Propagators

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Defn :– The <u>unit normal bundle</u> of a submanifold C in a smooth manifold A is the fiber bundle whose fiber over $x \in C$ is $SN_x = (N_x(C) \setminus \{0\}) / \mathbb{R}^{+*}$ where $\mathbb{R}^{+*}$ acts by scalar multiplication

Blowing up a submanifold C in a smooth manifold A transforms A into a smooth manifold $Bl(A, C)$ by replacing C by the total space of it's unit normal bundle.

Unlike AG blow-ups, the DG blow-ups amounts to remove an open tubular nbd of C which topologically creates boundaries.

Defn :– A smooth <u>submanifold transverse to the ridges</u> of a smooth manifold A is a subset of C of A

st for any point $\underline{x \in C}$ $\exists$ an smooth open embedding

$$\phi : \mathbb{R}^c \times \mathbb{R}^e \times [0,1)^d \hookrightarrow A$$

st $\phi(0) = x$

and $\operatorname{Im}\phi$ intersects C along

$\phi(0 \times \mathbb{R}^e \times [0,1)^d)$

$c, d, e \in \mathbb{Z}$

$\uparrow$ codomain of C

and d and e depends on x.

$\underline{\text{Defn}}$:- Let C be smooth manifold transverse to the ridges of a smooth manifold A. The blow up $Bl(A,C)$ is the unique smooth manifold $Bl(A,C)$ (with possible ridges) equipped with a cannonical smooth projection

$$p_b : Bl(A,C) \to A$$

called the blowdown map st

a, the restriction of p_b to $p_b^{-1}(A\setminus C)$ is a cannonical diffeo on $A\setminus C$ which identifies $p_b^{-1}(A\setminus C)$ with $A\setminus C$.

b, there is a cannonical identification of $p_b^{-1}(C)$ with the total space $SN(C)$ of the unit normal bundle to C in A.

c, the restriction of p_b to $p_b^{-1}(C) = SN(C)$ is the bundle projection from $SN(C)$ to C.

d, any smooth diffeomorphism $\phi: \mathbb{R}^c \times \mathbb{R}^e \times [0,1)^d$ onto an open subset $\phi(\mathbb{R}^c \times \mathbb{R}^e \times [0,1)^d) \subset A$ whose image intersects C exactly along $\phi(0 \times \mathbb{R}^e \times [0,1)^d)$ where $e, d, c \in \mathbb{Z}$ provides a smooth embedding $\sim$

$$[0,\infty) \times S^{c-1} \times (\mathbb{R}^{c} \times [0,1)^{d}) \xrightarrow{\ \phi\ } Bl(A,C)$$

$$(\lambda, v, x) \longmapsto \phi(\lambda v, x)$$

$$(0, v, x) \longmapsto T\phi(0,x)(v) \in SN_A(C)$$

with open image in $Bl(A,C)$.

