An Improvement of Rosenfeld-Gröbner Algorithm

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Hakim Abolfath Omar ebn Ibrahim Khayyám Nieshapuri (18 May 1048 -4 December 1131), born in Nishapur in North Eastern Iran, was a great Persian polymath, philosopher, mathematician, astronomer and poet, who wrote treatises on mechanics, geography, mineralogy, music, and Islamic theology.

Khayyám was famous during his times as a mathematician. He wrote the influential treatise on *demonstration of problems of algebra (1070)*, which laid down the principles of algebra, part of the body of Persian Mathematics that was eventually transmitted to Europe. In particular, he derived general methods for *solving cubic equations and even some higher orders*.

He wrote on *the triangular array of binomial coefficients* known as Khayyám-Pascal's triangle. In 1077, Khayyám also wrote a book published in English as "On the Difficulties of Euclid's Definitions". An important part of the book is concerned with Euclid's famous parallel postulate and his approach made their way to Europe, and may have contributed to the eventual development of non-Euclidean geometry.

Outline of talk

Differential algebra

Algebra techniques to analyze PDE's

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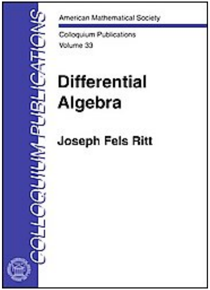
Algebra techniques to analyze PDE's



It was initiated mostly by French and American researchers and developed by the American teams J. F. Ritt (1893-1951) -which is the father of differential algebra- and E. R. Kolchin (1916-1991); a PhD student of Ritt. The aim is to study systems of polynomial nonlinear ordinary and partial differential equations (PDE's). from a purely algebraic point of view. Differential algebra is closer to ordinary commutative algebra than to analysis.

Notations

- K; an arbitrary field
- $R = K[x_1, \ldots, x_n]$; polynomial ring in the x_i's
- $F = \{f_1, \ldots, f_k\} \subset R$; a set of polynomials
- $I = \langle F \rangle = \{\sum_{i=1}^{k} p_i f_i \mid p_i \in R\}$; the ideal generated by the f_i's

Some important monomial orderings

Lex(Lexicographic) Ordering

$X^\alpha \prec_{lex} X^\beta$ if leftmost nonzero of $\alpha - \beta$ is < 0

Degree Reverse Lex Ordering

$X^\alpha \prec_{drl} X^\beta$ if

$$\begin{cases} |\alpha| < |\beta| \\ \quad \text{or} \\ |\alpha| = |\beta| \quad \text{and rightmost nonzero of } \alpha - \beta \text{ is } > 0 \end{cases}$$

- Example

  $x^{100} \prec_{lex(y,x)} y \qquad y \prec_{drl(y,x)} x^{100}$

Further notations

$R = K[x_1, \ldots, x_n], f \in R$

$\prec$ a monomial ordering on R

$I \subset R$ an ideal

$\mathrm{LM}(f)$: The greatest monomial (with respect to $\prec$) in f
$5x^3y^2 + 4x^2y^3 + xy + 1$

$\mathrm{LC}(f)$: The coefficient of $\mathrm{LM}(f)$ in f
$5x^3y^2 + 4x^2y^3 + xy + 1$

$\mathrm{LT}(f)$: $\mathrm{LC}(f)\mathrm{LM}(f)$
$5x^3y^2 + 4x^2y^3 + xy + 1$

$\mathrm{LT}(I)$: $\langle \mathrm{LT}(f) \mid f \in I \rangle$

Definition

- $I \subset K[x_1, \ldots, x_n]$
- $\prec$ A monomial ordering
- A finite set $G \subset I$ is a Gröbner Basis for I w.r.t. $\prec$, if

$$\mathrm{LT}(I) = \langle \mathrm{LT}(g) \mid g \in G \rangle$$

Existence of Gröbner bases

Each ideal has a Gröbner basis

Example

$I = \langle xy - x, x^2 - y \rangle$, $x \prec_{drl} y$
$\mathrm{LT}(I) = \langle x^2, xy, y^2 \rangle$
A Gröbner basis is: $\{xy - x, x^2 - y, y^2 - y\}$.

Buchberger's Criterion and Buchberger's Algorithm

Definition

S-polynomial

$$\text{Spoly}(f,g) = \frac{x^\gamma}{\text{LT}(f)}f - \frac{x^\gamma}{\text{LT}(g)}g$$

$$x^\gamma = lcm(\text{LM}(f), \text{LM}(g))$$

$$\text{Spoly}(x^3y^2+xy^3, xyz-z^3) = z(x^3y^2+xy^3)-x^2y(xyz-z^3) = zxy^3+x^2yz^3$$

Buchberger's Criterion

- ▷ G is a Gröbner basis for $\langle G\rangle$
- ▷ $\forall g_i, g_j \in G$, remainder$((\text{Spoly}(g_i,g_j), G) = 0$

Theorem

Suppose that $f, g \in K[x_1, \ldots, x_n]$ *are two polynomials so that* $lcm(\mathrm{LM}(f), \mathrm{LM}(g)) = \mathrm{LM}(f)\mathrm{LM}(g)$. *Then, the remainder of the division of* $\mathrm{Spoly}(f, g)$ *by* $\{f, g\}$ *is* 0.

