Missing Sets in Rational Parameterization of Surface of Revolution

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Plan

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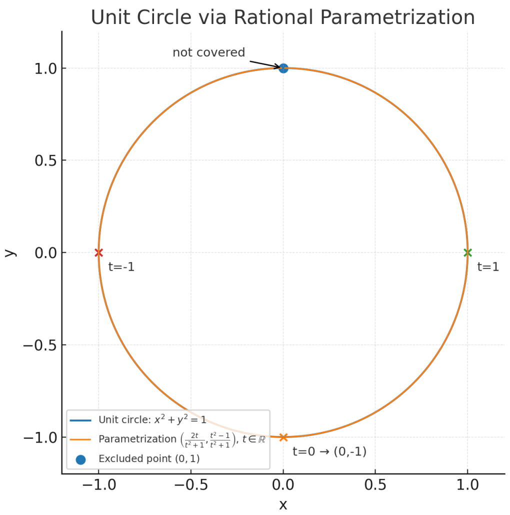
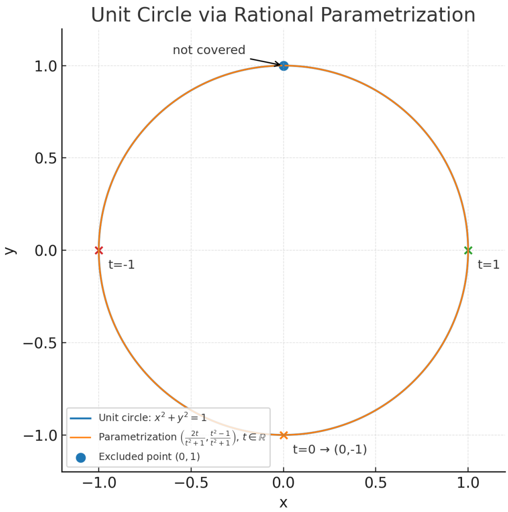
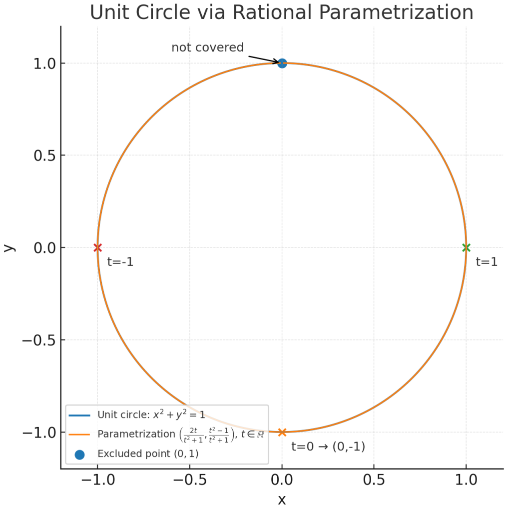


Unit Circle via Rational Parametrization
not covered
1.0
0.5
0.0
−0.5
−1.0
y
t=-1
t=1
t=0 → (0,-1)
Unit circle: $x^2 + y^2 = 1$
Parametrization $\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$, $t \in \mathbb{R}$
Excluded point $(0, 1)$
−1.0
−0.5
0.0
0.5
1.0
x



A rational parametrization of a variety $\mathcal{V}$, may *not cover* all $\mathcal{V}$.



A rational parametrization of a variety $\mathcal{V}$, may *not cover* all $\mathcal{V}$.

The missing part is a constructible set of $\mathcal{V}$ [3].

Constructible Set

A subset $C \subseteq \mathcal{V}$ is *constructible* if there exist finitely many Zariski open sets $U_1, \ldots, U_r \subseteq \mathcal{V}$ and Zariski closed sets $Z_1, \ldots, Z_r \subseteq \mathcal{V}$ such that

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Example ($\mathcal{V} = \mathbb{A}^1$ with the Zariski topology)

- $C = \mathbb{C} \setminus \{0\}$ is constructible, since

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- $C = \underbrace{\{0\}}_{\text{closed}} \cup \underbrace{\mathbb{C} \setminus \{1\}}_{\text{open}}$

Thus C is constructible but is *neither open nor closed.*

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- computer graphics [1, 4]
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Obvious solution:

- Find parametrizations that cover the whole object.
- For curves, see [7].
- For surfaces, open problem .

