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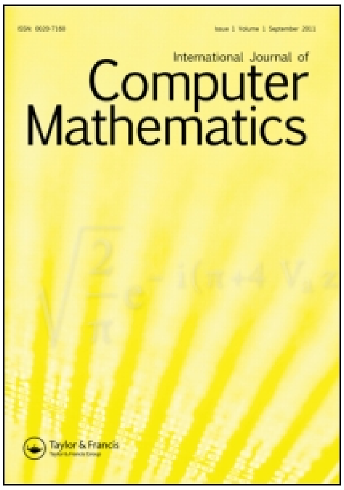

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A PARALLEL NUMERICAL ALGORITHM FOR FREDHOLM INTEGRO-DIFFERENTIAL TWO-POINT BOUNDARY VALUE PROBLEMS

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Numerical methods for second order differential equations with two-point boundary conditions are incorporated into a three part method for the solution of a second order nonlinear Fredholm integro-differential equation. The interest in this paper is the development of an algorithm for parallel processing the discrete nonlinear system. Numerical examples are given.

Keywords: Fredholm boundary value problems; Parallel processing; Toeplitz systems

C. R. Category: G.1.9

1. INTRODUCTION

In a paper by Garey, Gilmore and Gladwin [4], direct methods for both nonsingular and singular Fredholm type problems with two-point boundary conditions for linear problems were considered. A follow up paper [6] considered the nonlinear singular problem using an adaptation of the method in [3]. In this article, the work in [8] leads to the implementation of an iterative process for solving a nonlinear problem of the form of Eq. (1). Here the coefficient matrix for the numerical approximation is nonsymmetric but can be identified with a banded symmetric coefficient matrix and an

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additional sparse matrix. Taking advantage of this leads to the application of a parallel algorithm for the numerical solution. First, let us state the problem:

Consider a nonlinear second order Fredholm integro-differential equation (FIDE) of the form

$$y'' = f(x, y, z), y(0) = y_0, \quad y(a) = y_a \tag{1}$$

where

$$z(x) = \int_0^a K(x, t, y(t))dt, \quad 0 \le x \le a$$

The function $y(x)$ is unknown and f and K are given. For equations of the form (1) and defined for points S and T, where

$$S = \{(x, y, z): 0 \le x \le a, |y| < \infty, |z| < \infty\}$$

and

$$T = \{(x, t, y): 0 \le t, x \le a, |y| < \infty\},$$

we assume:

(i) f and K are uniformly continuous in each variable
(ii) for all (x, y, z), $(x, \bar{y}, z)$, and $(x, y, \bar{z})$ in S, f satisfies

$$|f(x, y, z) - f(x, \bar{y}, z)| \le L_1 |y - \bar{y}|$$
$$|f(x, y, z) - f(x, y, \bar{z})| \le L_2 |z - \bar{z}|$$

(iii) for all (x, t, y) and $(x, t, \bar{y})$ in T, K satisfies

$$|K(x, t, y) - K(x, t, \bar{y})| \le L_3 |y - \bar{y}|$$

and

(iv) the functions f_y, f_z and K_y are continuous and satisfy $f_y \ge 0, f_z \ge 0$ and $K_y \ge 0$ for all points in S and T.

To obtain numerical approximations, $[0, a]$ is partitioned with

$$I_N = \{x_n: x_n = nh, n = 0(1)N, h > 0, Nh = a\}.$$

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A general k-step method of solution is defined by

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i f(x_{n+i}, y_{n+i}, z_{n+i}), n = 0(1)N - k \tag{2}$$

with

$$z_n = h \sum_{j=0}^{N} w_j K(x_n, x_j, y_j), n > s, z_0 = 0$$

where y_i denotes an approximation to $y(x_i)$ and where $\{w_j\}$ denote the weights of the quadrature rule, s is related to the order of the method and $\{\alpha_i\}$ and $\{\beta_i\}$ are, respectively, the coefficients of the characteristics polynomials (ρ, σ). For Eq. (2),

$$\rho(z) = \sum_{i=0}^{k} \alpha_i z^i, \sigma(z) = \sum_{i=0}^{k} \beta_i z^i$$

