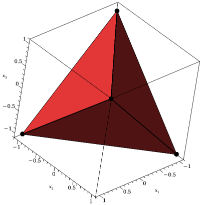

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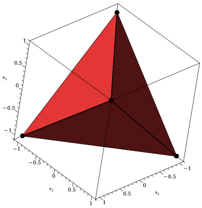

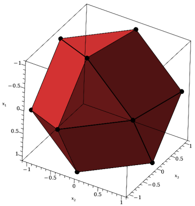

Integer hulls and lattices

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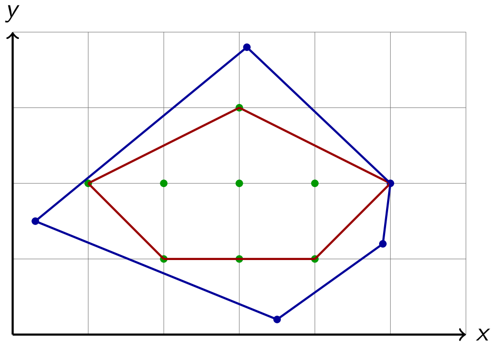

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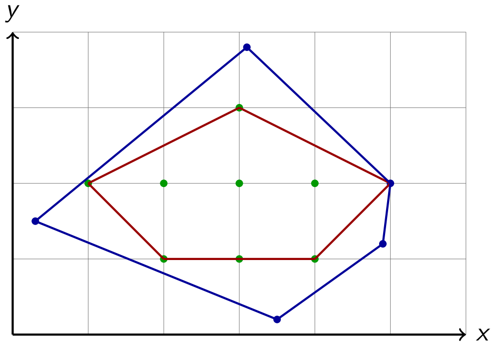

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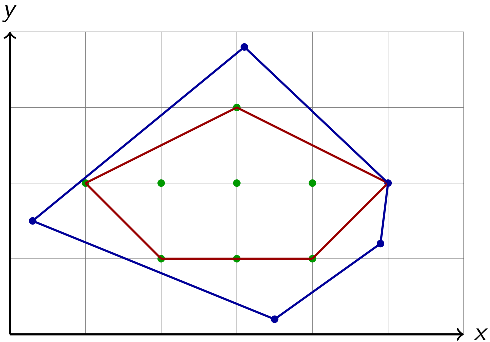

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3. It is convenient to see this lattice as the solution set of the systems of congruence relations $\mathbf{x} \equiv \mathbf{b} \mod A$.

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3. The algorithm IntegerPointDecomposition [14] decomposes any $\mathbb{Z}$-polyhedron into **normalized** $\mathbb{Z}$-polyhedra.

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Theorem 1 (parametric multivariate CRT)

The values of $(w_1, \ldots, w_\nu)$ for which the above system has solutions form a lattice of $\mathbb{Z}^\nu$. Moreover, for each value of $(w_1, \ldots, w_\nu)$, the $\mathbf{z}$-solutions form a lattice of $\mathbb{Z}^n$.

Proof.

Compute the Hermite normal forms of the appropriate matrices. □

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2. extended by the divisibility predicates $D_k : x \longmapsto k \mid x$, for all $k \in \mathbb{Z}_{>0}$.

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2. extended by the divisibility predicates $D_k : x \longmapsto k \mid x$, for all $k \in \mathbb{Z}_{>0}$.

A **Presburger formula F in prenex normal form** has the form:

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3. $\phi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a quantifier-free formula, where each **atom** (= formula free of quantifiers and connectives) is either
   a. a non-strict inequality $\ell(x_1, \ldots, x_m, y_1, \ldots, y_n) \leq 0$,
   b. or a divisibility relation $k \mid \ell(x_1, \ldots, x_m, y_1, \ldots, y_n)$,

   where:

   a. $k \in \mathbb{Z}_{>0}$ is a constant, and
   b. $\ell(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a **linear integer** polynomial, thus with total degree at most 1.

Quantifier elimination (1/2)

Theorem 2

Presburger arithmetic admits quantifier elimination.

Proof.

1. See the thesis of Mojżesz Presburger [26], the paper of David Cooper [6], and Christoph Haase's **Survival Guide to Presburger Arithmetic** [9].
2. See also our own proof in a few slides.

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Recall our Presburger formula

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Recall our Presburger formula

$$F = Q_1 x_1 \cdots Q_m x_m \; \phi(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

Our goal is to determine the set $D(y_1, \ldots, y_n) \subseteq \mathbb{Z}^n$ of **ALL** integer tuples of $(y_1, \ldots, y_n)$ for which the formula $F(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is true.

