Doing Algebraic Geometry with the RegularChains Library

Parisa Alvandi[1] Changbo Chen[2] Steffen Marcus[3]
Marc Moreno Maza[1] Éric Schost[1] Paul Vrbik[1]

[1]University of Western Ontario
[2]Chinese Academy of Science
[3]The College of New Jersay

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Driving application: computing intersection multiplicity

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- We will combine Fulton's Algorithm approach and the theory of regular chains.
- Our algorithm is complete in the bivariate case.
- We propose algorithmic criteria for reducing the case of n variables to the bivariate one. Experimental results are also reported.

The case of two plane curves

Given an arbitrary field $\mathbf{k}$ and two bivariate polynomials $f, g \in k[x, y]$, consider the affine algebraic curves $C := V(f)$ and $D := V(g)$ in $\mathbb{A}^2 = \overline{\mathbf{k}}^2$, where $\overline{\mathbf{k}}$ is the algebraic closure of k. Let p be a point in the intersection.

Definition

The intersection multiplicity of p in $V(f, g)$ is defined to be

$$I(p; f, g) = \dim_{\overline{k}}(\mathcal{O}_{\mathbb{A}^2,p}/\langle f, g\rangle)$$

where $\mathcal{O}_{\mathbb{A}^2,p}$ and $\dim_{\overline{k}}(\mathcal{O}_{\mathbb{A}^2,p}/\langle f, g\rangle)$ are the local ring at p and the dimension of the vector space $\mathcal{O}_{\mathbb{A}^2,p}/\langle f, g\rangle$.

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Remark

As pointed out by Fulton in his book *Algebraic Curves*, the intersection multiplicities of the plane curves C and D satisfy a series of 7 properties which uniquely define $I(p; f, g)$ at each point $p \in V(f, g)$.
Moreover, the proof is constructive, which leads to an algorithm.

Fulton's Properties

The intersection multiplicity of two plane curves at a point satisfies and is uniquely determined by the following.

(2-1) $I(p; f, g)$ is a non-negative integer for any C, D, and p such that C and D have no common component at p. We set $I(p; f, g) = \infty$ if C and D have a common component at p.

(2-2) $I(p; f, g) = 0$ if and only if $p \notin C \cap D$.

(2-3) $I(p; f, g)$ is invariant under affine change of coordinates on $\mathbb{A}^2$.

(2-4) $I(p; f, g) = I(p; g, f)$

(2-5) $I(p; f, g)$ is greater or equal to the product of the multiplicity of p in f and g, with equality occurring if and only if C and D have no tangent lines in common at p.

(2-6) $I(p; f, gh) = I(p; f, g) + I(p; f, h)$ for all $h \in k[x, y]$.

(2-7) $I(p; f, g) = I(p; f, g + hf)$ for all $h \in k[x, y]$.

Fulton's Algorithm

Algorithm 1: $\mathrm{IM}_2(p; f, g)$

Input: $p = (\alpha, \beta) \in \mathbb{A}^2(\mathbf{k})$ and $f, g \in \mathbf{k}[y \succ x]$ such that $\mathbf{gcd}(f, g) \in \mathbf{k}$
Output: $I(p; f, g) \in \mathbb{N}$ satisfying (2-1)–(2-7)

if $f(p) \neq 0$ or $g(p) \neq 0$ **then**
  return 0;

$r, s = \mathbf{deg}\,(f(x, \beta))\,, \mathbf{deg}\,(g(x, \beta))\,;$ assume $s \geq r$.

if $r = 0$ **then**
  write $f = (y - \beta) \cdot h$ and $g(x, \beta) = (x - \alpha)^m\,(a_0 + a_1(x - \alpha) + \cdots)$;
  return $m + \mathrm{IM}_2(p; h, g)$;
  $\mathrm{IM}_2(p; (y - \beta) \cdot h, g) = \mathrm{IM}_2(p; (y - \beta), g) + \mathrm{IM}_2(p; h, g)$
  $\mathrm{IM}_2(p; (y - \beta), g) = \mathrm{IM}_2(p; (y - \beta), g(x, \beta)) = \mathrm{IM}_2(p; (y - \beta), (x - \alpha)^m) = m$

if $r > 0$ **then**
  $h \leftarrow \mathrm{monic}\,(g) - (x - \alpha)^{s-r}\mathrm{monic}\,(f)$;
  return $\mathrm{IM}_2(p; f, h)$;