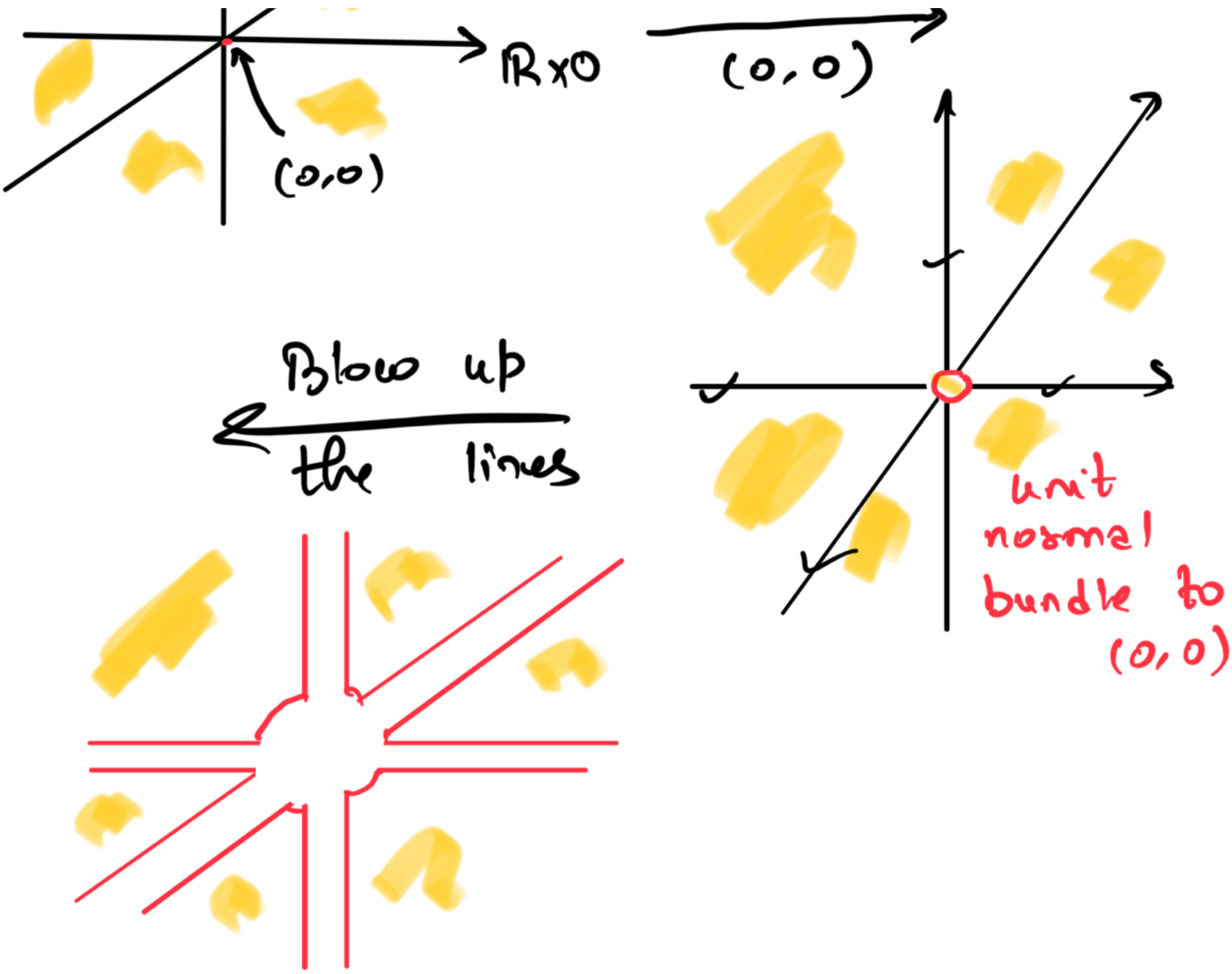
<u>Immediate Proposition</u>:- The blow-up manifold $Bl(A,C)$ is homeomorphic to the complement of A of an open tubular nbd of C.

In particular $Bl(A,C)$ is homotopy equiv to $A \setminus C$.

If C and A are compact $\Rightarrow Bl(A,C)$ is compact, it is a smooth compactification of $A \setminus C$.

<u>example</u>

Blow up



Proposition:- Let B and C be two smooth manifolds transvers to the ridges of a smooth manifold A.
Assume that C is a smooth manifold of B transverse to the ridges of B.

a) The closure $(\overline{B\setminus C})$ of $B\setminus C$ in $Bl(A,C)$ is a submanifold of $Bl(A,C)$ which intersects

$\mathrm{Bl}(A,C)$ ...

$$SN(C) \subseteq \partial \mathrm{Bl}(A,C)$$

as the unit normal bundle $SN_B(C)$ to C in B.

It is cannonically diffeomorphic to $\mathrm{Bl}(B,C)$

b, The blow-up $\mathrm{Bl}(\mathrm{Bl}(A,C), \overline{B\setminus C})$ of $\mathrm{Bl}(A,C)$ along $\overline{B\setminus C}$ has a cannonical diff. structure of a manifold with corners and the pre-image of $\overline{B\setminus C} \subset \mathrm{Bl}(A,C)$ in $\mathrm{Bl}(\mathrm{Bl}(A,C) \setminus \overline{B\setminus C})$ under the cannonical projection

$$\mathrm{Bl}(\mathrm{Bl}(A,C), \overline{B\setminus C}) \longrightarrow \mathrm{Bl}(A,C)$$

is the pull back via blow down projection $(\overline{B\setminus C} \longrightarrow B)$ of unit normal bundle to B in A.