Quantifier elimination (2/2)

Remark 1

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   b. *If $Q = \forall$, then we can replace $\forall x_1 \ F'$ with $\neg(\exists x_1 \ \neg(F'))$,*
   c. *Whenever possible, we should make use of rules like:*

$$\forall x_1 \cdots \forall x_m \ C \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \mathbf{q} \ \Rightarrow \ C = \mathbf{0} \ \bigwedge \ \mathbf{q} = \mathbf{0}, \qquad (3.1)$$

where

1. *$C \in \mathbb{Z}^{r \times m}$ is a matrix, and*

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   where
   1. *$C \in \mathbb{Z}^{r \times m}$ is a matrix, and*
   2. *$\mathbf{q} \in (\mathbb{Z}[y_1, \ldots, y_m])^r$ is a vector of linear polynomials.*

Coarsening the atoms

Remark 2

In Cooper's algorithm [6], when processing $\exists x_1\ F'$*, the formula* F' *uses the following four types of atoms:*

$$A_y < ax_1,\ ax_1 < A_y,\ k \mid (ax_1 + A_y), \text{ and } \neg(k \mid (ax_1 + A_y)), \qquad (3.2)$$

where $a \in \mathbb{Z}$ *and* $A_y \in \mathbb{Z}[y_1, \ldots, y_n]$ *is a linear polynomial.*

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Remark 3

We can rearrange our quantifier-free formula to:

$$\phi(x_1, \ldots, x_m, y_1, \ldots, y_n) = \bigvee_i Z_i(x_1, \ldots, x_m, y_1, \ldots, y_n), \tag{3.3}$$

where each Z_i is a predicate of the form

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{Z}\text{Polyhedron}(P_i, L_i), \tag{3.4}$$

for some polyhedron P_i and some integer lattice L_i.
We call such a predicate a **$\mathbb{Z}$-polyhedron predicate**.

Plan

Integer projection: $n = 1$

Remark 4

1. *From the above section, we consider the formula*

$$\exists x \ \phi(x, y_1, \ldots, y_n), \ \text{where} \ \phi(x, y_1, \ldots, y_n) = \bigvee_i \phi_i(x, y_1, \ldots, y_n), \tag{4.1}$$

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Remark 5

1. *Let $f_1, \ldots, f_s$, $g_1, \ldots, g_r \in \mathbb{Z}[x, \mathbf{y}]$ be linear and let $k_1, \ldots, k_r \in \mathbb{Z}_{>0}$.*

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3. *We shall determine the set $D(\mathbf{y})$ of integer tuples $(y_1, \dots y_n)$ for which $F(\mathbf{y})$ holds. We call $D(\mathbf{y})$ the* **integer projection** *of $F(\mathbf{y})$.*

Integer projection: $n = 1$, removing congruences

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6. *As a result, we now have a $\mathbb{Z}$-polyhedron predicate.*

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7. *If that process* **solves for** *x, then our problem becomes that of describing the points of a $\mathbb{Z}$-polyhedron, which can be done by our* IntegerPointDecomposition.

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8. *If that process* **does not solve for** x*, then go to next slide.*

Integer projection: $n = 1$, no congruences, 2 inequalities

Remark 7

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2. *We start with the case $s = 2$ and rename f_1, f_2 to A, B.*
3. *We also write:*
$$A = A_{\mathbf{y}} - a\,x, \quad \text{and} \quad B = -B_{\mathbf{y}} + b\,x, \tag{4.3}$$
where $a, b \in \mathbb{Z}$ are non-zero and where $A_{\mathbf{y}}, B_{\mathbf{y}} \in \mathbb{Z}[\mathbf{y}]$ are linear

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6. *Observe that Formula (4.2* null*) simplifies to:*
$$F(\mathbf{y}) : (\exists x \in \mathbb{Z})\ (A_{\mathbf{y}} \leq a\,x)\ \wedge\ (b\,x \leq B_{\mathbf{y}}), \tag{4.4}$$

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7. *We present a first formula for $D(\mathbf{y})$ based on Harris Williams [29, 30]*

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5. *With these assumptions, we call the inequalities* $A \leq 0$ *and* $B \leq 0$, *respectively a* **lower bound** *and an* **upper bound** *for* x.
6. *Observe that Formula (4.2* null*) simplifies to:*
$$F(\mathbf{y}) : (\exists x \in \mathbb{Z})\ (A_{\mathbf{y}} \leq a\,x)\ \wedge\ (b\,x \leq B_{\mathbf{y}}), \tag{4.4}$$
7. *We present a first formula for* $D(\mathbf{y})$ *based on Harris Williams [29, 30]*
8. *Then, we present a second one based on William Pugh's Omega test [23, 24].*