Methods for integro-differential equations are denoted by triples $((\rho, \sigma), Q, IP)$ where Q denotes the quadrature rule to approximate the integral and IP is the iteration procedure chosen. Here we consider, as an example, the five point rule [1, 2] for solving differential equations defined by

$$\rho(z) = z^4 - 16z^3 + 30z^2 - 16z + 1, \rho(z) = -12z^2$$

To approximate the integral term, the quadrature rule is chosen to be a Newton–Cotes or Newton–Gregory rule which is compatible with the order of the method defined by (ρ, σ). For this method and in addition to the two boundary conditions, two auxiliary conditions are required. From [9], the natural auxiliary conditions for this method are given by

$$10y_0 - 15y_1 - 4y_2 + 14y_3 - 6y_4 + y_5 = 12h^2 f(x_1, y_1, z_1)$$

and

$$y_{N-5} - 6y_{N-4} + 14y_{N-3} - 4y_{N-2} - 15y_{N-1} + 10y_N = 12h^2 f(x_{N-1}, y_{N-1}, z_{N-1}).$$

With these equations added, respectively, as the first and $(N-1)$th equation in the system, we have the following

$$CY = 12h^2 F(Y) + R \tag{3}$$

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where

$$C = \begin{pmatrix} -15 & -4 & 14 & -6 & 1 & & \\ 16 & -30 & 16 & -1 & 0 & & \\ -1 & 16 & -30 & 16 & -1 & & \\ & & \ddots & & \ddots & & \\ & & 1 & -6 & 14 & -4 & -15 \end{pmatrix}$$

$$F(Y) = \begin{pmatrix} f_1 \\ f_2 \\ \\ \vdots \\ \\ f_{N-2} \\ f_{N-1} \end{pmatrix}, \quad R = \begin{pmatrix} -10y_0 \\ y_0 \\ 0 \\ \vdots \\ 0 \\ y_N \\ -10y_N \end{pmatrix}$$

Here, we have an $N-1 \times N-1$ system as the boundary conditions have been substituted, leading to the terms in R on the right hand side. We note that the above method solves the problem in its given form. Expressing the problem (1) as a system of two first order equations and subsequently finding a numerical approximation to the unknown function $y(x)$ and its derivative are considered in [6]. System (3) is solved by an iteration procedure (*IP*). The coefficient matrix C is banded and near Toeplitz. A number of recent articles have appeared which discuss the solution of linear systems which are banded and Toeplitz. A paper by Yan and Chung [12] provides a LU factorization into a pair of Toeplitz matrices following a perturbation of the given system. Another method based on the method of odd/even reduction [11] offers an approach to solve the system by parallel processing. In addition, we see in [5] a treatment of five-band systems which can be factored. Once factored, the possibilities for solution include both the odd/even method and a parallel version of the work in [12]. For this iteratively solved problem, the work by Shaw [10] provides some insight regarding the processing of the right hand side of system (3). In the following sections, we consider how the iterative solution of system (3) is enhanced in these two ways–processing the approximations to the integral terms on the right hand side and implementing a parallel version of the method in [12] as part of the iterative solution of the factored system. Two numerical examples are presented and solved in the final section.

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2. THE PARALLEL ALGORITHM

Let us consider a system to be solved iteratively of the form

$$Ax^{(i+1)} = \mathbf{b}_i \equiv \mathbf{b}(x^{(i)}) \tag{4}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & & & \\ \beta & \alpha & \beta & \gamma & 0 & & & \\ \gamma & \beta & \alpha & \beta & \gamma & & & \\ & & \ddots & & \ddots & & & \\ & & 0 & \gamma & \beta & \alpha & \beta \\ & & a_{15} & a_{14} & a_{13} & a_{12} & a_{11} \end{pmatrix} \tag{5}$$

where A is nonsingular. The matrix A can be written as $A = B + E$ with

$$B = \begin{pmatrix} \alpha' & \beta & \gamma & & & & \\ \beta & \alpha & \beta & \gamma & 0 & & \\ \gamma & \beta & \alpha & \beta & \gamma & & \\ & & \ddots & & \ddots & & \\ & & 0 & \gamma & \beta & \alpha & \beta \\ & & & & \gamma & \beta & \alpha' \end{pmatrix}$$