<u>Proof</u>:- (a) Choose an embedding ϕ into A st $\phi(\mathbb{R}^c \times \mathbb{R}^e \times [0,1)^d)$

intersects C exactly along
$\phi(0\times \mathbb{R}^e \times [0,1)^d)$ and intersects
B exactly along
$\phi(0\times \mathbb{R}^k \times [0,1)^d)$ $k>e$

The induced chart $\tilde{\phi}$ of $Bl(A,C)$
near a point of $\partial Bl(A,C)$.

The intersection of $(B\setminus C)$ with the
image of $\tilde{\phi}$ is
$\tilde{\phi}((0,\infty)\times(0\times S^{k-e-1}\subset S^{c-1})\times\mathbb{R}^e\times[0,1)^d)$

Thus, the closure of $(B\setminus C)$ intersects
the image of $\tilde{\phi}$ as
$\tilde{\phi}([0,\infty)\times(0\times S^{k-e-1}\subset S^{c-1})\times[0,1)^d)$.

b) Along with the charts in (a) of
$\overline{B\setminus C}$, the smooth injective map

$$\mathbb{R}^{e-k+e}\times S^{k-e-1}\longrightarrow S^{c-1}$$

$$(u,y)\longmapsto\frac{(u,y)}{\|(u,y)\|}$$

identifies $\mathbb{R}^{c-k+e}$ with fibres of

the normal bundle $\overline{B\setminus C}$ in $Bl(A, C)$.
The blow-up process replaces $(\overline{B\setminus C})$ by the quotient of $(\mathbb{R}^{c-k+e} \setminus \{0\})$ bundle by $(0, \infty)$, which is the pull back under the blow down projection $(\overline{B\setminus C} \to B)$ of the unit normal bundle to B in A.

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The fiber $SN_e(C) = (N_c(C) \setminus \{0\})/\mathbb{R}^{+*}$ is oriented as the boundary of a unit ball of $N_c(C)$.

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R is the rational homology sphere.
$\check{R} = R \setminus \{\infty\}$

Def<u>n</u>:- The <u>configuration space</u> $C_2(R)$ is the compact G manifold with boundary and ridges obtained from R^2 by blowing up $\underline{(\infty, \infty)}$ in R^2 $\overline{\{\infty\} \times \check{R}}$, and the closures of

$\breve{R} \times \{\infty\}$ and the diagonal of $\breve{R}^2$ in $Bl(R^2, (\infty, \infty))$ succesively.

$\Rightarrow \partial C_2(R)$ contains unit normal bundle $\left(\dfrac{T\breve{R}^2}{\Delta(T\breve{R}^2)} \setminus \{0\} \right) / \mathbb{R}^{+*}$

to the diagonal of $\breve{R}^2$.

This bundle is cannonically isomorphic to the unit tangent bundle $U\breve{R}$ to $\breve{R}$ via $[x, y] \longmapsto [y - x]$.

<u>Lemma</u>:- Let $\breve{C}_2(R) = \breve{R}^2 \setminus \Delta(\breve{R}^2)$.
The open manifold $C_2(R) \setminus \partial C_2(R)$ is $\breve{C}_2(R)$ and $\breve{C}_2(R) \hookrightarrow C_2(R)$ is a homotopy equivalence.
In particular, $C_2(R)$ is a compactification of $\breve{C}_2(R)$ homotopy equiv to $\breve{C}_2(R)$ and it has the same rational homology as a sphere S^2.

The manifold $C_2(R)$ is a smooth compact 6-manifold with boundaries and ridges. There is a cannonical smooth proj.

$$p_{R^2} : C_2(R) \longrightarrow R^2$$

$$\partial C_2(R) = p_{R^2}^{-1}(\infty,\infty) \cup (S^2_\infty \times \check{R}) \cup (-\check{R} \times S^2_\infty) \cup U\check{R}$$

Proof// Let $B_{1,\infty}$ be the complement of open ball of radius 1 of $\mathbb{R}^3$ in S^3. Blowing up (∞,∞) in $B_{1,\infty}$ transforms a nbd of (∞,∞) into a product $[0,1) \times S^5$.

