Prenex normal form

It is often more convenient to deal with formulas in which all quantifiers have been moved to the front of the expression. These types of formulas are said to be in prenex normal form.

Definition A formula is in *prenex normal form* if it is of the form

$$Q_1x_1\,Q_2x_2\ldots Q_nx_nB$$

where $Q_i (i = 1, \ldots n)$ is $\forall$ or $\exists$ and the formula B is quantifier free.

The string $Q_1x_1\,Q_2x_2\ldots Q_nx_n$ is called the *prefix* and B is called the *matrix*.

A formula with no quantifiers is regarded as a trivial case of a prenex normal form.

Example

Which of the following expressions are in prenex normal form?

$$\forall x P(x) \vee \forall x Q(x)$$

$$\forall x \forall y \neg (P(x) \rightarrow Q(y))$$

$$\forall x \exists y R(x, y)$$

$$R(x, y)$$

$$\neg \forall x R(x, y)$$

Algorithm for prenex normal form

Any expression can be converted into prenex normal form. To do this, the following steps are needed:

1. Eliminate all occurrences of $\rightarrow$ and $\leftrightarrow$ from the formula in question.

2. Move all negations inward such that, in the end, negations only appear as part of literals.

3. Standardize the variables apart (when necessary).

4. The prenex normal form can now be obtained by moving all quantifiers to the front of the formula.

To accomplish Step 1 (eliminate the $\rightarrow, \leftrightarrow$), make use of the following logical equivalences:

- $A \rightarrow B \;\; \models\!\!\!\dashv \neg A \vee B$.
- $A \leftrightarrow B \;\; \models\!\!\!\dashv (\neg A \vee B) \wedge (A \vee \neg B)$.
- $A \leftrightarrow B \;\; \models\!\!\!\dashv (A \wedge B) \vee (\neg A \wedge \neg B)$.

To accomplish Step 2 (move all negations inward, such that negations only appear as parts of literals), use the logical equivalences:

- De Morgan's Laws.
- $\neg\neg A \;\; \models\!\!\!\dashv A$.
- $\neg \exists x A(x) \;\; \models\!\!\!\dashv \forall x \neg A(x)$.
- $\neg \forall x A(x) \;\; \models\!\!\!\dashv \exists x \neg A(x)$.

To accomplish Step 3 (standardize variables apart), make use of the following result.

The following theorem allows one to rename the variables in order to make them distinct. Renaming the variables in a formula such that distinct variables have distinct names is called *standardazing the variables apart.*

Theorem (Replaceability of bound variable symbols)

Suppose A results from A' by replacing in A some (not necessarily all) occurrences of $Q(x)B(x)$ by $Q(y)B(y)$. Then $A \models\!\mid A'$ and $A \vdash\!\dashv A'$.

Note: As before, Q denotes either the existential or the universal quantifier.

Example

Standardize all variables apart in the following formula:

$$\forall x(P(x) \rightarrow Q(x)) \wedge \exists x Q(x) \wedge \exists z P(z) \wedge$$
$$\wedge \exists z(Q(z) \rightarrow R(x))$$

Solution: Use y for x in $\forall x$, u for x in $\exists x Q(x)$, and w for z in $\exists z P(z)$ to obtain:

$$\forall y(P(y) \rightarrow Q(y)) \wedge \exists u Q(u) \wedge \exists w P(w) \wedge$$
$$\wedge \exists z(Q(z) \rightarrow R(x))$$

To accomplish Step 4 (move all quantifiers in front of the formula) make use of the following logical equivalences:

- $A \wedge \exists x B(x) \;\; \models\!\!\!\!\dashv \; \exists x(A \wedge B(x))$, x not occurring in A.
- $A \wedge \forall x B(x) \;\; \models\!\!\!\!\dashv \; \forall x(A \wedge B(x))$, x not occurring in A.
- $A \vee \exists x B(x) \;\; \models\!\!\!\!\dashv \; \exists x(A \vee B(x))$, x not occurring in A.
- $A \vee \forall x B(x) \;\; \models\!\!\!\!\dashv \; \forall x(A \vee B(x))$, x not occurring in A.

(These equivalences essentially show that if a formula A has a truth value that does not depend on x, then one is allowed to quantify over x.)

More logical equivalences for Step 4

- $\forall x A(x) \land \forall x B(x) \;\models\!\mid\; \forall x (A(x) \land B(x))$.
- $\exists x A(x) \lor \exists x B(x) \;\models\!\mid\; \exists x (A(x) \lor B(x))$.
- $\forall x \forall y A(x, y) \;\models\!\mid\; \forall y \forall x A(x, y)$.
- $\exists x \exists y A(x, y) \;\models\!\mid\; \exists y \exists x A(x, y)$.
- $Q_1 x A(x) \land Q_2 y B(y) \;\models\!\mid\; Q_1 x Q_2 y (A(x) \land B(y))$.
- $Q_1 x A(x) \lor Q_2 y B(y) \;\models\!\mid\; Q_1 x Q_2 y (A(x) \lor B(y))$.

Where $Q_1, Q_2 \in \{\forall, \exists\}$.

For example, if $Q_1 = \forall$ and $Q_2 = \exists$, we have
$\forall x A(x) \land \exists y B(y) \;\models\!\mid\; \forall x \exists y (A(x) \land B(y)$.

Example

Find the prenex normal form of

$$\forall x(\exists y R(x,y) \wedge \forall y \neg S(x,y) \rightarrow \neg(\exists y R(x,y) \wedge P))$$

Solution:

- According to Step 1, we must eliminate $\rightarrow$, which yields

$$\forall x(\neg(\exists y R(x,y) \wedge \forall y \neg S(x,y)) \vee \neg(\exists y R(x,y) \wedge P))$$

- We move all negations inwards, which yields:

$$\forall x(\forall y \neg R(x,y) \vee \exists y S(x,y) \vee \forall y \neg R(x,y) \vee \neg P).$$

- Next, all variables are standardized apart:

$$\forall x(\forall y_1 \neg R(x,y_1) \vee \exists y_2 S(x,y_2) \vee \forall y_3 \neg R(x,y_3) \vee \neg P)$$

- We can now move all quantifiers in front, which yields

$$\forall x \forall y_1 \exists y_2 \forall y_3 (\neg R(x,y_1) \vee S(x,y_2) \vee \neg R(x,y_3) \vee \neg P).$$

Exercise

Transform the following formula into prenex normal form:

$$\neg[\forall x \exists y F(u,x,y) \rightarrow \exists x(\neg \forall y G(y,v) \rightarrow H(x))]$$

Solution:

$$\forall x \exists y \exists z [F(u,x,y) \wedge \neg G(z,v) \wedge \neg H(x))]$$