Our goal: extending Fulton's Algorithm

Limitations of Fulton's Algorithm

Fulton's Algorithm

- does not generalize to $n > 2$, that is, to n polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ since $k[x_1, \ldots, x_{n-1}]$ is no longer a PID.
- is limited to computing the IM at a single point with rational coordinates, that is, with coordinates in the base field k. (Approaches based on standard or Gröbner bases suffer from the same limitation)

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Our contributions

- We adapt Fulton's Algorithm such that it can work at any point of $V(f_1, f_2)$, rational or not.
- For $n > 2$, we propose an algorithmic criterion to reduce the n-variate case to that of $n - 1$ variables.

A first algorithmic tool: *regular chains* (1/2)

Definition

$T \subset k[x_n > \cdots > x_1]$ is a triangular set if $T \cap k = \emptyset$ and $\mathrm{mvar}(p) \neq \mathrm{mvar}(q)$ for all $p, q \in T$ with $p \neq q$.

For all $t \in T$ write $\mathrm{init}(t) := \mathrm{lc}(t, \mathrm{mvar}(t))$ and $h_T := \prod_{t \in T} \mathrm{init}(t)$. The saturated ideal of T is:

$$\mathrm{sat}(T) = \langle T \rangle : h_T^{\infty}.$$

Theorem (J.F. Ritt, 1932)

Let $V \subset \overline{k}^n$ be an irreducible variety and $F \subset k[x_1, \ldots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.

$$(\forall\ g \in \langle \mathbf{F} \rangle)\ \ \mathrm{prem}(g, T) = 0.$$

Therefore, we have

$$V = V(\mathrm{sat}(T)).$$

A first algorithmic tool: *regular chains* (2/2)

Definition (M. Kalkbrner, 1991 - L. Yang, J. Zhang 1991)

T is a regular chain if $T = \emptyset$ or $T := T' \cup \{t\}$ with $\mathrm{mvar}(t)$ maximum s.t.

- T' is a regular chain,
- $\mathrm{init}(t)$ is regular modulo $\mathrm{sat}(T')$

Kalkbrener triangular decomposition

For all $F \subset k[x_1, \ldots, x_n]$, one can compute a family of regular chains $T_1, \ldots, T_e$ of $k[x_1, \ldots, x_n]$, called a Kalkbrener triangular decomposition of $V(F)$, such that we have

$$V(F) = \cup_{i=1}^{e} V(\mathrm{sat}(T_i)).$$

A second algorithmic tool: *the D5 Principle*

Original version (Della Dora, Discrescenzo & Duval)

Let $f, g \in k[x_1]$ such that f is squarefree. Without using irreducible factorization, one can compute $f_1, \ldots, f_e \in k[x_1]$ such that

- $f = f_1 \ldots f_e$ holds and,
- for each $i = 1 \cdots e$, either $g \equiv 0 \mod f_i$ or g is invertible modulo f_i.

Multivariate version

Let $T \subset k[x_1, \ldots, x_n]$ be a regular chain such that $\mathrm{sat}(T)$ is zero-dimensional, thus $\mathrm{sat}(T) = \langle T \rangle$ holds. Let $f \in k[x_1, \ldots, x_n]$.

The operation Regularize (f, T) computes regular chains $T_1, \ldots, T_e \subset k[x_1, \ldots, x_n]$ such that

- $V(T) = V(T_1) \cup \cdots \cup V(T_e)$ holds and,
- for each $i = 1 \cdots e$, either $V(T_i) \subseteq V(f)$ or $V(T_i) \cap V(f) = \emptyset$ holds.

Moreover, only polynomial GCDs and resultants need to be computed, that is, irreducible factorization is not required.