Explicitly,

$$\psi : [0,1) \times S^5 \longrightarrow Bl(B^2_{1,\infty}, (\infty,\infty))$$

$$(\lambda, (x \neq 0, y \neq 0) \in S^5 \subset (\mathbb{R}^3)^2) \mapsto \left(\frac{1}{\lambda \|x\|^2} x, \frac{1}{\lambda \|y\|^2} y\right)$$

$$(\lambda, (0, y \neq 0) \in S^5 \subset (\mathbb{R}^3)^2) \mapsto$$

$$\left(\infty, \frac{1}{\lambda \|y\|^2} y\right)$$

$$(\lambda, (x \neq 0, 0)) \longmapsto \left(\frac{1}{\lambda \|x\|^2} x, \infty\right)$$

which is a diffeomorphism onto its open image.

The explicit image of $\left(\lambda \in (0,1), (x \neq 0, y \neq 0) \in S^5 \subset (\mathbb{R}^3)^2\right)$ is written as $(\mathring{B}_{1,\infty} \setminus \{\infty\})^2 \subset Bl(\mathring{B}^2_{1,\infty}, (\infty,\infty))$ where $(\mathring{B}_{1,\infty} \setminus \{\infty\}) \subset \mathbb{R}^3$.

The image of ψ is a nbd of the preimage of (∞,∞) under the blowdown map $Bl(R^2, (\infty,\infty)) \xrightarrow{p_1} R^2$

This nbd respectively intersects $\infty \times \breve{R}$, $\breve{R} \times \infty$ and $\Delta(\breve{R}^2)$ as $\psi((0,1) \times 0 \times S^2)$, $\psi((0,1) \times S^2 \times 0)$ and $\Delta(\breve{R}^2)$ in $Bl(R^2, (\infty,\infty))$ intersects the boundary $\psi(0 \times S^5)$ of $Bl(R^2, (\infty,\infty))$ as the disjoint spheres in S^5 and form $\infty \times Bl(R,\infty)$, $Bl(R,\infty) \times \infty$ and $\Delta(Bl(R,\infty)^2)$.

These blow ups preserves the product structure $\psi([0,1) \times \_\_)$.

Thus, $C_2(R)$ is a smooth compact 6-manifold with boundary, with 3 ridges $S^2 \times S^2$ in $p_{R^2}^{-1}(\infty,\infty)$. A nbd of these ridges in $C_2(R)$ is a diffeomorphism to $[0,1)^2 \times S^2 \times S^2$.

<u>Defn</u> Let τ_s denote the standard parallelization of $\mathbb{R}^3$.

By a <u>parallelization</u>

$$\tau : \breve{R} \times \mathbb{R}^3 \longrightarrow T\breve{R}$$

of $\breve{R}$ that coincide with τ_s on $\mathring{B}_{2,\infty} \setminus \{\infty\}$ is asymptotically standard.

<u>Proposition:-</u> For any asymptotically standard parallelization τ of $\breve{R}$ $\exists$ a smooth map
$p_\tau : \partial C_2(R) \longrightarrow S^2$ st

$$p_\tau = \begin{cases} p_{S^2} & \text{on } p_{R^2}^{-1}(\infty,\infty) \\ -p_\infty \circ p_1 & \text{on } S^2_\infty \times \breve{R} \\ p_\infty \circ p_2 & \text{on } \breve{R} \times S^2_\infty \\ p_\tau & \text{on } U\breve{R} = \breve{R} \times S^2 . \end{cases}$$

where p_1 and p_2 denotes the first & second proj.

$\because$ $C_2(R)$ is homotopy equiv to $(\breve{R}^2 \setminus \Delta(\breve{R}^2))$

$\Rightarrow$ $H_2(C_2(R) ; \mathbb{Q}) = \mathbb{Q}[S]$ where the cannonical generator $[S]$ is the homology class of fiber of

$\cup \check{R} \subset \partial C_2(R)$.

