Computing the Integer Points of a Polyhedron

Complexity Estimates

Rui-Juan Jing[1,2] and Marc Moreno Maza[2,3]

[1]Key Laboratoty of Mathematics Mechnization,
Academy of Mathematics and Systems Science,
Chinese of Academy of Sciences
[2]University of Western Ontario, London, Ontario
[3]IBM Center for Advanced Studies, Markham, Ontario

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Plan

Motivations

1. Data dependence analysis and scheduling of for-loop nests of computer programs,
2. support for decision problems in Presburger arithmetic,
3. manipulation of $\mathbb{Z}$-polyhedra.

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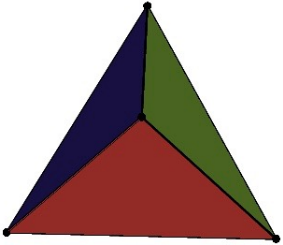
Procedure 3: GreyShadow($\mathbf{Mt} \leq \mathbf{v}$)
Output the grey shadow parts of polyhedron represented by $\mathbf{Mt} \leq \mathbf{v}$, each integer point in every grey shadow part corresponding to one integer point satisfying $\mathbf{Mt} \leq \mathbf{v}$ and the number of variables to be dealt with is less than the length of $\mathbf{t}$.

Plan

Complexity-FM elimination

K : polyhedron in $\mathbb{R}^d$, defined by m inequalities



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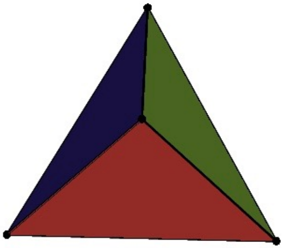
($\mathbf{Ax} \leq \mathbf{b}$ has no implicit equations.)



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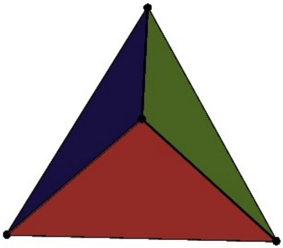
k-dimentional face of K $\leftrightarrow$ $\begin{cases} \mathbf{A}_{I_k}\mathbf{x} = \mathbf{b}_{I_k}, \\ \mathbf{A}_{I \setminus I_k} \leq \mathbf{b}_{I \setminus I_k} \end{cases}$
($I = \{1, \ldots, m\}$, $I_k \subset I$ with $d - k$ elements.)



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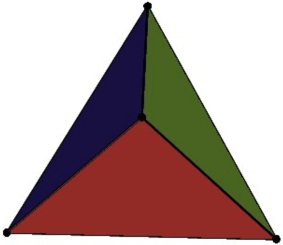

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Lemma

Let $f_{d,m,k}$ be the number of k-dimensional faces of K. Then, we have

$$f_{d,m,k} \leq \binom{m}{d-k}.$$

Therefore, we have $f_{d,m,0} + f_{d,m,1} + \cdots + f_{d,m,d-1} \leq m^d$.

Complexity-FM elimination

Proposition

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$$\|\mathbf{M}, \mathbf{v}\|_\infty \leq (k+1)^{\frac{k+1}{2}} L^{k+1}.$$

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Notation

Given a linear program with total bit size H and with d variables
$\mathrm{LP}(d, H)$*: the number of bit operations required for solving it.*
Karmarkar's algorithm: $\mathrm{LP}(d, H) \in O(d^{3.5} H^2 \cdot \log H \cdot \log\log H)$.

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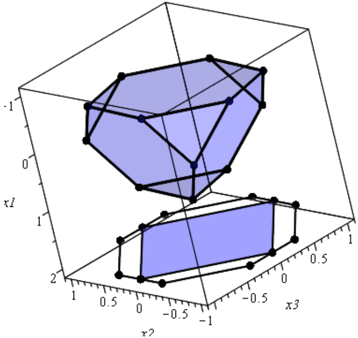
Proposition

Given a polyhedron K in $\mathbb{R}^d$, which is defined by m inequalities and with coefficient maximum bit size h, one can perform Fourier-Motzkin elimination within $O(d^2\, m^{2d}\, \mathrm{LP}(d, 2^d h d^2 m^d))$ bit operations.

Complexity of our algorithm

Hypothesis

During the execution of the function call IntegerSolve(K), *for any polyhedral set* K', *input of a recursive call, each facet of the dark shadow of* K' *is parallel to some facet of the real shadow of* K'.