Buchberger's Criterion

- G is a Gröbner basis for $\langle G \rangle$
- $\forall g_i, g_j \in G$ which do not satisfy above criterion we have remainder$((\mathrm{Spoly}(g_i, g_j), G) = 0$

Example

Let $f = x^2 + x$ and $g = y^2 + y$. Then, $\{f, g\}$ is a Gröbner basis of $\langle f, g \rangle$ w.r.t. any ordering.

Definition

An operator $\delta : R \to R$ is a *derivation operator*, if

$$\begin{aligned} \delta(a+b) &= \delta(a) + \delta(b) \\ \delta(ab) &= \delta(a)b + a\delta(b). \end{aligned}$$

A *differential ring* is a pair (R, Δ) satisfying

- $\Delta = \{\delta_1, \ldots, \delta_m\}$
- $\delta_i \delta_j a = \delta_j \delta_i a$

If $m = 1$, then R is called an *ordinary* differential ring; otherwise it will be called *partially*.

Definition

An algebraic ideal I of R is called a *differential ideal* if $\delta a \in I$ for each $\delta \in \Delta$ and $a \in I$.

Example

$\mathbb{C}[x_1, \ldots, x_m]$ together with the set of operators $\partial/\partial x_1, \ldots, \partial/\partial x_m$ is a differential ring.

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Definition

- $\Theta := \left\{\delta_1^{t_1}\delta_2^{t_2} \ldots \delta_m^{t_m} \mid t_1, \ldots, t_m \in \mathbb{N}\right\}$
- $\theta = \delta_1^{t_1} \cdots \delta_m^{t_m} \in \Theta$ is called a *derivation operator* of R
- $\mathrm{ord}(\theta) := \sum_{i=1}^{m} t_i$ is called the *order* of θ
- a *differential polynomial ring*:

$$R := K\{u_1, \ldots, u_n\} := K\left[\Theta U\right]$$

is the usual commutative polynomial ring generated by ΘU over K, where $U := \{u_1, \ldots, u_n\}$ is the set of *differential indeterminate*.

Ranking

- ☞ $R = K\{u_1, \ldots, u_n\}$: a differential polynomial ring
- ☞ $U = \{u_1, \ldots, u_n\}$
- ☞ $\Theta := \left\{\delta_1^{t_1}\delta_2^{t_2}\ldots\delta_m^{t_m} \mid t_1, \ldots, t_m \in \mathbb{N}\right\}$
- ☞ $\Theta := \left\{\delta_1^{t_1}\delta_2^{t_2}\ldots\delta_m^{t_m} u \mid t_1, \ldots, t_m \in \mathbb{N},\ u \in U\right\}$

Definition

ranking $>$: $\forall \delta \in \Theta, \forall v, w \in \Theta U$: $\delta v > v$ and $v > w \Longrightarrow \delta v > \delta w$

Example

$R := \mathbb{Q}(x, y)\{u, v\}, v <_{lex} u, y <_{lex} x$

$p := v_{x,y,y}^2 u_{x,y} + v_x u_{x,x}^2 + u_x;$

leader(p)$=u_{x,x}$;

rank(p)$=u_{x,x}^2$

initial(p)$=v_x$

separant(p)$:=\partial p/\partial leader(p) = 2u_{x,x}$

Definition

Let us consider two differential polynomials p_1 and p_2 with $ld(p_i) = \theta_i \, u_i, \; i = 1, 2$.

$$\Delta(p_1, p_2) = \begin{cases} s_{p_2} \frac{\theta_{1,2}}{\theta_1} p_1 - s_{p_1} \frac{\theta_{1,2}}{\theta_2} p_2 & u_1 = u_2, \\ 0 & u_1 \neq u_2, \end{cases}$$

where $\theta_{1,2} = lcm(\theta_1, \theta_2)$.