Dealing with non-rational points

Working with regular chains

To deal with non-rational points, we extend Fulton's Algorithm to compute $\mathrm{IM}_2(T; f_1, f_2)$, where $T \subset k[x_1, x_2]$ is a regular chain such that we have $V(T) \subseteq V(f_1, f_2)$.

- This makes sense thanks to the theorem below, which is non-trivial since intersection multiplicity is really a local property.
- For an arbitray zero-dimensional regular chain T, we apply the D5 Principle to Fulton's Algorithm in order to reduce to the case of the theorem.

Theorem 1

Recall that $V(f_1, f_2)$ is zero-dimensional. Let $T \subset k[x_1, x_2]$ be a regular chain such that we have $V(T) \subset V(f_1, f_2)$ and the ideal $\langle T \rangle$ is maximal. Then $\mathrm{IM}_2(p; f_1, f_2)$ is the same at any point $p \in V(T)$.

`TriangularizeWithMultiplicity`

We specify `TriangularizeWithMultiplicity` for the bivariate case.

Input $f, g \in \mathbf{k}[x, y]$ such that $V(f, g)$ is zero-dimensional.

Output Finitely many pairs $[(T_1, m_1), \ldots, (T_\ell, m_\ell)]$ of the form $(T_i :: \text{RegularChain}, m_i :: \text{nonnegint})$ such that for all $p \in V(T_i)$

$$\mathrm{I}(p; f, g) = m_i \quad \text{and} \quad V(f, g) = V(T_1) \uplus \cdots \uplus V(T_\ell).$$

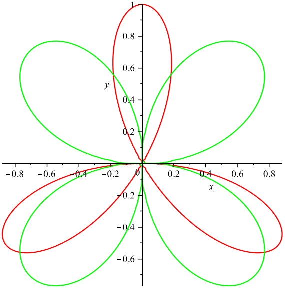
Implementating `TriangularizeWithMultiplicity` is done by

- first calling `Triangularize` (which encode the points of $V(f, g)$ with regular chains, and
- secondly calling $\mathrm{IM}_2(T; f, g)$ for all $T \in \texttt{Triangularize}(f, g)$.

This approach allows optimizations such that using the Jacobian criterion to quickly discover points of IM equal to 1.

$> Fs := \left[\left(x^2+y^2\right)^2+3x^2y-y^3, \left(x^2+y^2\right)^3-4x^2y^2\right]:$

> plots[implicitplot](Fs,x=-2..2,y=-2..2);



$> R := PolynomialRing\left([x, y], 101\right):$

$> rcs := Triangularzie\left(Fs, R, normalized = `yes`\right):$

$> \mathrm{seq}\left(\mathrm{TriangularizeWithMultiplicity}\left(Fs, T, R\right), T \ \mathbf{in} \ rcs\right):$

$$\left[\left[1, \begin{cases} x-1=0 \\ y+14=0 \end{cases}\right]\right], \left[\left[1, \begin{cases} x+1=0 \\ y+14=0 \end{cases}\right]\right], \left[\left[1, \begin{cases} x-47=0 \\ y-14=0 \end{cases}\right]\right],$$

$$\left[\left[1, \begin{cases} x+47=0 \\ y-14=0 \end{cases}\right]\right], \left[\left[14, \begin{cases} x=0 \\ y=0 \end{cases}\right]\right]$$

$> Fs := \left[x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1\right]$:

$> R := PolynomialRing([x, y, z], 101)$:

$>$ TriangularizeWithMultiplicity(Fs, R):

$$\left[\left[1, \begin{cases} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0 \end{cases}\right]\right], \left[\left[2, \begin{cases} x = 0 \\ y = 0 \\ z - 1 = 0 \end{cases}\right]\right],$$

$$\left[\left[2, \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}\right]\right], \left[\left[2, \begin{cases} x - 1 = 0 \\ y = 0 \\ z = 0 \end{cases}\right]\right]$$