Williams-style Projection

Theorem 3

Let $\ell = \mathrm{lcm}(a, b)$*,* $b' = \ell/a$ *and* $a' = \ell/b$*. For* $0 \le k < b$*, define*

$$E_k := \{\mathbf{y} \mid \mathrm{rem}(B_{\mathbf{y}}, b) = k\}.$$

Then, the following two formulas are equivalent:

1. $F(\mathbf{y})$: $(\exists x \in \mathbb{Z})\ (A_{\mathbf{y}} \le a\,x) \ \wedge\ (b\,x \le B_{\mathbf{y}})$,
2. $\bigvee_{k=0}^{k=b-1} (\mathbf{y} \in E_k) \wedge (a' k \ \le\ a' B_{\mathbf{y}} - b' A_{\mathbf{y}})$.

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Proof.

1. If $a = 1$ or $b = 1$ holds, then: $F(\mathbf{y}) \iff b'\,A_{\mathbf{y}} \le a'\,B_{\mathbf{y}}$.

□

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Proof.

1. If $a = 1$ or $b = 1$ holds, then: $F(\mathbf{y}) \iff b'\,A_{\mathbf{y}} \leq a'\,B_{\mathbf{y}}$.
2. From now on, assume $a > 1$ and $b > 1$ both hold. Observe that

$$F(\mathbf{y}) \iff (\exists x \in \mathbb{Z})\ (b'A_{\mathbf{y}} \leq \ell\,x)\ \wedge\ (\ell\,x \leq a'B_{\mathbf{y}}).$$

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$$F(\mathbf{y}) \iff (\exists x \in \mathbb{Z})\ (b' A_{\mathbf{y}} \le \ell\,x) \wedge (\ell\,x \le a' B_{\mathbf{y}}).$$

3. Hence, $F(\mathbf{y})$ says that a multiple of ℓ lies between $b' A_{\mathbf{y}}$ and $a' B_{\mathbf{y}}$).

□

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Then, the following two formulas are equivalent:

1. *F(**y**):* $(\exists x \in \mathbb{Z})\ (A_{\mathbf{y}} \le a\,x)\ \wedge\ (b\,x \le B_{\mathbf{y}})$*,*
2. $\bigvee_{k=0}^{k=b-1} (\mathbf{y} \in E_k) \wedge (a'k \ \le\ a' B_{\mathbf{y}} - b' A_{\mathbf{y}})$*.*

Proof.

1. If $a = 1$ or $b = 1$ holds, then: $F(\mathbf{y}) \iff b'\,A_{\mathbf{y}} \le a'\,B_{\mathbf{y}}$.
2. From now on, assume $a > 1$ and $b > 1$ both hold. Observe that
$$F(\mathbf{y}) \iff (\exists x \in \mathbb{Z})\ (b' A_{\mathbf{y}} \le \ell\,x)\ \wedge\ (\ell\,x \le a' B_{\mathbf{y}}).$$
3. Hence, $F(\mathbf{y})$ says that a multiple of ℓ lies between $b' A_{\mathbf{y}}$ and $a' B_{\mathbf{y}})$.
4. Thus, $F(\mathbf{y}) \iff b'\,A_{\mathbf{y}} \ \le\ a'\,B_{\mathbf{y}} - \mathrm{rem}(a'\,B_{\mathbf{y}}, \ell)$.
5. That is, $F(\mathbf{y}) \iff b' A_{\mathbf{y}} \ \le\ a'(B_{\mathbf{y}} - \mathrm{rem}(B_{\mathbf{y}}, b))$.

□

Pugh's omega test (1/2)

Lemma 4 (William Pugh)

If we have:

$$aB_{\mathbf{y}} - bA_{\mathbf{y}} \;\geq\; (a-1)(b-1) \tag{4.5}$$

then $F(\mathbf{y})$ holds.