Example

$R := \mathbb{Q}(x, y)\{u, v\}, v <_{lex} u, y <_{lex} x$
$p := u_y + v;$
$q := vu_x^2 + v_{y,y};$
$\Delta(p, q) = 2vu_x v_x - v_y ux^2 - v_{y,y,y}$

Rosenfeld-Gröbner algorithm:

- Constructing the Δ-polynomials
- Reducing each Δ-polynomial (partially and algebraically) w.r.t. the current set of polynomials
- Discussing the nullity of initials

$$\Sigma : \begin{cases} (\frac{\partial^2 u}{\partial x^2})^2 v + \frac{\partial^2 u}{\partial x^2} v + \frac{\partial u}{\partial x} = 0 \\ \frac{\partial^2 u}{\partial x \partial y} = 0 \\ \frac{\partial^2 u}{\partial y^2} - 1 = 0 \end{cases}$$

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Step 1: Defining a *differential ring*: $\mathbb{Q}(x, y)[u, v]$

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Step 1: Defining a *differential ring*: $\mathbb{Q}(x, y)[u, v]$

Step 2: Constructing the *differential ideal*

$$I = \langle u_{x,x}^2 v + u_{x,x} v + u_x, u_{x,y}, u_{y,y}^2 - 1 \rangle$$

Rosenfeld-Gröbner algorithm:

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Step 1: Defining a *differential ring*: $\mathbb{Q}(x,y)[u,v]$

Step 2: Constructing the *differential ideal*
$I = \langle u_{x,x}^2 v + u_{x,x} v + u_x, u_{x,y}, u_{y,y}^2 - 1 \rangle$

Step 3: Decomposing $\sqrt{I}$ by *Rosenfeld-Gröbner Algorithm*:
$\sqrt{I} = \langle u_{y,y}^2 - 1, u_x \rangle \cap$
$\langle u_{y,y}^2 - 1, 4u_x - v, v_x + 2, v_y \rangle \cap$
$\langle u_{x,x}^2 v + u_{x,x} v + u_x, u_{x,y}, u_{y,y}^2 - 1, v_y \rangle : \langle v, 2u_{x,x} + 1 \rangle^\infty$

$$\langle u_{x,x}^2 v + u_{x,x} v + u_x, u_{x,y}, u_{y,y}^2 - 1, v_y \rangle : \langle v, 2u_{x,x} + 1 \rangle^\infty$$

We impose the following initial values to find the Taylor series up to order 3:

$$v(x,y) = 1 + xf(x)$$

and,

$$u(0,0) = c_0, \frac{\partial^2 u}{\partial y^2}(0,0) = 1, \frac{\partial^2 u}{\partial x^2}(0,0) = -1, \frac{\partial u}{\partial y}(0,0) = c_1, \frac{\partial u}{\partial x}(0,0) = 0$$

$$\langle u_{x,x}^2 v + u_{x,x} v + u_x, u_{x,y}, u_{y,y}^2 - 1, v_y \rangle : \langle v, 2u_{x,x} + 1 \rangle^\infty$$

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$$\Rightarrow \begin{cases} u(x, y) & = & c_0 + c_1 y + 1/2y^2 - 1/2x^2 - 1/6x^3 \\ v(x, y) & = & 1 + f(0)x + f'(0)x^2 + 1/2f''(0)x^3 \end{cases}$$

Carra-Ferro and Ollivier, 87 :

- Developing the concept of differential Gröbner bases

Mansfield, 91 :

- Developing another concept of differential Gröbner bases
- MAPLE package `diffgrob2` to solve systems of linear or nonlinear PDEs

Boulier et al., 09 :

- Presenting Rosenfeld-Gröbner algorithm
- MAPLE package `DifferentialAlgebra`

Theorem (Boulier et al., 09)

Let p and q be two differential polynomials which are linear, homogeneous, in one differential indeterminate and with constant coefficients. Further, suppose that we have $lcm(\theta u, \phi u) = \theta\phi u$ where $ld(p) = \theta u$ and $ld(q) = \phi u$. Then the full differential remainder of $\Delta(p, q)$ w.r.t $\{p, q\}$ is zero.

Example

$R := \mathbb{Q}(x, y)\{u\}, y <_{lex} x$

$p := u_{x,x} + u_{x,y}; \rightsquigarrow x^2 + xy$

$q := u_{y,y} + u_y, \rightsquigarrow y^2 + y;$

Theorem

Let p and q be two differential polynomials which are products of differential polynomials which are linear, homogeneous, in the same differential indeterminate and with constant coefficients. Furthermore, suppose that we have $lcm(\theta u, \phi u) = \theta\phi u$ where $ld(p) = \theta u$ and $ld(q) = \phi u$. Then the full differential remainder of $\Delta(p, q)$ w.r.t $\{p, q\}$ is zero.

MAPLE package `RosenfeldGrobner.mpl`:
`http://amirhashemi.iut.ac.ir/software.html`

Example

$R := \mathbb{Q}(x, y)\{u\}, y <_{lex} x$
$p := (u_{x,x,x,x} - u)^2(u_{x,x} + u_x)^3$
$q := (u_{y,y,y} - u_{y,y})^2(u_y + u)^3$
Time and memory by `DifferentialAlgebra`: 25.8 sec, 8.8Gb
Time and memory by `RosenfeldGrobner.mpl`: 17.2 sec, 8.4Gb

Thanks for your attention!