and

$$E = \begin{pmatrix} a_{11}-\alpha' & a_{12}-\beta & a_{13}-\gamma & a_{14} & a_{15} & & \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & & \\ & & \ddots & & \ddots & & \\ & & 0 & 0 & 0 & 0 & 0 \\ & & a_{15} & a_{14} & a_{13}-\gamma & a_{12}-\beta & a_{11}-\alpha' \end{pmatrix}$$

where $\alpha' = \alpha - \gamma$. B is a near-Toeplitz, symmetric, sparse and banded matrix. In our example, $\gamma = 1$. In general, for $\gamma \neq 0$, let $\gamma D = B$ and write

$$D\bar{x}^{(i+1)} = \frac{1}{\gamma}\mathbf{b}_i \equiv \hat{\mathbf{b}}_i \tag{6}$$

where

$$D = \begin{pmatrix} \hat{\alpha}' & \hat{\beta} & 1 & & & \\ \hat{\beta} & \hat{\alpha} & \hat{\beta} & 1 & & \\ 1 & \hat{\beta} & \hat{\alpha} & \hat{\beta} & 1 & \\ & & & & \ddots & \\ & & 1 & \hat{\beta} & \hat{\alpha} & \hat{\beta} \\ & & & 1 & \hat{\beta} & \hat{\alpha}' \end{pmatrix}$$

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It is known (see [5]) that D can be factored into two tridiagonal matrices $D_1=\{1,d_1,1\}$ and $D_2=\{1,d_2,1\}$ with

$$d_{1,2}=\frac{\hat{\beta}\pm\sqrt{\hat{\beta}^2-4(\hat{\alpha}-2)}}{2}.$$

For the coefficient matrix C in system (3), we have $C_1=\{-1,2,-1\}$ and $C_2=\{1,-14,1\}$. For the matrix D, let D_2 be a diagonally dominant Toeplitz matrix. Then

$$D_2D_1\bar{x}^{(i+1)}=\hat{\mathbf{b}}_i$$

and we solve a pair of systems

$$D_2x_1^{(i+1)}=\hat{\mathbf{b}}_i$$

and

$$D_1\bar{x}^{(i+1)}=x_1^{(i+1)}$$

The matrix D_2 can be written in the form

$$D_2=D_{21}+D_{22}$$

where

$$D_{22}=\left[\begin{array}{cc|cc} -b & & & 0 \\ & & 1 & \\ \hline & 1 & -b & \\ 0 & & & 0 \end{array}\right]$$

and

$$D_{21}=\left[\begin{array}{ccccc|ccccc} a & 1 & & & & & & & & \\ 1 & d & 1 & & & & & & & \\ & & \ddots & & & & & & & \\ & & 1 & d & 1 & & & & & \\ & & & 1 & d & & & & & \\ \hline & & & & & a & 1 & & & \\ & & & & & 1 & d & 1 & & \\ & & & & & & & \ddots & & \\ & & & & & & & 1 & d & 1 \\ & & & & & & & & 1 & d \end{array}\right]$$

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Again, a perturbed system is to be solved

$$D_{21}\bar{x}_1^{(i+1)} = \hat{\mathbf{b}}_i \tag{7}$$

By its description, this can be written as two systems and they can be solved in parallel. Each diagonal block in D_{21} is taken to be of size k if $n=2k$, otherwise the sizes are m and k, where $m=k+1$. Thus, we have

$$\begin{aligned} D_{21\cdot m}\bar{\mathbf{x}}_{1\cdot m}^{(i+1)} &= \hat{\mathbf{b}}_{i\cdot m} \\ D_{21\cdot k}\bar{\mathbf{x}}_{1\cdot k}^{(i+1)} &= \hat{\mathbf{b}}_{i\cdot k}. \end{aligned} \tag{8}$$

Each system is solved independently.

With reference to the FIDE, recall that the νth component of $\hat{\mathbf{b}}_i$ is given by

$$\hat{b}_\nu = \frac{1}{\gamma}h^2\sum_{j=0}^{k}\beta_j f\left(x_{\nu+j-1}, y_{\nu+j-1}^{(i)}, z_{\nu+j-1}\right)$$

with

$$z_{\nu+j-1} = h\sum_{m=0}^{N} w_m K\left(x_{\nu+j-1}, x_m, y_m^{(i)}\right).$$

Note that $y_0^{(i)} = y_0$ and $y_N^{(i)} = y_a \forall i$.