Alternative solution:

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These techniques produces *huge critical sets* and requires solving *systems of algebraic equations*.

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- In this talk, we analyze the case of surfaces of revolution given by means of a *real plane curve parametrization*.

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- For structured surfaces, a preliminary analysis of the structure can help to describe the critical set.
- Any rational ruled surface can be parametrized so that the critical set is a line [6], known as *profile curve*.
- In this talk, we analyze the case of surfaces of revolution given by means of a *real plane curve parametrization*.
- The *critical set* for the real part is, in the worst-case, the union of a curve (the mirror curve of the profile curve) and a circle passing through the critical point of the profile curve.

Plan

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- $circ(\alpha, p)$ is the circle of radius $|\alpha|$ centered at $(0, 0, p)$

$$P = \left(\frac{2s}{1+s^2} \alpha, \frac{1-s^2}{1+s^2} \alpha, p \right).$$

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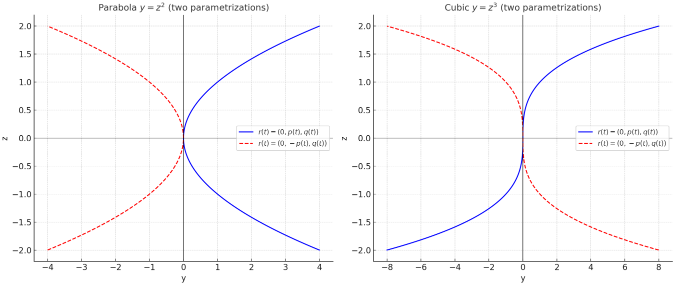
Example

- the parabola $y = z^2$ is *equal* to its mirror curve
- the cubic $y = z^3$ is *not equal* to its mirror curve



- r is *normal* if all points on the curve are reachable by $r(t)$ when t takes values in the field of the complex numbers.

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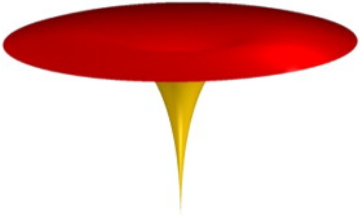
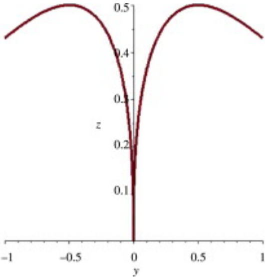
Non-symmetric real case:

- r is normal, C_M is a real-critical set of P.
- r is not normal, and $(0, b, c)$ is its critical point, then a real-critical set of P is $\begin{cases} C_M & \text{, if} (0, -b, c) \in C \\ C_M \cup circ(b, c), & \text{otherwise} \end{cases}$.

Plan

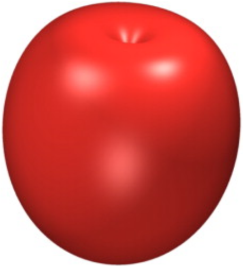
Example 1

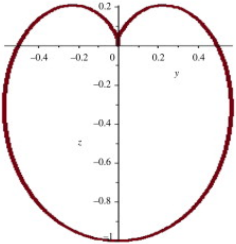




- $r(t) = \left(0, \frac{t^5}{t^4+1}, \frac{t^2}{t^4+1}\right)$
- $C = C_M$
- r is normal
- critical set is empty
- all real part of S is covered by P

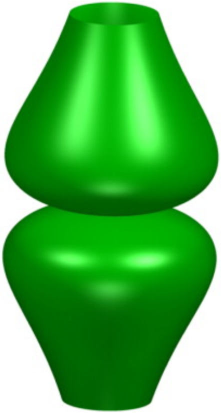
Example 2

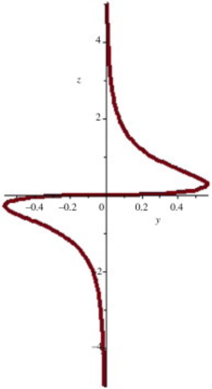




- $r(t) = \left(0, \frac{t}{t^4+1}, \frac{t^2-1}{t^4+1}\right)$
- $C = C_M$
- r is not normal
- critical point is $(0, 0, 0)$
- all real part of $S \setminus \{(0, 0, 0)\}$ is covered by P