For two component links (J, K) of $\check{R}$, the homology class $[J \times K]$ of $J \times K$ in $H_2(C_2(R); \mathbb{Q})$ is $lk(J, K)[S]$.

<u>Defn</u> :- An <u>assymptotic rational homology $\mathbb{R}^3$</u> is a pair $(\check{R}, \tau)$ where $\check{R}$ is a 3-manifold which the union of $(1,2] \times S^2$ of a rational homology ball B_R and the complement $\mathring{B}_{1,\infty} \setminus \{\infty\}$ of the unit ball of $\mathbb{R}^3$ and τ is the assymptotical standard parallelization of $\check{R}$.

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<u>Defn</u> :- A <u>volume 1-form</u> of S^2 is a 2-form ω_S of S^2 st $\int_S \omega_S = 1$

Defn :- A propagating chain of $(C_2(R), \tau)$ is a 4-chain P of $C_2(R)$ st $\partial P = p_\tau^{-1}(a)$ for $a \in S^2$.

Defn :- A propagating form of $(C_2(R), \tau)$ is a closed 2-form ω on $C_2(R)$ whose restriction to $\partial C_2(R)$ is $p_\tau^*(\omega_S)$ for some volume 1 form ω_S of S^2.

Note :- Propagating chains & propagating forms will be called just propagators!

Lemma :- Let $(\check{R}, \tau)$ be an assymptotic rational homology $\mathbb{R}^3$. Let C be a 2-cycle of $C_2(R)$. For any propagating chains P of $(C_2(R), \tau)$ transverse to C and for any propagating forms ω of $(C_2(R), \tau)$

$$[C] = \int_C \omega[S] = \langle C, P\rangle_{C_2(R)}[S]$$

in $H_2(C_2(R); \mathbb{Q}) = \mathbb{Q}[S]$.

In particular for any two component link (J,K) of $\breve{R}$,

$$lk(J,K) = \int_{J\times K} \omega = \langle J\times K, P\rangle_{C_2(R)}$$

<u>Proof</u>:- Fix a propagating chain P, the algebraic intersection $\langle C, P\rangle_{C_2(R)}$ only depends on the homology class $[C]$ of C in $C_2(R)$.

Similarly, $\because$ ω is closed, $\int_C \omega$ only depends on $[C]$.

Further, the dependence on $[C]$ is linear!

$\therefore$ It suffices to check for the lemma for a chain that represents the cannonical generator $[S]$ of $H_2(C_2(R);\mathbb{Q})$.

Any fiber $U\breve{R}$ is such a chain.

<u>**Defn**</u> :- A propagating form ω of (C_2R, τ) is <u>homogeneous</u> if its restriction to $\partial C_2 R$ is $p_\tau^*(\omega_{S^2})$ for the homogeneous volume form ω_{S^2} of S^2 of total volume 1.

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Let ι be the involution of $C_2(R)$ that exchanges two coordinates in $\breve{R}^2 \setminus \Delta(\breve{R}^2)$.

<u>Lemma</u>:- If ω_0 is a propagating form of $(C_2(R), \tau)$, then $(-\iota^*(\omega_0))$ and $\omega = \frac{1}{2}(\omega_0 - \iota^*(\omega_0))$ are propagating forms of $(C_2(R), \tau)$. Further, $\iota^*(\omega_0) = -\omega$ and if ω_0

is homogeneous $\Rightarrow (-\iota^*(\omega_0))$ and
$\omega = \frac{1}{2}(\omega_0 - \iota^*(\omega_0))$ are homogeneous.

Proof:-

There is a volume 1-form of S^2

st $\omega_0|_{\partial C_2(R)} = p_2^*(\omega_s)$

st $(-\iota^*(\omega_0))|_{\partial C_2(R)} = p_2^*(-\iota_{S^2}^*(\omega_s))$

when ι_{S^2} is the antipodal map

which sends x to $\iota_{S^2}(x) = -x$

and $(-\iota_{S^2}^*(\omega_s))$ is a volume 1-form

of S^2