The two systems in (8) can be solved separately, but the solution components must be shared before the process is repeated. The idea of splitting the system can be extended to four or more subsystems to be solved separately and in parallel. As we shall see in the next section, correcting for the perturbations requires that each subsystem be of a certain size in order to accommodate the desired accuracy of the final approximation.

3. CORRECTING THE PERTURBED SYSTEMS

Two systems have been perturbed. We consider first correcting for D_{22}. Denoting the solution for the pair of systems (8), by $\bar{\mathbf{x}}_1^{(i+1)}$, we have

$$D_2\bar{\mathbf{x}}_1^{(i+1)} = D_{21}\bar{\mathbf{x}}_1^{(i+1)} + D_{22}\bar{\mathbf{x}}_1^{(i+1)}$$

and

$$D_2\bar{\mathbf{x}}_1^{(i+1)} = D_2\mathbf{x}_1^{(i+1)} + D_{22}\bar{\mathbf{x}}_1^{(i+1)}.$$

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When this method is used, we express $\mathbf{x}_1^{(i+1)}$ in terms of $\bar{\mathbf{x}}_1^{(i+1)}$ and a correction term before the second system involving D_1 is solved. Multiplying through by D_2^{-1} yields

$$\mathbf{x}_1^{(i+1)} = \bar{\mathbf{x}}_1^{(i+1)} - D_2^{-1} D_{22} \bar{\mathbf{x}}_1^{(i+1)}.$$

The matrix D_{22} has only four nonzero elements so we can write

$$\mathbf{x}_1^{(i+1)} = \bar{\mathbf{x}}_1^{(i+1)} - D_2^{-1}\left(- b\bar{x}_{1\cdot 1}^{(i+1)}\mathbf{e}_1 - b\bar{x}_{1\cdot m+1}^{(i+1)}\mathbf{e}_{m+1} + \bar{x}_{1\cdot m}^{(i+1)}\mathbf{e}_{m+1} + \bar{x}_{1\cdot m+1}^{(i+1)}\mathbf{e}_m\right).$$

To avoid calculating D_2^{-1}, vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ need to be determined such that

$$D_2\mathbf{p} = \mathbf{e}_1$$
$$D_2\mathbf{q} = \mathbf{e}_m$$
$$D_2\mathbf{r} = \mathbf{e}_{m+1}$$

To this end, let

$$\mathbf{p} = \{b, b^2, \ldots, b^t, 0, \ldots, 0\}^T$$

where $|b| < 1$.

We determine that

$$D_2\mathbf{p} = -e_1 + b^t(e_{t+1} - be_t).$$

Details are given in [12]. For the vectors $\mathbf{q}$ and $\mathbf{r}$, let

$$\mathbf{q} = \{\underbrace{0, \ldots, 0}_{m-t}, \underbrace{b^t, b^{t-1}, \ldots, b}_{t}, 0, \ldots, 0\}^T$$

and

$$\mathbf{r} = \{\underbrace{0, \ldots, 0}_{m}, \underbrace{b, b^2, \ldots, b^t}_{t}, 0, \ldots, 0\}^T.$$

From [8], we find

$$D_2\mathbf{q} = -\mathbf{e}_m + b\mathbf{e}_{m+1} + b^t(\mathbf{e}_{m-t} - b\mathbf{e}_{m-t+1})$$

and

$$D_2\mathbf{r} = -\mathbf{e}_{m+1} + b\mathbf{e}_m + b^t(\mathbf{e}_{m+t+1} - b\mathbf{e}_{m+t}).$$

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By considering expressions for

$$\left(\frac{1}{1-b^2}\right)D_2(\mathbf{r}+b\mathbf{q})$$

and

$$\left(\frac{1}{1-b^2}\right)D_2(\mathbf{q}+b\mathbf{r}),$$

we can write the approximate solution as

$$\mathbf{x}_1^{(i+1)} = \bar{\mathbf{x}}_1^{(i+1)} - b\bar{x}_{1\cdot 1}^{(i+1)}\mathbf{p} - \frac{1}{1-b^2} \left(\left(b\bar{x}_{1\cdot m+1}^{(i+1)} - \bar{x}_{1\cdot m}^{(i+1)}\right)(\mathbf{r}+b\mathbf{q}) - \bar{x}_{1\cdot m+1}^{(i+1)}(\mathbf{q}+b\mathbf{r})\right). \tag{9}$$