Experiments

System	Degree	Time($\triangle$ize)	#rc's	Time(rc_im)
$\langle 1, 3\rangle$	888	9.7	20	19.2
$\langle 1, 4\rangle$	1456	226.0	8	9.023
$\langle 1, 5\rangle$	1595	169.4	8	25.4
$\langle 3, 5\rangle$	1413	22.5	27	28.6
$\langle 4, 5\rangle$	1781	218.4	9	13.9
$\langle 5, 1\rangle$	1759	113.0	10	15.8
$\langle 6, 8\rangle$	1680	99.7	12	37.6
$\langle 6, 9\rangle$	2560	299.3	10	22.9
$\langle 6, 10\rangle$	1320	131.9	7	8.4
$\langle 6, 11\rangle$	1440	59.8	17	27.5
$\langle 7, 8\rangle$	1152	32.8	12	16.2
$\langle 7, 9\rangle$	756	18.5	16	11.2
$\langle 7, 10\rangle$	595	8.1	17	13.0
$\langle 7, 11\rangle$	648	9.2	25	11.1
$\langle 8, 9\rangle$	1984	374.5	10	11.3
$\langle 8, 10\rangle$	1362	232.5	7	9.3
$\langle 8, 11\rangle$	1256	49.6	17	45.7
$\langle 9, 10\rangle$	2080	504.9	12	34.812
$\langle 9, 11\rangle$	1792	115.1	16	17.2
$\langle 10, 11\rangle$	1180	40.9	17	21.3

Reducing from dim n to dim $n-1$: using transversality (1/2)

Definition

The intersection multiplicity of p in $V(f_1, \ldots, f_n)$ is given by

$$I(p; f_1, \ldots, f_n) := \dim_{\overline{k}} \left(\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle \right).$$

where $\mathcal{O}_{\mathbb{A}^n,p}$ and $\dim_{\overline{k}}(\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle)$ are respectively the local ring at the point p and the dimension of the vector space $\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle$.

The next theorem reduces the n-dimensional case to $n-1$, under assumptions which state that f_n does not contribute to $I(p; f_1, \ldots, f_n)$.

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Theorem 2

Assume that $h_n = V(f_n)$ is non-singular at p. Let v_n be its tangent hyperplane at p. Assume that h_n meets each component (through p) of the curve $\mathcal{C} = V(f_1, \ldots, f_{n-1})$ transversely (that is, the tangent cone $TC_p(\mathcal{C})$ intersects v_n only at the point p). Let $h \in k[x_1, \ldots, x_n]$ be the degree 1 polynomial defining v_n. Then, we have

$$I(p; f_1, \ldots, f_n) = I(p; f_1, \ldots, f_{n-1}, h).$$

Reducing from dim n to dim $n-1$: using transversality (2/2)

The theorem again:

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$$I(p; f_1, \ldots, f_n) = I(p; f_1, \ldots, f_{n-1}, h).$$

How to use this theorem in practise?

Assume that the coefficient of x_n in h is non-zero, thus $h = x_n - h'$, where $h' \in k[x_1, \ldots, x_{n-1}]$. Hence, we can rewrite the ideal $\langle f_1, \ldots, f_{n-1}, h\rangle$ as $\langle g_1, \ldots, g_{n-1}, h\rangle$ where g_i is obtained from f_i by substituting x_n with h'. Then, we have

$$I(p; f_1, \ldots, f_n) = I(p|_{x_1,\ldots,x_{n-1}}; g_1, \ldots, g_{n-1}).$$

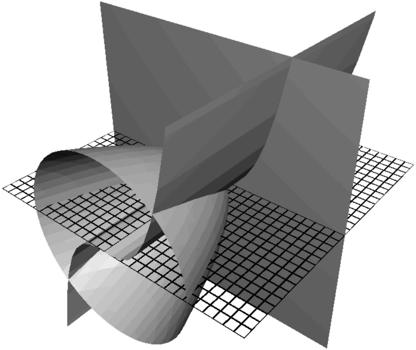
Reducing from dim n to dim $n-1$: a simple case (1/3)

Example

Consider the system

$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$

near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$



Reducing from dim n to dim $n-1$: a simple case (2/3)

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Recall the system

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near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$.

Computing the IM using the definition

Let us compute a basis for $\mathcal{O}_{\mathbb{A}^3,o}/\langle f_1, f_2, f_3\rangle$ as a vector space over $\overline{k}$.

