Twisted Arrow Construction for Segal Spaces

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Content

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$$\begin{array}{ccc} C & \xleftarrow{k} & C' \\ f\downarrow & & \downarrow f' \\ D & \xrightarrow{h} & D' \end{array}$$

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Morphisms

$$\begin{array}{ccccc} C & \xleftarrow{k} & C' & \xleftarrow{k'} & C'' \\ f\big\downarrow & & \big\downarrow f' & & \big\downarrow f'' \\ D & \xrightarrow{h} & D' & \xrightarrow{h'} & D'' \end{array}$$

with $kk' : C'' \to C$ and $h'h : D \to D''$

Composition

A simplicial space W is a functor,

$$W\colon \Delta^{op} \times \Delta^{op} \to \mathrm{Set}.$$

By Yoneda Lemma, $W_{n,l} \cong sS(F(n) \times \Delta[l], W)$.

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Applying Tw on generators,

$$\begin{gathered}\mathrm{Tw} : sS \to sS \\ F(n) \times \Delta[l] \mapsto F(2n+1) \times \Delta[l].\end{gathered}$$

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Hence,

$$Tw(W)_{n,l} \cong sS(F(2n+1) \times \Delta[l], W) \cong W_{2n+1,l}.$$

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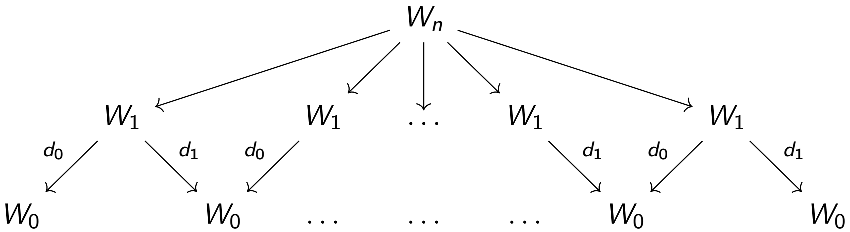

Lemma

If W is a Segal space then $\mathrm{Tw}(W)$ is a Segal space.

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Consider the case $n = 2$,

$$\mathrm{Tw}(W_1) \underset{\mathrm{Tw}(W_0)}{\times} \mathrm{Tw}(W_1) \cong W_3 \underset{W_1}{\times} W_3 \longrightarrow W_3 \xrightarrow{\simeq} W_1 \underset{W_0}{\times} W_1 \underset{W_0}{\times} W_1$$

$$\begin{array}{ccccc} W_3 \underset{W_1}{\times} W_3 & \longrightarrow & W_3 & & W_1 \underset{W_0}{\times} W_1 \underset{W_0}{\times} W_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{Tw}(d_1)} & \overset{\pi_1}{\swarrow} & \\ W_3 & \xrightarrow{\mathrm{Tw}(d_0)} & W_1 & & \\ {\scriptstyle \simeq}\downarrow & \overset{\pi_3}{\nearrow} & & & \\ W_1 \underset{W_0}{\times} W_1 \underset{W_0}{\times} W_1 & & & & \end{array}$$

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\end{array}
.
$$

From 2-out-of-3 property,

$$
\begin{array}{ccc}
\mathrm{Tw}(W_2) \cong W_5 & \xrightarrow{\simeq} & W_3 \underset{W_1}{\times} W_3 \\
& \underset{\simeq}{\searrow} \qquad \underset{\simeq}{\swarrow} & \\
& W_1 \underset{W_0}{\times} W_1 \underset{W_0}{\times} W_1 \underset{W_0}{\times} W_1 \underset{W_0}{\times} W_1 &
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For a Segal space W the space of homotopy equivalences $W_{hoequiv} \subset W_1$ is such that every map is a homotopy equivalence.

A Segal space W is a complete Segal space if,

$$\begin{array}{ccc} W_0 & \xrightarrow{s_0} & W_1 \\ & \searrow & \uparrow \simeq \\ & & W_{hoequiv} \end{array} .$$

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- $\mathrm{TwHo}(W) \to \mathrm{HoTw}(W)$ is an equivalence.
- $$\begin{array}{ccc} \mathrm{Tw}(W)_{hoequiv} & \hookrightarrow & \mathrm{Tw}(W)_1 \\ \downarrow & \lrcorner & \downarrow \\ W^{op}_{hoequiv} \times W_{hoequiv} & \hookrightarrow & W^{op}_1 \times W_1 \end{array}.$$

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- $\mathrm{TwHo}(W) \to \mathrm{HoTw}(W)$ is an equivalence.
- $$\begin{array}{ccccc} \mathrm{Tw}(W)_0 & \xrightarrow{\simeq} & \mathrm{Tw}(W)_{hoequiv} & \hookrightarrow & \mathrm{Tw}(W)_1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ W_0^{op} \times W_0 & \xrightarrow{\simeq} & W_{hoequiv}^{op} \times W_{hoequiv} & \hookrightarrow & W_1^{op} \times W_1 \end{array}.$$

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1. $\mathrm{Tw}(W) \to W^{op} \times W$ is a Reedy fibration.
2. $Tw(W)$ is a Segal space and,

$$\begin{array}{ccc} \mathrm{Tw}(W)_1 & \longrightarrow & \mathrm{Tw}(W)_0 \\ \downarrow & \lrcorner & \downarrow \\ W_1^{op} \times W_1 & \longrightarrow & W_0^{op} \times W_0 \end{array} .$$

Hence the result follows from by [Ras17, Lemma 3.29].