Multiplying through by D_2 and collecting the terms which give $(D_{21}+D_{22})\bar{x}_1^{(i+1)}$, we can replace this by $\hat{\mathbf{b}}_i$ and write

$$D_2\mathbf{x}_1^{(i+1)} - \hat{\mathbf{b}}_i = \frac{b^t}{1-b^2}\Big(-b(1-b^2)\bar{x}_{1\cdot 1}^{(i+1)}\mathbf{e}_{t+1} + b^2(1-b^2)\bar{x}_{1\cdot 1}^{(i+1)}\mathbf{e}_t + \bar{x}_{1\cdot m}^{(i+1)}\mathbf{e}_{m+t+1} - b\bar{x}_{1\cdot m}^{(i+1)}\mathbf{e}_{m+t} - \left((b^2-1)\bar{x}_{1\cdot m+1}^{(i+1)} - b\bar{x}_{1\cdot m+1}^{(i+1)}\right)\mathbf{e}_{m-t} + \left(b(b^2-1)\bar{x}_{1\cdot m+1}^{(i+1)} - b^2\bar{x}_{2\cdot m}^{(i+1)}\right)\mathbf{e}_{m-t+1}\Big)$$

THEOREM 1

$$\|D_2\boldsymbol{x}^{(i+1)} - \hat{\boldsymbol{b}}_i\| \le \frac{|b|^t}{1-b^2}\frac{2}{|d^2|-2}\|\hat{\boldsymbol{b}}_i\|$$

Proof See [8] for details.

The second correction relates to the matrix E in system (4). With that system, The solution $\bar{\mathbf{x}}^{(i+1)}$ coming from Eq. (6) is the solution to

$$Bx = \mathbf{b}.$$

Consider

$$A\bar{\mathbf{x}}^{(i+1)} = B\bar{\mathbf{x}}^{(i+1)} + E\bar{\mathbf{x}}^{(i+1)} \tag{10}$$

and recall that $A\mathbf{x}^{(i+1)} = \mathbf{b}(\bar{\mathbf{x}}^{(i)})$ and $B\bar{\mathbf{x}}^{(i+1)} = \mathbf{b}(\bar{\mathbf{x}}^{(i)})$ so that (10) can be rewritten

$$\mathbf{x}^{(i+1)} = \bar{\mathbf{x}}^{(i+1)} - A^{-1}E\bar{\mathbf{x}}^{(i+1)} \tag{11}$$

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From the definition of E, $E\bar{\mathbf{x}}^{(i+1)}$ contains only nonzero elements in the first and last row. Thus

$$E\bar{x}^{(i+1)} = z_1^{(i+1)}\mathbf{e}_1 + z_n^{(i+1)}\mathbf{e}_n$$

where $\mathbf{e}_1$ and $\mathbf{e}_n$ are the usual n-dimensional unit vectors and

$$\begin{aligned} z_1^{(i+1)} = {} & (a_{11} - \alpha')\bar{x}_1^{(i+1)} + (a_{12} - \beta)\bar{x}_2^{(i+1)} + (a_{13} - \gamma)\bar{x}_3^{(i+1)} \\ & + a_{14}\bar{x}_4^{(i+1)} + a_{15}\bar{x}_5^{(i+1)} \end{aligned}$$

and

$$\begin{aligned} z_n^{(i+1)} = {} & a_{15}\bar{x}_{n-4}^{(i+1)} + a_{14}\bar{x}_{n-3}^{(i+1)} + (a_{13} - \gamma)\bar{x}_{n-2}^{(i+1)} \\ & + (a_{12} - \beta)\bar{x}_{n-1}^{(i+1)} + (a_{11} - \alpha')\bar{x}_n^{(i+1)}. \end{aligned}$$

