The Quickhull Algorithm for Higher Dimensional Convex Hulls

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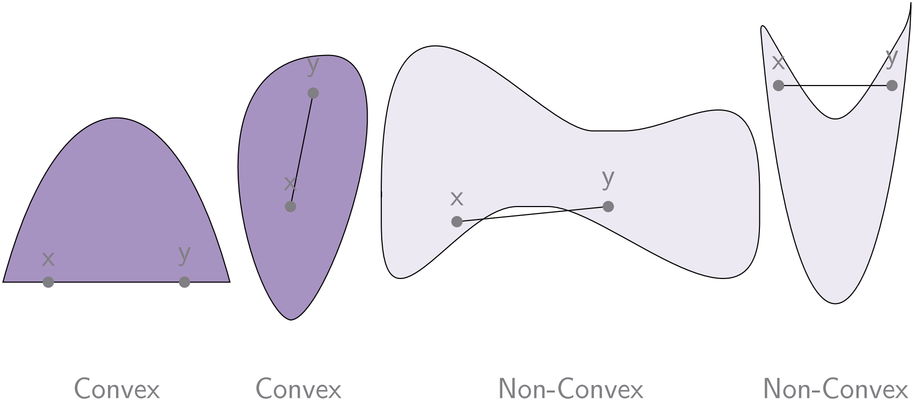
Saskatchewan is flat as a pancake, this is first time seeing a hill. In Québec, July 1st is also colloquially called 'moving day', because many leases for apartments get over in this time. Alaska is being tied and gagged, because it's very Canadian dream to kidnap Alaska!

2013 - r/polandball

Content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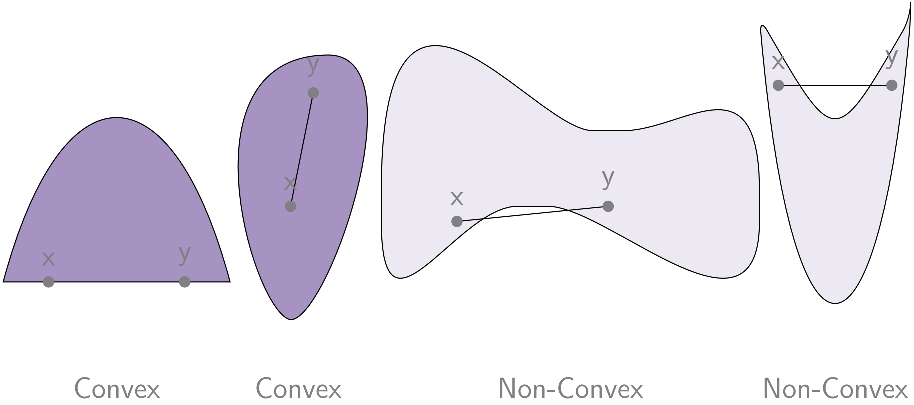
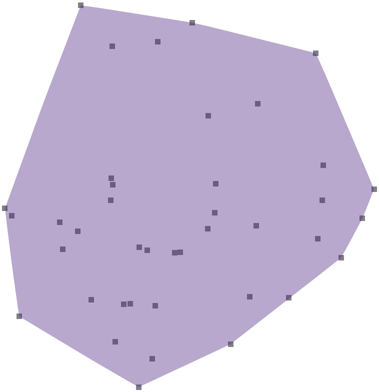


A set S is called convex if the line joining any two points in S is in S, i.e.,

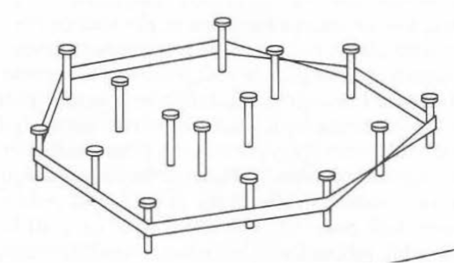
$$\forall x, y \in S, \forall \lambda \in [0, 1], \lambda x + (1 - \lambda) y \in S.$$

The convex hull of a set S the set of all convex combination of S, i.e.,

$$\textbf{conv}(S) := \{\Sigma_{i=1}^{n}\lambda_i x_i \mid \Sigma_{i=1}^{n}\lambda_i = 1, \lambda_i \in [0,1]\}.$$



Existence and uniqueness of a convex hull.