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2. If I does not contain an integer, then we have:

$$i < \frac{A_{\mathbf{y}}}{a} \leq \frac{B_{\mathbf{y}}}{b} < i+1, \quad \text{where} \quad i \;=\; \left\lfloor \frac{A_{\mathbf{y}}}{a} \right\rfloor. \tag{4.6}$$

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3. Let $\rho := \mathrm{rem}(A_{\mathbf{y}}, a)$. Since $i < \frac{A_{\mathbf{y}}}{a}$ holds, we have:
$$A_{\mathbf{y}} \ = \ i\,a + \rho \ \text{ and } 0 < \rho < a, \tag{4.7}$$
4. from which we deduce: $\frac{A_{\mathbf{y}}}{a} - i \ \geq \ \frac{1}{a}$.

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5. Similarly, we obtain: $i + 1 - \frac{B_{\mathbf{y}}}{b} \ \geq \ \frac{1}{b}$.

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Proof.

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5. Similarly, we obtain: $i + 1 - \frac{B_{\mathbf{y}}}{b} \;\geq\; \frac{1}{b}$.
6. From the above two inequalities, elementary manipulations yield:
$$aB_{\mathbf{y}} - bA_{\mathbf{y}} \;\leq\; ab - a - b. \tag{4.8}$$

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5. Similarly, we obtain: $i + 1 - \frac{B_{\mathbf{y}}}{b} \ \geq \ \frac{1}{b}$.
6. From the above two inequalities, elementary manipulations yield:
$$aB_{\mathbf{y}} - bA_{\mathbf{y}} \ \leq \ ab - a - b. \tag{4.8}$$
7. Therefore, if the above inequality does not hold, that is, if $aB_{\mathbf{y}} - bA_{\mathbf{y}} \ \geq \ (a-1)(b-1)$ does hold, then I contains an integer.

Pugh's omega test (2/2)

Theorem 5

Define $\kappa(a,b) := \lceil \frac{(a-1)(b-1)}{a'} \rceil$*. Then, Formula* $F(\mathbf{y})$ *is equivalent to:*

$$((a-1)(b-1) \leq aB_{\mathbf{y}} - bA_{\mathbf{y}}) \bigvee_{k=\kappa(a,b)}^{k=b-1} (\mathbf{y} \in E_k) \wedge (a'k \leq a'B_{\mathbf{y}} - b'A_{\mathbf{y}}) \,. \quad (4.9)$$

Proof.

This is a direct consequence of William Pugh's lemma and Harris Williams' projection formula □

Remark 8

1. *William Pugh's lemma reduces significantly the number of "cuts"*
2. *To take a concrete example, say with* $a = 7$ *and* $b = 11$*:*
   - a. *with Williams' projection alone* k *ranges from* 0 *to* 10*,*
   - b. *with William Pugh's lemma,* k *ranges from* 9 *to* 10*.*

Integer projection: $n = 1$, s inequalities

We now describe a procedure Projection$(f_1, \ldots, f_s; x)$ computing $D(\mathbf{y})$.

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We now describe a procedure Projection($f_1, \ldots, f_s; x$) computing $D(\mathbf{y})$.

1. If $f_1, \ldots, f_s$ only count lower (resp. upper) bounds for x, then return true.
2. Initialize $D(\mathbf{y})$ to true.
3. For each pair (A, B) consisting of a lower bound and an upper bound of x, replace $D(\mathbf{y})$ with $D(\mathbf{y}) \wedge$ Projection(A, B), where Projection(A, B) is given by Pugh's omega test.

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4. Convert $D(\mathbf{y})$ to DNF yielding a formula of the form
$$S_0 \vee (C_1 \wedge S_1) \vee \cdots \vee (C_e \wedge S_e) \tag{4.10}$$
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   a. $S_0, S_1, \ldots, S_e$ are systems of non-strict linear inequalities in the variables $\mathbf{y}$,
   b. $C_1, \ldots, C_e$ are systems of congruences in the variables $\mathbf{y}$, and

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where
   - a. $S_0, S_1, \ldots, S_e$ are systems of non-strict linear inequalities in the variables $\mathbf{y}$,
   - b. $C_1, \ldots, C_e$ are systems of congruences in the variables $\mathbf{y}$, and
   - c. S_0 is the conjunction of the $((a-1)(b-1) \leq aB_{\mathbf{y}} - bA_{\mathbf{y}})$, for all pairs (A, B) of lower and upper bounds of x.

Plan

1. Overview
2. Basic concepts
3. Quantifier elimination over the integers
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5. Experimentation
6. Concluding remarks

Experimentation

Two strategies for integer projection

1. **IPD:** the one presented in this paper and based on the IntegerPointDecomposition algorithm [13],
2. **NIP:** one based on NumberOfIntegerPoints [13] and thus a parametric adaptation of **Barvinok's algorithm** [3, 17, 28].

