Products

— Chirantan Mukherjee

Suppose we have K a G-complex and K' another G'-complex.
$\Rightarrow$ $K \times K'$ with the product cell structure and the weak topology is a $\underline{G \times G'}$-$\underline{\text{complex}}$.

$K \times K'$ has a product cell structure as, we choose a CW-complex structure for K using cells e_K and attaching maps φ_K. Similarly for K', we choose a CW-complex structure using cells $e_{K'}$ and attaching maps $\varphi_{K'}$.
$\Rightarrow$ the product $e_K \times e_{K'}$ are cells and $\varphi_K \times \varphi_{K'}$ are attaching maps for a CW-complex structure on $K \times K'$

<u>example</u>

$\mathbb{Z}_n$ acts on S^1 and we have a G-CW structure by taking $\mathbb{Z}_n$ acting on n-gon by rotation.

If we take the product of two such S^1's one with $\mathbb{Z}_n$ action and the

$\mathbb{Z}_n$ action and the other with $\mathbb{Z}_m$ action then $\mathbb{Z}_n \times \mathbb{Z}_m$-complex structure on the torus.

<u>example</u>:

The universal cover of a connected CW-complex X has a structure $\pi_1(X)$-CW complex.

If X and Y are connected CW-complex their product has fundamental group isomorphic to $\pi_1(X) \times \pi_1(Y)$ and it's universal cover $\tilde{X}, \tilde{Y}$ respectively of X and Y.

On this there is a $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ - action.

Now we can equipp on this a $\pi_1(X) \times \pi_1(Y)$ - CW structure.

If $\mathcal{L}$ and $\mathcal{L}'$ are local coefficient system on K and K' respectively, then

$$\mathcal{L} \hat{\otimes} \mathcal{L}' \in \mathcal{LC}_{K \times K'}$$

by $(\mathcal{L} \hat{\otimes} \mathcal{L}')(W) = \mathcal{L}(\pi_1 W) \otimes$

$\mathcal{L}'(\pi_2 W)$

$f \in C^p(K;\mathcal{L})$ and $f' \in C^p(K';\mathcal{L}')$

We can define $f \times f' \in C^{p+q}(K \times K'; \mathcal{L} \hat{\otimes} \mathcal{L}')$

by $(f \times f')(\sigma \times \tau) = f(\sigma) \otimes f'(\tau)$

where σ and τ are oriented p and q-cells resp.

If $g \in G$ and $g' \in G'$

$\Rightarrow (g \times g')(f \times f') = g(f) \times g'(f')$

Also, $\delta(f \times f') = (\delta f) \times f' + (-1)^p f \times (\delta f')$

$\Rightarrow \times$ induces a chain map

$$C^p_G(K;\mathcal{L}) \otimes C^p_{G'}(K';\mathcal{L}') \longrightarrow C^{p+q}_{G \times G'}(K \times K'; \mathcal{L} \hat{\otimes} \mathcal{L}')$$

and hence

$$H^p_G(K;\mathcal{L}) \otimes H^q_{G'}(K';\mathcal{L}') \longrightarrow H^{p+q}(K \times K'; \mathcal{L} \hat{\otimes} \mathcal{L}')$$

$\to H_{G\times G'}(K\times K'; \ldots)$

We define an element,

$$\underline{C}_n(K; Z) \in \mathcal{C}_G$$

by $\underline{C}_n(K; Z)(G/H) = C_n(K^H; Z)$

So, $\forall n$ these objects form a chain complex $\underline{C}_*(K; Z)$ in the abelian category $\mathcal{C}_G$

Homology $\underline{H}_n(K; Z) = H_n(\underline{C}_*(K, Z)) \in \mathcal{C}_G$ of this chain complex is again

$$\underline{H}_n(K; Z)(G/H) = H_n(K^H; Z)$$

Let, $f \in C^n_G(K; M)$ where $M \in \mathcal{C}_G$

$\Rightarrow$ for n-cell σ, $f(\sigma) \in M(G/G_\sigma)$

Suppose that $\sigma \in K^H \Rightarrow H \subset G_\sigma$

so that $\underbrace{M(G/H \to G/G_\sigma) f(\sigma)}_{\hat{f}(G/H)(\sigma)} \in M(G/H)$

Th… extends to a homomorphism

This map extends to a homomorphism

$\longrightarrow \hat{f}(G/H) : C_n(K^H; \mathbb{Z}) \longrightarrow M(G/H)$

It is natural with respect to morphism of $\mathcal{O}_G$ so that

$\hat{f} : \underline{C}_n(K; \mathbb{Z}) \longrightarrow M$ is a natural transformation of functors.

i.e. $\hat{f} \in \mathrm{Hom}(\underline{C}_n(K; \mathbb{Z}), M)$

morphism is a morphism of the abelian category $\mathcal{C}_G$

Conversely, suppose we are given an element $\hat{f} \in \mathrm{Hom}(\underline{C}_n(K; \mathbb{Z}), M)$

Let σ be an n-cell of K and we regard $\sigma \in C_n(K^{G_\sigma}, \mathbb{Z})$

We define $f(\sigma) = \hat{f}(G/G_\sigma)(\sigma)$

$\in M(G/G_\sigma)$

so that $f \in C^n(K; M)$

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We check if f is equivariant

$\hat{f}$ is natural to $\hat{g}$ of $\mathcal{O}_G$

$$\hat{g} : G/G_{g\sigma} = G/gG_\sigma g^{-1} \longrightarrow G/G_\sigma$$

we see,

$$\begin{tikzcd}
C_n(K^{G_\sigma}; Z) \arrow[r, "\hat{f}(G/G_\sigma)"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\circlearrowright"] & M(G/G_\sigma) \arrow[d, "g_* = M(\hat{g})"] \\
C_n(K^{G_{g\sigma}}; Z) \arrow[r, "\hat{f}(G/G_\sigma)"] & M(G/G_{g\sigma})
\end{tikzcd}$$

$$\begin{aligned}
\Rightarrow f(g\sigma) &= \hat{f}(G/G_{g\sigma})(g\sigma) \\
&= g_*\left(\hat{f}(G/G_\sigma)(\sigma)\right) \\
&= g_*\left(f(\sigma)\right)
\end{aligned}$$

$\Rightarrow$ f is equivariant

We have shown $C^n_G(K; M)$

$$\cong \mathrm{Hom}(\underline{C_n}(K; Z), M)$$

and this isomorphism is given by

$$f \longrightarrow \hat{f} .$$

This isomorphism also preserves coboundary operators.

To see this, we apply homology

$$H^n_G(K; M) \approx H^n(\mathrm{Hom}(\underline{C}_*(K; Z), M))$$

Now, since Hom is left exact on $\mathcal{C}_G$ we obtain cannonical homomoophism

$$H^n_G(K; M) \longrightarrow \mathrm{Hom}(\underline{H}_n(K; Z), M)$$

If K has no $(n-1)$ cells

$$\Rightarrow \underline{C}_{n-1}(K; Z) = 0$$

and $\Rightarrow$ $H^n_G(K; M) \approx \mathrm{Hom}(\underline{H}_n(K; Z), M)$.