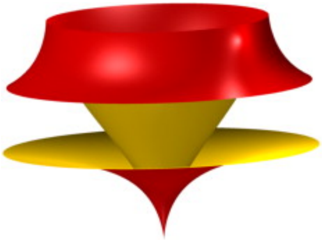
Example 3

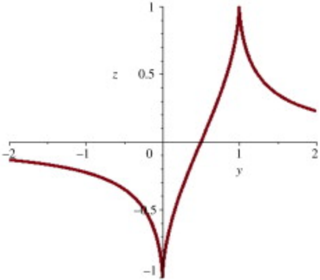




- $r(t) = \left(0, \frac{t}{t^4+1}, \frac{t^3}{t^2+1}\right)$
- $C \neq C_M$
- r is normal
- critical set is r_M
- all real part of $S \setminus r_M$ is covered by P

Example 4





- $r(t) = \left(0, \frac{t^3}{t^3+1}, \frac{t^2-1}{t^2+1}\right)$
- $C \neq M$
- r is not normal
- critical point is $(0, 1, 1)$
- $(0, -1, 1) \notin C$
- critical set is $r_M \cup circ(1, 1)$
- all real part of $S \setminus (r_M \cup circ(1, 1))$ is covered by P

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Possible real critical sets:

Symmetric	Normal	Real critical set
Yes	Yes	Empty set, ex. 14

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Yes	No	Critical point, ex. 15
No	Yes	Mirror curve, ex. 16

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Yes	Yes	Empty set, ex. 14
Yes	No	Critical point, ex. 15
No	Yes	Mirror curve, ex. 16
No	No	Mirror curve union cross section circle at the critical point, ex. 17

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Future work:

- We have analyzed the missing area of S, when P takes values in $\mathbb{C}^2$.

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- We have analyzed the missing area of S, when P takes values in $\mathbb{C}^2$.
- Study the missing sets of S when P takes values in $\mathbb{R}^2$.

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- Study the missing sets of S when P takes values in $\mathbb{R}^2$.
- Study the missing sets of other surface constructions in CAD, for instance swung surfaces [5, 12].

References I

[1] M. K. Agoston. *Computer graphics and geometric modeling: mathematics*. Springer, 2005.

[2] C. L. Bajaj and A. V. Royappa. "Finite representations of real parametric curves and surfaces". In: *International Journal of Computational Geometry & Applications* 5.03 (1995), pp. 313–326.

[3] J. Harris. *Algebraic geometry, vol. 133 of Graduate Texts in Mathematics*. 1992.

[4] D. Marsh. *Applied geometry for computer graphics and CAD*. Springer, 2006.

[5] H. Qin and D. Terzopoulos. "Dynamic NURBS swung surfaces for physics-based shape design". In: *Computer-Aided Design* 27.2 (1995), pp. 111–127.

References II

[6] J. Sendra, D. Sevilla, and C. Villarino. “Covering rational ruled surfaces”. In: *Mathematics of Computation* 86.308 (2017), pp. 2861–2875.

[7] J. R. Sendra. “Normal parametrizations of algebraic plane curves”. In: *Journal of Symbolic Computation* 33.6 (2002), pp. 863–885.

[8] J. R. Sendra, D. Sevilla, and C. Villarino. “Covering of surfaces parametrized without projective base points”. In: *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*. 2014, pp. 375–380.

[9] J. R. Sendra, D. Sevilla, and C. Villarino. “Some results on the surjectivity of surface parametrizations”. In: *Computer Algebra and Polynomials: Applications of Algebra and Number Theory*. Springer, 2015, pp. 192–203.

References III

[10] J. R. Sendra, F. Winkler, and S. Pérez-Díaz. *Rational algebraic curves: a computer algebra approach*. Springer, 2008.

[11] N. Shragai and G. Elber. “Geometric covering”. In: *Computer-Aided Design* 45.2 (2013), pp. 243–251.

[12] Y. Zhao, Y. Zhou, J. L. Lowther, and C.-K. Shene. “Cross-sectional design: A tool for computer graphics and computer-aided design courses”. In: *FIE’99 Frontiers in Education. 29th Annual Frontiers in Education Conference. Designing the Future of Science and Engineering Education. Conference Proceedings (IEEE Cat. No. 99CH37011)*. Vol. 2. IEEE. 1999, 12B3–1.