As in the case of the earlier correction, we attempt to find vectors $\mathbf{P}$ and $\mathbf{Q}$ satisfying $A\mathbf{P} = \mathbf{e}_1$ and $A\mathbf{Q} = \mathbf{e}_n$. Starting with $P = \{P_1, P_2, \ldots, P_n\}$, we note that it must satisfy each equation in the system and

$$\begin{gathered} a_{11}P_1 + a_{12}P_2 + a_{13}P_3 + a_{14}P_4 + a_{15}P_5 = 1 \\ \beta P_1 + \alpha P_2 + \beta P_3 + \gamma P_4 = 0 \\ \gamma P_i + \beta P_{i+1} + \alpha P_{i+2} + \beta P_{i+3} + \gamma P_{i+4} = 0;\ i = 1(1)n-4 \\ \gamma P_{n-3} + \beta P_{n-2} + \alpha P_{n-1} + \beta P_n = 0 \\ a_{15}P_{n-4} + a_{14}P_{n-3} + a_{13}P_{n-2} + a_{12}P_{n-1} + a_{11}P_n = 0 \end{gathered}$$

All equations with the exception of the first two and last two are satisfied by taking

$$P_i = \sum_{j=1}^{s} k_j r_j^i, s \leq 4$$

for arbitrary constants k_j and where r_j, $j = 1(1)s$ are the characteristic roots of

$$\gamma\delta^4 + \beta\delta^3 + \alpha\delta^2 + \beta\delta + \gamma = 0.$$

By choosing the arbitrary constants so that the remaining four equations are satisfied, the definition of $\mathbf{P}$ is complete. For the example in this paper, the four roots are $1, 1, 7 \pm 4\sqrt{3}$. In fact we only require three of these because

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the second and $(n-1)$st equations are dependent, so the largest root $7+4\sqrt{3}$ is not used.

In a similar manner, we can choose a vector $\mathbf{Q}$ such that $A\mathbf{Q}=\mathbf{e}_n$. It is easily seen that the components of $\mathbf{Q}$ are the same as those of $\mathbf{P}$ but in reverse order. As a result, Eq. (11) becomes

$$\mathbf{x}^{(i+1)} = \bar{\mathbf{x}}^{(i+1)} - z_1^{(i+1)}\mathbf{P} - z_n^{(i+1)}\mathbf{Q}. \tag{12}$$

The values for $k_j, j=1,2,3$ are only calculated once although they are used repeatedly with each iteration.

Remarks With reference to Eq. (6), we saw that $D=D_1D_2$ where $-D_1=\{1,-2,1\}$ and $D_2=\{1,-14,1\}$. This iteration procedure converges if

$$\begin{aligned} \|x^{(i+1)}-x\| &\leq 12h^2\|D_1^{-1}\|\|D_2^{-1}\|\|\hat{b}(x^{(i)})-\hat{b}\| \\ &\leq 12h^2\|D_1^{-1}\|\|D_2^{-1}\|L\|x^{(i)}-x\| \end{aligned}$$

where L is a Lipschitz constant. From this we see that convergence follows if

$$12h^2L\|D_1^{-1}\|\|D_2^{-1}\| \leq 1.$$

In particular, we note that D_1 and $-D_2$ are monotone matrices with $\|D_2^{-1}\| < \|D_1^{-1}\| \leq (n^2/8)$ (see [7], p. 360–371).

4. NUMERICAL EXAMPLES

In this section, the five-band method is used to solve two nonlinear examples. Two methods are employed to solve the systems that arise in the iteration process. The weights for the integration of the integral term are the Newton–Gregory fourth order weights.

Example 1

$$\begin{gathered} y'' = 2y^2/(x+1) - x + \int_0^1 x(t+1)y(t)dt \\ y(0)=0, y(1)=1/2 \end{gathered}$$

Exact solution: $y(x)=1/(x+1)$.

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Example 2

$$y'' = y + \frac{2x - (2x+1)e^2 + 3}{4e^2} + \int_0^1 (x+t)y(t)^2 dt$$
$$y(0) = 1, y(1) = 1/e$$

Exact solution: $y(x) = e^{-x}$.