1984 – Dewdney's Analog Gadgets

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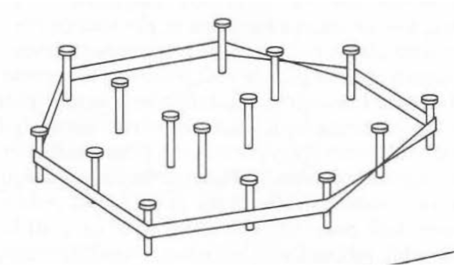

1984 – Dewdney's Analog Gadgets

The **conv**(S) is the smallest convex set containing S, i.e. it is the intersection of all convex sets containing S.

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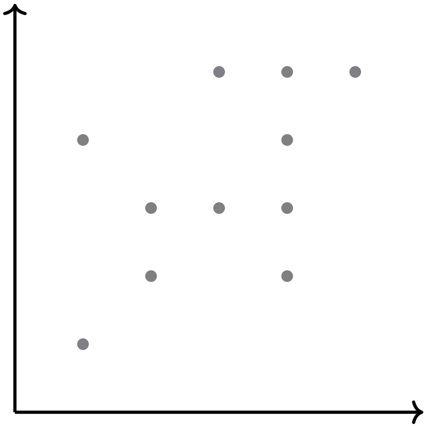
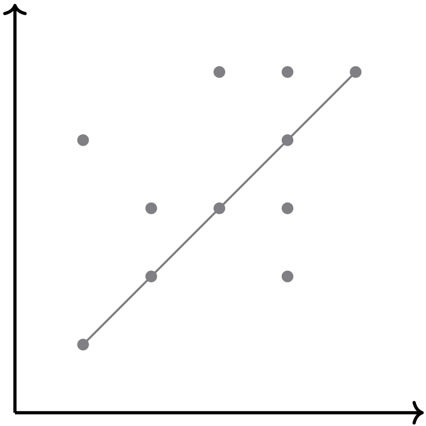
- Best optimal output-sensitive algorithm created by Chan in 1996.
- Higher dimensional Quickhull was invented in 1996 by C. Bradford Barber, David P. Dobkin, and Hannu Huhdanpaa.

1. Find the point min-x, and another point max-x.



1. Find the point min-x, and another point max-x.
2. Join these two points with a line L that divides the shape into two parts.



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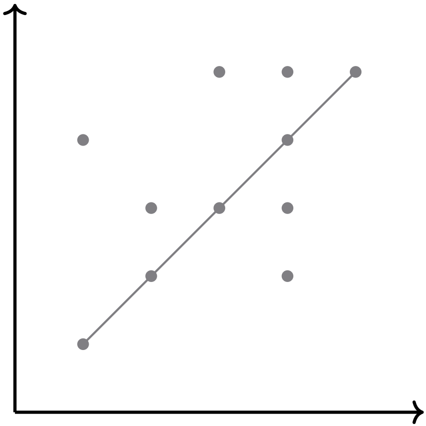

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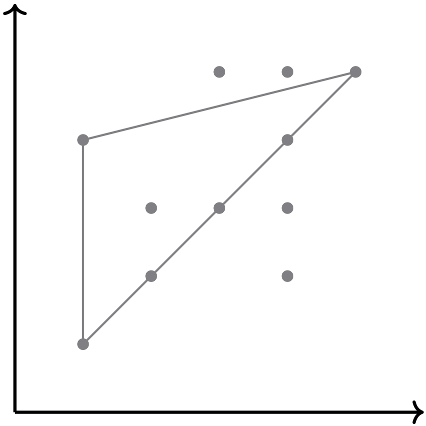
3. Find the point P outwards with the maximum distance from L.
4. Join P with min-x and max-x, ignore all other points inside the triangle (convex hull).



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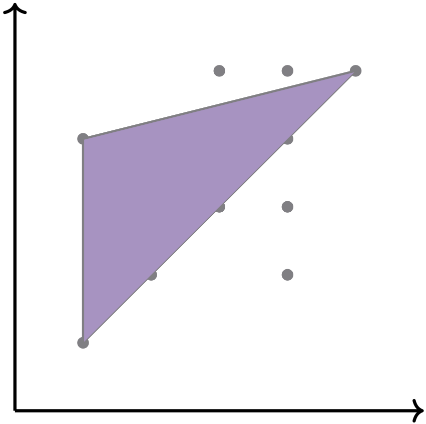
5. Repeat step 3 and step 4 on the two lines formed by the two new sides of the triangle until all points are inside the convex hull.