Sources of test cases

1. examples from the literature (mainly from compiler theory)
2. examples from the SMT-LIA category of the SMT-LIB data-base [2],

Our code can be accessed here.

test	IPD MEMORY(MB)	IPD TIME(s)	NIP MEMORY(MB)	NIP TIME(s)
T1[**BoGoWo17**]	24.232	0.121	33.811	0.193
T2[**BoGoWo17**]	57.843	0.281	59.136	0.344
T3[**BoGoWo17**]	121.978	0.671	189.439	1.256
T4[**BoGoWo17**]	42.531	0.240	65.162	0.378
T5[**BoGoWo17**]	22.114	0.110	31.725	0.167
T6[**SeLoMe12**]	97.739	0.481	64.456	0.333
T7[**St23**]	671.154	3.506	1066.889	6.608
T8[**KVeWo08**]	69.087	0.338	58.668	0.328
T9 [**KVeWo08**]	245.156	1.235	979.964	6.462
T17[**BoGoWo17**]	5.315	0.043	12.771	0.060
T18[**CaLiZh22**]	39.055	0.200	48.237	0.205
T19[**Fe88**]	355.466	1.786	1715.958	10.941
T20[**Ve24**]	25.453	0.154	28.667	0.180
T32[**SeLoMe12**]	28216.613	156.989	$>$ 10 GB	$>$ 600
T33[**SeLoMe12**]	70.135	0.351	345.340	1.920
T34[**SeLoMe12**]	178.657	0.928	366.935	2.487
T35[**SeLoMe12**]	121.098	0.645	165.582	1.053
T36[**SeLoMe12**]	1243.682	6.209	798.004	4.822
T44[**Ve24**]	1.549	0.013	1.550	0.014
T45[**Ve24**]	1.549	0.014	1.551	0.013
T46[**Ve24**]	1.551	0.017	1.552	0.013
T47[**Ve15**]	49.726	0.236	45.779	0.219
T48[**Ve15**]	53.819	0.260	98.094	0.540
T49[**Ve15**]	32.190	0.197	27.997	0.153

Table: Maple 2024, Ubuntu 24.04.1 LTS, 16GB RAM and 12th Gen Intel(R) Core(TM) i5-1235U processor

1. **IPD = IntegerPointDecomposition**
2. **NIP = NumberOfIntegerPoints**

Using the SMT-LIB data-base

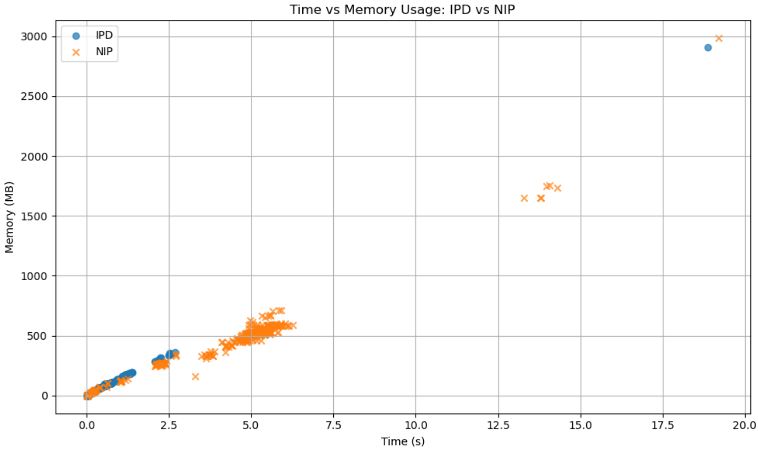

Figure: Time vs Memory for the SMT-LIA examples.

We have tested all the examples (about 400) from SMT-LIA that are Presburger formulas.

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Summary and notes

We have discussed algebraic issues for the problem of quantifier elimination in Presbuger arithmetic. Our findings are:

1. Cooper's algorithm [6] is equivalent to Williams' projection [29].
2. The Omega test [24] is a substantial optimization of the latter two projections.
3. The algorithm IntegerPointDecomposition [12], which is based on the Omega test, seems experimentally superior to algorithms based on parametric versions of Barvinok's algorithm.

Work in progress

1. We shall continue investigating **heuristics to bypass double negation** when dealing with universal quantifiers.
2. In the presence of free variables, QE tend to split computations more than necessary and we are developing algorithms dealing with this **expression swell** issue.
3. We are extending our implementation of Presburger arithmetic to support certain classes of **non-linear expressions** that are of practical interest in compiler theory [5].

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