4.1. Algorithm: Solving a Nonlinear FIDE

Given: The non symmetric coefficient matrix A

The right hand vector expression – nonlinear in the unknown vector **y**

Determine: The symmetric 5-band matrix B from (5)

Factor $B = D_2 D_1$, $D_1 = \{1, d_1, 1\}$, $D_2 = \{1, d_2, 1\}$

For D_2, calculate

$$a = \frac{-d_2 \pm \sqrt{(d_2^2 - 4)}}{2}, |a| > 1$$
$$b = a - d_2$$

Rewrite $D_2 = D_{21} + D_{22}$

Select a tolerance τ and determine

$$t(\tau) = \left\lceil \frac{\log(|d_2| - 2) + \log(\tau) + \log(|b^2 - 1|) - \log(2)}{\log(|b|)} \right\rceil.$$

For $i = 0$, select an initial vector $\mathbf{y}^{(i)} = \bar{\mathbf{y}}^{(i)}$

Loop:

Evaluate in parallel

$$\hat{\mathbf{b}}_i$$

Solve in parallel

$$D_{21\cdot m}\bar{\mathbf{y}}_{1\cdot m}^{(i+1)} = \hat{\mathbf{b}}_{i\cdot m}$$
$$D_{21\cdot k}\bar{\mathbf{y}}_{1\cdot k}^{(i+1)} = \hat{\mathbf{b}}_{i\cdot k}$$

Correct $\bar{\mathbf{y}}_1^{(i+1)}$ using

$$\mathbf{y}_1^{(i+1)} = \bar{\mathbf{y}}_1^{(i+1)} - b\bar{y}_{1\cdot 1}^{(i+1)}\mathbf{p}$$
$$- \frac{1}{1-b^2}\left(\left(b\bar{y}_{1\cdot m+1}^{(i+1)} - \bar{y}_{1\cdot m}^{(i+1)}\right)(\mathbf{r} + b\mathbf{q}) - \bar{y}_{1\cdot m+1}^{(i+1)}(\mathbf{q} + b\mathbf{r})\right).$$

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Solve $D_1\bar{\mathbf{y}}^{(i+1)} = \mathbf{y}_1^{(i+1)}$
Correct $\bar{\mathbf{y}}^{(i+1)}$ using

$$\mathbf{y}^{(i+1)} = \bar{\mathbf{y}}^{(i+1)} - z_1^{(i+1)}\mathbf{P} - z_{N-1}^{(i+1)}\mathbf{Q}$$

Test the Approximation
if $(|\mathbf{y}^{(i+1)} - \mathbf{y}^{(i)}| \geq \tau)$,
$i \leftarrow i+1$
$\bar{\mathbf{y}}^{(i)} \leftarrow \mathbf{y}^{(i)}$
Repeat from Loop
else
$\mathbf{y} \leftarrow \mathbf{y}^{(i+1)}$

Table I contains some results for these two numerical examples. Each was solved for three different values of N and the iteration procedure terminated using a tolerance of h^4. In addition, the problems were solved sequentially and using the parallel processing ideas in this article. The last column gives us the efficiency of the implemented algorithm. The algorithms were coded in Fortran 77 and parallel processed using the Portland Group's pgf 77 compiler on a Linux operating system. The multiprocessor is a Hewlett Packard four processor Pentium Pro. The method of solution was essentially based on a parallel implementation of the Yan and Chung [12] work. Although the split discussed in this work was based on a partition of the system into two parts, the four processor computer allowed us to test up to a four part split. It must be added that the right hand side of the system (3) involved the integration of the integral term over [0, 1] for each equation in the system. This allowed us the opportunity to implement a parallel approach for processing the integral terms as well.

In summary, the original system (3) has been replaced by a two part procedure consisting of Eqs. (6) and (12). In particular, (6) has been solved by a parallel version of Yan and Chung's [12] method and the final correction followed using (12). In [8], it was argued that for linear systems

TABLE I Sequential *verses* parallel

Example	*N*	*Max. error*	*Seq. time*	*Par. time*	*Efficiency*
1	512	−2.0079(−12)	0.7827	0.2263	88.47%
	1024	−1.5500(−13)	3.3270	0.9383	88.64%
	4096	−4.2538(−15)	64.317	17.497	91.90%
2	512	−1.7938(−12)	0.8400	0.2393	87.76%
	1024	−1.5128(−13)	3.8163	1.0243	93.14%
	4096	−4.1131(−15)	72.516	18.857	96.14%

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this was more efficient. Here a two part approach has provided an iterative solution for the nonlinear examples.

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