Setting $x = 0$ and $y = z^3$, we must have $z^2(z^4+1) = 0$ in $\mathcal{O}_{\mathbb{A}^3,o} = \overline{k}[x,y,z]_{(z,y,z)}$.

Since $z^4 + 1$ is a unit in this local ring, we see that

$$\mathcal{O}_{\mathbb{A}^3,o}/\langle f_1, f_2, f_3\rangle = \langle 1, z\rangle$$

as a vector space, so $I(o; f_1, f_2, f_3) = 2$.

Reducing from dim n to dim $n-1$: a simple case (3/3)

Example

Recall the system again
$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$
near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$.

Computing the IM using the reduction

We have
$$\mathcal{C} := V(x, x + y^2 - z^2) = V(x, (y-z)(y+z)) = TC_o(\mathcal{C})$$
and we have
$$h = y.$$
Thus $\mathcal{C}$ and $V(f_3)$ intersect transversally at the origin. Therefore, we have
$$I_3(p; f_1, f_2, f_3) = I_2((0,0); x, x - z^2) = 2.$$

Reducing from dim n to dim $n-1$: via cylindrification (1/3)

In practise, this reduction from n to $n-1$ variables does not always apply. For instance, this is the case for *Ojika 2*:

$$x^2 + y + z - 1 = x + y^2 + z - 1 = x + y + z^2 - 1 = 0.$$

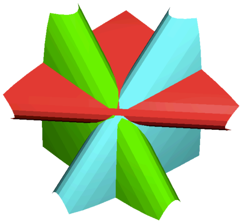


Figure: The real points of $V(x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1)$.

Reducing from dim n to dim $n-1$: via cylindrification (2/3)

Recall the system

$$x^2 + y + z - 1 = x + y^2 + z - 1 = x + y + z^2 - 1 = 0.$$

If one uses the first equation, that is $x^2 + y + z - 1 = 0$, to eliminate z from the other two, we obtain two bivariate polynomials $f, g \in k[x, y]$.

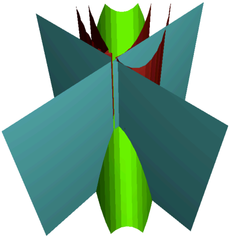


Figure: The real points of
$V(x^2 + y + z - 1, x + y^2 - x^2 - y, x - y + x^4 + 2\,x^2 y - 2\,x^2 + y^2)$ near the origin.

Reducing from dim n to dim $n-1$: via cylindrification (3/3)

At any point of $p \in V(h, f, g)$ the tangent cone of the curve $V(f, g)$ is independent of z; in some sense it is "vertical". On the other hand, at any point of $p \in V(h, f, g)$ the tangent space of $V(h)$ is not vertical.

Thus, the previous theorem applies without computing any tangent cones.

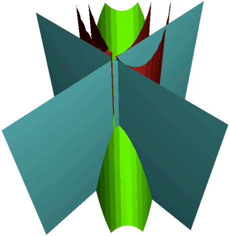
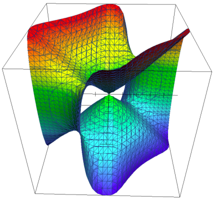
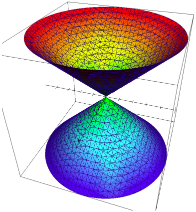


Figure: The real points of
$V(x^2 + y + z - 1, x + y^2 - x^2 - y, x - y + x^4 + 2\,x^2 y - 2\,x^2 + y^2)$ near the origin.

Tangent cone computation without standard bases





Assume $\overline{k} = \mathbb{C}$ and none of the $V(f_i)$ is singular at p. For each component $\mathcal{G}$ through p of $\mathcal{C} = V(f_1, \ldots, f_{n-1})$,

- There exists a neighborhood B of p such that $V(f_i)$ is not singular at all $q \in (B \cap \mathcal{G}) \setminus \{p\}$, for $i = 1, \ldots, n-1$.
- Let $v_i(q)$ be the tangent hyperplane of $V(f_i)$ at q. Regard $v_1(q) \cap \cdots \cap v_{n-1}(q)$ as a parametric variety with q as parameter.
- Then, $TC_p(\mathcal{G}) = v_1(q) \cap \cdots \cap v_{n-1}(q)$ when q approaches p.
- Finally, $TC_p(\mathcal{C})$ is the union of all $TC_p(\mathcal{G})$. This approach avoids standard basis computation and extends for working with $V(T)$ instead of p.

