# Master Theorem: Practice Problems and Solutions 

## Master Theorem

The Master Theorem applies to recurrences of the following form:

$$
T(n)=a T(n / b)+f(n)
$$

where $a \geq 1$ and $b>1$ are constants and $f(n)$ is an asymptotically positive function.
There are 3 cases:

1. If $f(n)=O\left(n^{\log _{b} a-\epsilon}\right)$ for some constant $\epsilon>0$, then $T(n)=\Theta\left(n^{\log _{b} a}\right)$.
2. If $f(n)=\Theta\left(n^{\log _{b} a} \log ^{k} n\right)$ with $^{1} k \geq 0$, then $T(n)=\Theta\left(n^{\log _{b} a} \log ^{k+1} n\right)$.
3. If $f(n)=\Omega\left(n^{\log _{b} a+\epsilon}\right)$ with $\epsilon>0$, and $f(n)$ satisfies the regularity condition, then $T(n)=\Theta(f(n))$.

Regularity condition: $a f(n / b) \leq c f(n)$ for some constant $c<1$ and all sufficiently large $n$.

## Practice Problems

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

1. $T(n)=3 T(n / 2)+n^{2}$
2. $T(n)=4 T(n / 2)+n^{2}$
3. $T(n)=T(n / 2)+2^{n}$
4. $T(n)=2^{n} T(n / 2)+n^{n}$
5. $T(n)=16 T(n / 4)+n$
6. $T(n)=2 T(n / 2)+n \log n$

[^0]7. $T(n)=2 T(n / 2)+n / \log n$
8. $T(n)=2 T(n / 4)+n^{0.51}$
9. $T(n)=0.5 T(n / 2)+1 / n$
10. $T(n)=16 T(n / 4)+n!$
11. $T(n)=\sqrt{2} T(n / 2)+\log n$
12. $T(n)=3 T(n / 2)+n$
13. $T(n)=3 T(n / 3)+\sqrt{n}$
14. $T(n)=4 T(n / 2)+c n$
15. $T(n)=3 T(n / 4)+n \log n$
16. $T(n)=3 T(n / 3)+n / 2$
17. $T(n)=6 T(n / 3)+n^{2} \log n$
18. $T(n)=4 T(n / 2)+n / \log n$
19. $T(n)=64 T(n / 8)-n^{2} \log n$
20. $T(n)=7 T(n / 3)+n^{2}$
21. $T(n)=4 T(n / 2)+\log n$
22. $T(n)=T(n / 2)+n(2-\cos n)$

## Solutions

1. $T(n)=3 T(n / 2)+n^{2} \Longrightarrow T(n)=\Theta\left(n^{2}\right)($ Case 3$)$
2. $T(n)=4 T(n / 2)+n^{2} \Longrightarrow T(n)=\Theta\left(n^{2} \log n\right)($ Case 2$)$
3. $T(n)=T(n / 2)+2^{n} \Longrightarrow \Theta\left(2^{n}\right)($ Case 3$)$
4. $T(n)=2^{n} T(n / 2)+n^{n} \Longrightarrow$ Does not apply ( $a$ is not constant)
5. $T(n)=16 T(n / 4)+n \Longrightarrow T(n)=\Theta\left(n^{2}\right)($ Case 1$)$
6. $T(n)=2 T(n / 2)+n \log n \Longrightarrow T(n)=n \log ^{2} n($ Case 2$)$
7. $T(n)=2 T(n / 2)+n / \log n \Longrightarrow$ Does not apply (non-polynomial difference between $f(n)$ and $n^{\log _{b} a}$ )
8. $T(n)=2 T(n / 4)+n^{0.51} \Longrightarrow T(n)=\Theta\left(n^{0.51}\right)($ Case 3)
9. $T(n)=0.5 T(n / 2)+1 / n \Longrightarrow$ Does not apply $(a<1)$
10. $T(n)=16 T(n / 4)+n!\Longrightarrow T(n)=\Theta(n!)($ Case 3$)$
11. $T(n)=\sqrt{2} T(n / 2)+\log n \Longrightarrow T(n)=\Theta(\sqrt{n})$ (Case 1)
12. $T(n)=3 T(n / 2)+n \Longrightarrow T(n)=\Theta\left(n^{\lg 3}\right)($ Case 1$)$
13. $T(n)=3 T(n / 3)+\sqrt{n} \Longrightarrow T(n)=\Theta(n)$ (Case 1)
14. $T(n)=4 T(n / 2)+c n \Longrightarrow T(n)=\Theta\left(n^{2}\right)$ (Case 1$)$
15. $T(n)=3 T(n / 4)+n \log n \Longrightarrow T(n)=\Theta(n \log n)$ (Case 3)
16. $T(n)=3 T(n / 3)+n / 2 \Longrightarrow T(n)=\Theta(n \log n)($ Case 2$)$
17. $T(n)=6 T(n / 3)+n^{2} \log n \Longrightarrow T(n)=\Theta\left(n^{2} \log n\right)($ Case 3$)$
18. $T(n)=4 T(n / 2)+n / \log n \Longrightarrow T(n)=\Theta\left(n^{2}\right)($ Case 1$)$
19. $T(n)=64 T(n / 8)-n^{2} \log n \Longrightarrow$ Does not apply $(f(n)$ is not positive $)$
20. $T(n)=7 T(n / 3)+n^{2} \Longrightarrow T(n)=\Theta\left(n^{2}\right)($ Case 3$)$
21. $T(n)=4 T(n / 2)+\log n \Longrightarrow T(n)=\Theta\left(n^{2}\right)$ (Case 1)
22. $T(n)=T(n / 2)+n(2-\cos n) \Longrightarrow$ Does not apply. We are in Case 3, but the regularity condition is violated. (Consider $n=2 \pi k$, where $k$ is odd and arbitrarily large. For any such choice of $n$, you can show that $c \geq 3 / 2$, thereby violating the regularity condition.)

[^0]:    ${ }^{1}$ most of the time, $k=0$