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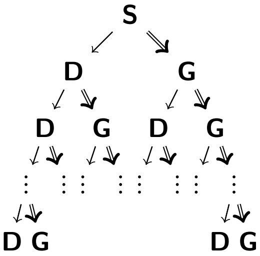

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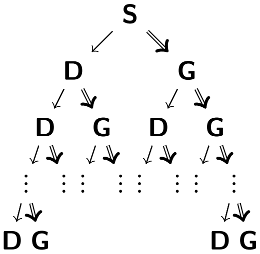

- number of pathes T:

$$T \leq m^{d^2} d^{3d^3} L^{3d^3}$$

- coefficient bound M in any node in a path:

$$M \leq d^{3d^2} d^{4d^3} L^{6d^3}$$

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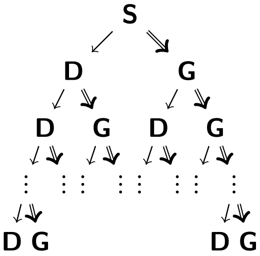

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Theorem

Under our Hypothesis, the function call IntegerSolve(K) *runs within* $O(m^{2d^2} d^{4d^3} L^{4d^3} \mathrm{LP}(d, m^d d^4 (\log d + \log L)))$ *bit operations.*

Plan

Experiments

IntegerSolve is implemented in the `Polyhedra` library and available from www.regularchains.org

Example	m	d	L	m_o	L_o	?Hyp	t_H	t_P
Tetrahedron	4	3	1	1	1	yes	0.695	0.697
TruncatedTetrahedron	8	3	1	1	1	yes	1.461	1.468
Presburger 4	3	4	5	2	12	yes	0.706	0.871
Presburger 6	4	5	89	6	35	yes	0.893	0.746
Bounded 7	8	3	19	3	190	no	138.448	239.637
Bounded 8	4	3	25	5	67	yes	6.462	3.821
Bounded 9	6	3	18	6	74	no	23.574	16.763
Unbounded 2	3	4	10	61	2255	no	0.547	0.600
Unbounded 5	5	4	8	1	8	no	1.321	1.319
Unbounded 6	10	4	8	1	8	no	1.494	1.479
P91	12	3	96	5	96	no	19.318	15.458
Sys_1	6	3	15	2	67	yes	2.413	1.915
Sys_3	8	3	1	1	1	yes	1.481	1.479
Automatic	8	2	999	1	999	yes	0.552	0.549
Automatic2	6	4	1	1	2	yes	1.115	1.113

Table: Implementation

Plan

Application

Solve integer programming:

$$\min_{\text{lex}}(x_1, \ldots, x_d)$$
$$\mathbf{Ax} \leq \mathbf{b},$$
$$\mathbf{x} \in \mathbb{Z}^d$$

Example

Problem:

$$\min_{\text{lex}}(x_3, x_2, x_1)$$
$$3x_1 - 2x_2 + x_3 \leq 7$$
$$-2x_1 + 2x_2 - x_3 \leq 12$$
$$-4x_1 + x_2 + 3x_3 \leq 15$$
$$-x_2 \leq -25$$
$$x_1, x_2, x_3 \in \mathbb{Z}$$

Application

Example

Input: $K_1 : \begin{cases} 3x_1 - 2x_2 + x_3 \le 7 \\ -2x_1 + 2x_2 - x_3 \le 12 \\ -4x_1 + x_2 + 3x_3 \le 15 \\ -x_2 \le -25 \end{cases}$, assume $x_1 > x_2 > x_3$.

Output: $K_1^1, K_1^2, K_1^3, K_1^4, K_1^5$ given by:

$$\begin{cases} 3x_1 - 2x_2 + x_3 \le 7 \\ -2x_1 + 2x_2 - x_3 \le 12 \\ -4x_1 + x_2 + 3x_3 \le 15 \\ 2x_2 - x_3 \le 48 \\ -5x_2 + 13x_3 \le 67 \\ -x_2 \le -25 \\ 2 \le x_3 \le 17 \end{cases}, \begin{cases} x_1 = 15 \\ x_2 = 27 \\ x_3 = 16 \end{cases}, \begin{cases} x_1 = 18 \\ x_2 = 33 \\ x_3 = 18 \end{cases}, \begin{cases} x_1 = 14 \\ x_2 = 25 \\ x_3 = 15 \end{cases}, \begin{cases} x_1 = 19 \\ x_2 = 50 + t \\ x_3 = 50 + 2t \\ -25 \le t \le -16. \end{cases}$$

Application

$$\min(x_3, x_2, x_1)$$
$$K_1 \cap \mathbb{Z}^3$$

$\min(x_3, x_2, x_1)$	$\min(x_3, x_2, x_1)$	$\min(x_3, x_2, x_1)$	$\min(x_3, x_2, x_1)$	$\min(x_3, x_2, x_1)$
$K_1^1 \cap \mathbb{Z}^3$	$K_1^2 \cap \mathbb{Z}^3$	$K_1^3 \cap \mathbb{Z}^3$	$K_1^4 \cap \mathbb{Z}^3$	$K_1^5 \cap \mathbb{Z}^3$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$(2, -8, -4)$	$(16, 27, 15)$	$(18, 33, 18)$	$(15, 25, 14)$	$(0, 25, 19)$

$$\Downarrow$$

$$(0, 25, 19)$$

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- Assuming that each facet of the dark shadow of a polyhedron is parallel to some facet of its real shadow, we prove that our algorithm runs in time single exponential in the dimension d of the ambient space.

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Works in progress

- A `CilkPlus` version of the `Polyhedra` library
- Parametric integer programming (PIP) in support of automatic parallelization.