But hhow to compute the limit of $v_1(q) \cap \cdots \cap v_{n-1}(q)$ when approaches p?

Tangent cone computation with regular chains (1/2)

Algorithm principle

- Let $m(x_1, \ldots, x_n)$ be a point on the curve $\mathcal{C} = V(f_1, \ldots, f_{n-1})$,
- Let $\vec{u}$ be a unit vector directing the line (pm)
- The set $\{\lim_{m \to p, m \neq p} \ \vec{u}\}$ describes $TC_p(\mathcal{C})$

Step 1

- Let T de a 0-dim regular chain defining the point p; rename its variables to $y_1, \ldots, y_n$.

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- Consider the polynomial system (S) defined by T and $f_1 = \cdots = f_{n-1} = 0$.

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- This is a 1-dim system in the variables $y_1, \ldots, y_n, x_1, \ldots, x_n$.
- Let $R_1, \ldots, R_e$ be regular chains decomposing the zero set V of (S).

Tangent cone computation with regular chains (2/2)

Recall

- The set $\{\lim_{m \to p, m \neq p} \vec{u}\}$ describes $TC_p(\mathcal{C})$
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Step 2

- We divide each component of $p\vec{m}$ by $x_1 - y_1$. This works only if $x_1 - y_1$ vanishes finitely many times in V.

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- Fix $i = 1 \cdots e$. If $x_1 - y_1$ is regular modulo the saturated ideal of R_i, then each compliant of $p\vec{m}$ can be divided by $x_1 - y_1$.

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- Assume $x_1 - y_1$ is regular modulo the saturated ideal of R_i. Define $s_i = \frac{x_i - y_i}{x_1 - y_1}$. We have $\vec{u} = (1, s_2, \ldots, s_n)$.

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- Let $s_2, \ldots, s_n$ be variables; extend R_j with the polynomials $s_2(x_1 - y_1) - (x_2 - y_2), \ldots, s_n(x_1 - y_1) - (x_n - y_n)$ to a chain S_j.

Tangent cone computation with regular chains (2/2)

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- Let $s_2, \ldots, s_n$ be variables; extend R_j with the polynomials $s_2(x_1 - y_1) - (x_2 - y_2), \ldots, s_n(x_1 - y_1) - (x_n - y_n)$ to a chain S_j.
- Finally $\{\lim_{m \to p, m \neq p} \ \vec{u}\}$ is given by the limit points of the S_j's, that is, the sets $\overline{W(S_j)} \setminus W(S_j)$.

Limit points of a quasi-component

Input

- Let $R \subset \mathbb{C}[X_1, \ldots, X_s]$ be a regular chain.
- Let h_R be the product of initials of polynomials of R.
- Let $W(R)$ be the quasi-component of R, that is $V(R) \setminus V(h_R)$.

Desired output

The non-trivial limit points of $W(R)$, that is

$$\lim(W(R)) \; := \; \overline{W(R)}^{Z} \setminus W(R).$$

Puiseux expansions of a regular chain

Notation

- Let $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\} \subset \mathbb{C}[X_1 < \cdots < X_s]$ be a 1-dim regular chain.
- Assume R is strongly normalized, that is, $\mathrm{init}(R) \in \mathbb{C}[X_1]$.
- Let $\mathbf{k} = \mathbb{C}(\langle X_1^* \rangle)$.
- Then R generates a zero-dimensional ideal in $\mathbf{k}[X_2, \ldots, X_s]$.
- Let $V^*(R)$ be the zero set of R in $\mathbf{k}^{s-1}$.

Definition

We call *Puiseux expansions* of R the elements of $V^*(R)$.

Remarks

- The *strongly normalized assumption* is only for presentation ease.
- Generically, The 1-*dim* assumption extends to dimension $d \leq 2$.
- Higher dimension requires the Jung-Abhyankar theorem.