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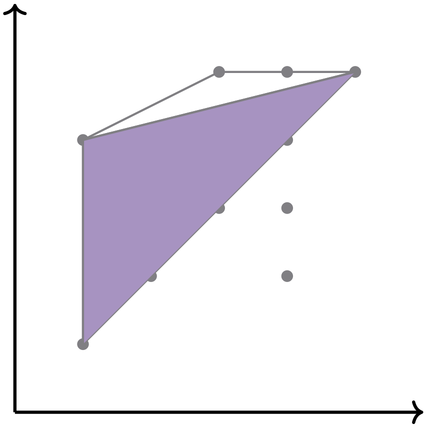

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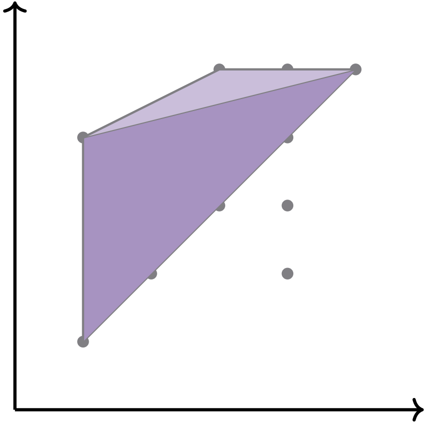

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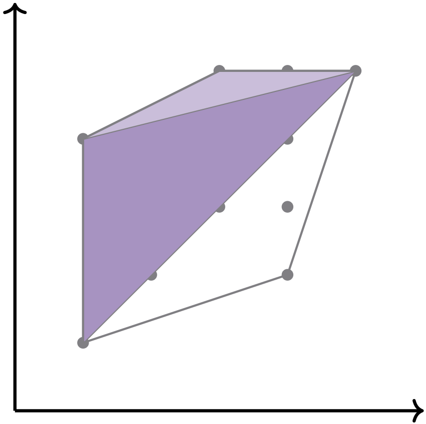

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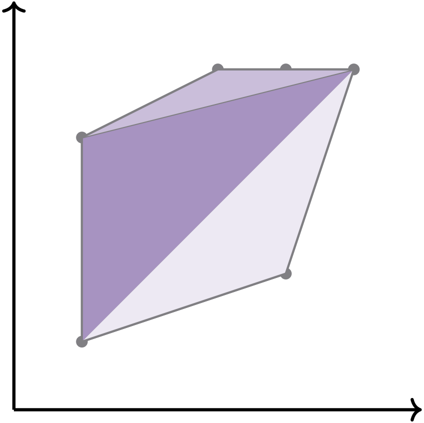

A nice example of QuickHull algorithm in Python, which randomly generates a set of points and finds the convex hull can be found at AnantJoshiCZ/QuickHull.git.

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A face is an intersection of convex polytope with supporting hyperplane.
An $(n-1)$-dimensional face of a polytope is called a facet and $(n-2)$-dimensional face is called a ridge.

The main geometric operation used in the quickhull algorithm is the signed distance from a point $\vec{p} \in \mathbb{R}^n$ to a hyperplane:

$$\frac{\vec{n} \cdot \vec{p} - b}{||\vec{n}||}$$

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Normal vector $\vec{n} = (a_1, \ldots, a_n)$. Let, $\vec{x_0}$ be a point in the hyperplane then, $\vec{q} = \vec{p} - \vec{x_0}$. Hence, $proj_{\vec{n}}\vec{q} = \frac{\vec{n} \cdot \vec{q}}{||\vec{n}||} = \frac{\vec{n} \cdot (\vec{p} - \vec{x_0})}{||\vec{n}||} = \frac{\vec{n} \cdot \vec{p} - \vec{n} \cdot \vec{x_0}}{||\vec{n}||} = \frac{\vec{n} \cdot \vec{p} - b}{||\vec{n}||}$, as $\vec{n} \cdot (\vec{x} - \vec{x_0}) = 0 \iff \vec{n} \cdot x = \vec{n} \cdot \vec{x_0} = b$.

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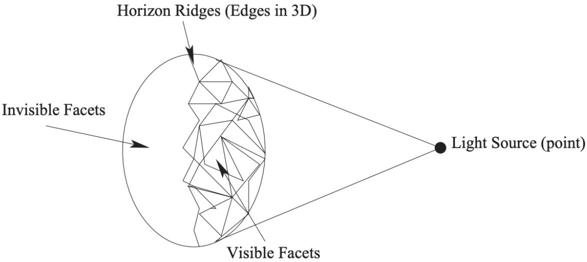
If the distance is positive, we say the point is above the hyperplane, else the point is below the hyperplane.

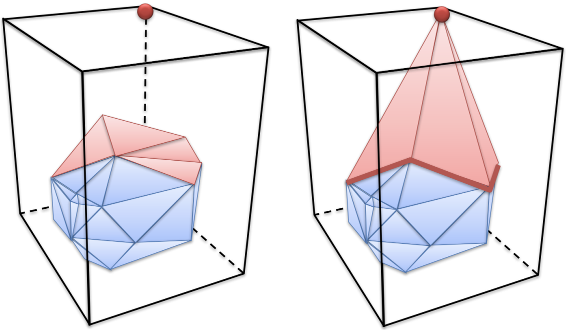


2005 - Agarwal

Theorem (Simplified Beneath-Beyond, 1961 - Grünbaum)

If S is a convex hull in $\mathbb{R}^n$ and P be a point in $\mathbb{R}^n \setminus S$, then F is a facet of $\textbf{conv}(P \cup S)$ *if and only if*:

- *F is a facet of S, and P is below F, or*
- *F is not a facet of S, and its vertices are P and the vertices of the ridges of S with one facet below P and other facet above P.*



2010 - Farag

1. Create initial convex hull Hull by choosing $(n+1)$-points which do not share a plane or a hyperplane.
2. For each facet F in Hull, find all unassigned points above it and add them to F's outside set F^c.
3. For each F with non-empty F^c,
   1. Find the point P with maximum distance from F and add it to Fac.
   2. Create a visible set V and initialize it to F.
   3. The boundary of V froms the set of horizon ridges H.
   4. Add the facets created from P and H to Fac.
   5. Delete all the points of F^c which are created from facets of V.
   6. For each Fac\Hull go to step 3.2
   7. Delete the internal facets from Hull and add Fac\Hull to Hull.
4. Repeat step 2 and step 3 until all points are points are inside the convex hull.

A nice visual implementation of QuickHull algorithm in 3-dimension can be found at carolhmj/quickhull-3d.git.

Lemma

Every extreme point of the input is added to the convex hull irrespective of which outside set it has been assigned to.

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Prove by contradiction: assume an extreme point P which does not belong to any outisde set and hence do not belong to Hull. Since, P is an extreme point it must belong to atleast one outside set for F in the initial Hull. By assumption, there is a point Q with P in its visible outside sets but not in its new outside sets. By definition P is above a visible facet and below all new visible facets for Q, which implies that P is in Hull and hence not an extreme point.

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The quickhull algorithm produces a convex hull of a set of points in $\mathbb{R}^n$.

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The algorithm begins with a convex hull of $(n+1)$-points and partitions points into outside sets and chooses the point with maximum distance. By last lemma and the Simplified Beneath-Beyond theorem quickhull algorithm produces convex hull of processed points.

Grahm Scan	$\mathcal{O}(nlogn)$
sorted points	$\mathcal{O}(n)$
Jarvis March or Gift wrapping	$\mathcal{O}(nh)$
all points on convex hull	$\mathcal{O}(n^2)$
Quickhull for dimension ≤ 3	$\mathcal{O}(nlogh)$
for dimension $d > 3$ [1996 - Klee]	$\mathcal{O}(\frac{n}{h}(\mathcal{O}(h^{\lfloor d/2 \rfloor}/\lfloor d/2 \rfloor!)))$
Kirkpatrick–Seidel algorithm	$\mathcal{O}(nlogh)$
Chan's algorithm	$\mathcal{O}(nlogh)$

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- Data Visualization: Graham's scan, Quickhull, and Chan's algorithm are popular choices due to their efficiency and ease of implementation.