An example

A regular chain R

$$R := \begin{cases} X_1 X_3^2 + X_2 \\ X_1 X_2^2 + X_2 + X_1 \end{cases}$$

Puiseux expansions of R

$$\begin{cases} X_3 &= 1 + O(X_1^2) \\ X_2 &= -X_1 + O(X_1^2) \end{cases} \qquad \begin{cases} X_3 &= -1 + O(X_1^2) \\ X_2 &= -X_1 + O(X_1^2) \end{cases}$$

$$\begin{cases} X_3 &= {X_1}^{-1} - \frac{1}{2}X_1 + O(X_1^2) \\ X_2 &= -{X_1}^{-1} + X_1 + O(X_1^2) \end{cases} \qquad \begin{cases} X_3 &= -{X_1}^{-1} + \frac{1}{2}X_1 + O(X_1^2) \\ X_2 &= -{X_1}^{-1} + X_1 + O(X_1^2) \end{cases}$$

Relation between $\lim_0(W(R))$ and Puiseux expansions of R

Theorem

For $W \subseteq \mathbb{C}^s$, denote

$$\lim_0(W) := \{x = (x_1, \ldots, x_s) \in \mathbb{C}^s \mid x \in \lim(W) \text{ and } x_1 = 0\},$$

and define

$V^*_{\geq 0}(R) := \{\Phi = (\Phi^1, \ldots, \Phi^{s-1}) \in V^*(R) \mid \mathrm{ord}(\Phi^j) \geq 0, j = 1, \ldots, s-1\}.$

Then we have

$$\lim_0(W(R)) = \cup_{\Phi \in V^*_{\geq 0}(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}.$$

$$V^*_{\geq 0}(R) := \left\{ \begin{array}{lcl} X_3 & = & 1 + O(X_1^2) \\ X_2 & = & -X_1 + O(X_1^2) \end{array} \right. \cup \left\{ \begin{array}{lcl} X_3 & = & -1 + O(X_1^2) \\ X_2 & = & -X_1 + O(X_1^2) \end{array} \right.$$

Thus the limit ponts are $\lim_0(W(R)) = \{(0, 0, 1), (0, 0, -1)\}$.

Limit points of a quasi-component

```
> with(AlgebraicGeometryTools):
> R := PolynomialRing([x, y, t]);
> F := [t*y^2 + y + 1, (t + 2)*t*x^2 + (y +1)* (x + 1)];
> C := Chain(F, Empty(R), R);
> lm := LimitPoints(C, R, false, true);
> Display(lm, R);
```

$$R := polynomial\_ring$$

$$F := \left[t\, y^2 + y + 1,\ (t+2)\, t\, x^2 + (y+1)\,(x+1)\right]$$

$$C := regular\_chain$$

$$lm := [regular\_chain,\ regular\_chain,\ regular\_chain,\ regular\_chain]$$

$$\left[\begin{cases} x+1=0 \\ y+\frac{1}{2}=0 \\ t+2=0 \end{cases},\ \begin{cases} x+1=0 \\ y-1=0 \\ t+2=0 \end{cases},\ \begin{cases} x+\frac{1}{2}=0 \\ y+1=0 \\ t=0 \end{cases},\ \begin{cases} x-1=0 \\ y+1=0 \\ t=0 \end{cases}\right]$$

Conclusions

Let $f_1, \ldots, f_n \in k[x_1, \ldots, k_n]$ such that $V(f_1, \ldots, f_n)$ is zero-dimensional.

- For $n = 2$, in all cases, and for $n > 2$, under genericity assumptions, we saw how to compute the intersection multiplicity $I(p; f_1, \ldots, f_n)$ at any $p \in V(f_1, \ldots, f_n)$.
- In some cases, the tangent cone of a curve at a point is computed.
- When this happens, computing limit points of constructible sets may be computed as well.
- All these operations rely on regular chain manipulations instead of standard basis computation.
- They are part of the new module `AlgebraicGeometryTools` of the next release the `RegularChains` library.
- The latest `RegularChains.mla` library archive can be downloaded from www.regularchains.org