Computing Limits of Real Multivariate Rational Functions

Parisa Alvandi, Mahsa Kazemi, Marc Moreno Maza

Western University, Canada

July 22, 2016

Outline

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Question

Let $q \in \mathbb{Q}(X_1, \ldots, X_n)$ be a multivariate rational function. Assume that the origin is an isolated zero of the denominator of q.

$$\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1, \ldots, x_n) = ?$$

Previous works: part I

Univariate functions (including transcendental ones)

D. Gruntz (1993, 1996), B. Salvy and J. Shackell (1999)

- Corresponding algorithms are available in popular computer algebra systems

Multi variables rational functions

S.J. Xiao and G.X. Zeng (2014)

- Given $q \in \mathbb{Q}(X_1, \ldots, X_n)$, they proposed an algorithm deciding whether or not: $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q$ *exists and is zero*.
- No assumptions on the input multivariate rational function
- Techniques used:
  - triangular decomposition of algebraic systems,
  - rational univariate representation,
  - adjoining infinitesimal elements to the base field.

Lagrange multipliers (1/2)

Let q and t be real bivariate functions of class C^1 .

Problem

$$\begin{array}{c}\text{optimize } q(x,y)\\ \text{subject to } t(x,y)=0\end{array}$$

Solution

1. Assuming $\nabla t(x,y)$ does not vanish on $t(x,y)=0$, solve the following system of equations:
$$\left\{\begin{array}{rcl}\nabla q(x,y) & = & \lambda \nabla t(x,y)\\ t(x,y) & = & 0\end{array}\right.$$
2. Plug in all (x,y) solutions obtained at Step (1) into $q(x,y)$ and identify the minimum and maximum values, provided that they exist.

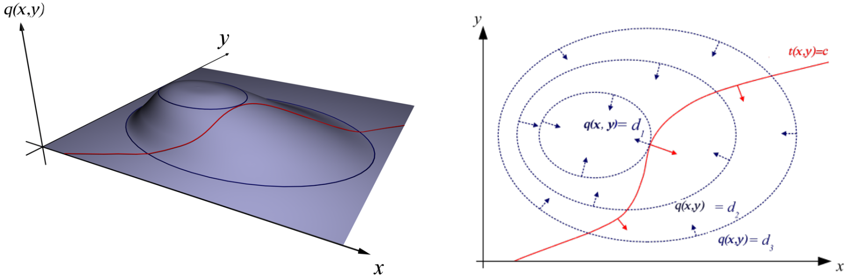


Figure: Optimizing $q(x, y)$ under $t(x, y) = c$

Previous works: bivariate rational functions

C. Cadavid, S. Molina, and J. D. Vélez (2013):

- Assumes that the origin is an isolated zero of the denominator
- Maple built-in command `limit/multi`

Discriminant variety

$$\chi(q) \;=\; \{(x,y) \in \mathbb{R}^2 \;\mid\; y\frac{\partial q}{\partial x} - x\frac{\partial q}{\partial y} = 0\}.$$

Key observation

For determining the existence and possible value of

$$\lim_{(x,y)\to(0,0)} q(x,y),$$

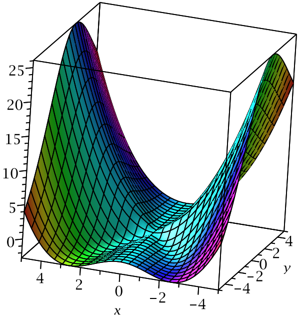
it is sufficient to compute

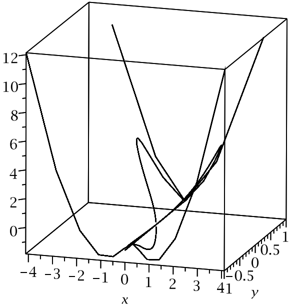
$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in\chi(q)}} q(x,y).$$

Example

Let $q \in \mathbb{Q}(x, y)$ be a rational function defined by $q(x,y) = \frac{x^4+3x^2y-x^2-y^2}{x^2+y^2}$.

$$\chi(q) = \left\{ \begin{array}{rcl} x^4 + 2x^2y^2 + 3y^3 & = & 0 \\ y & < & 0 \end{array} \right. \cup \left\{ \begin{array}{rcl} x & = & 0 \end{array} \right.$$





Previous works: trivariate rational functions

J.D. Vélez, J.P. Hernández, and C.A Cadavid (2015).

- Assumes that the origin is an isolated zero of the denominator
- Ad-hoc methods reduce to the case of bivariate rational functions

Similar key observation

For determining the existence and possible value of

$$\lim_{(x,y,z)\to(0,0,0)} q(x,y,z),$$

it is sufficient to compute

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\(x,y,z)\in\chi(q)}} q(x,y,z).$$

Techniques used

- Computation of singular loci
- Variety decomposition into irreducible components

Outline

Our contribution

* Generalize the trivariate algorithm of J.D. Vélez, J.P. Hernández, and C.A Cadavid to arbitrary number of variables
* Avoiding the computation of singular loci and irreducible decompositions

How?

Triangular decomposition of semi- algebraic systems

Outline

Regular semi-algebraic system

Notation

- Let $T \subset \mathbb{Q}[X_1 < \ldots < X_n]$ be a regular chain with $\mathbf{Y} := \{\mathrm{mvar}(t) \mid t \in T\}$ and $\mathbf{U} := \mathbf{X} \setminus \mathbf{Y} = U_1, \ldots, U_d$.
- Let P be a finite set of polynomials, s.t. every $f \in P$ is regular modulo $\mathrm{sat}(T)$.
- Let $\mathcal{Q}$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{U}]$.

Definition

We say that $R := [\mathcal{Q}, T, P_>]$ is a regular semi-algebraic system if:

(i) $\mathcal{Q}$ defines a non-empty open semi-algebraic set $\mathcal{O}$ in $\mathbb{R}^d$,

(ii) the regular system $[T, P]$ specializes well at every point u of $\mathcal{O}$

(iii) at each point u of $\mathcal{O}$, the specialized system $[T(u), P(u)_>]$ has at least one real solution.

Define

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

Regular semi-algebraic system

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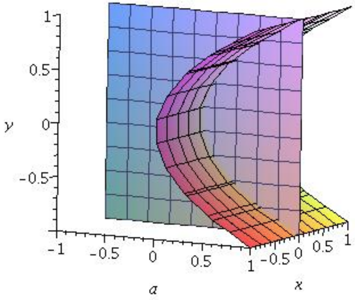
Example

The system $[\mathcal{Q}, T, P_>]$, where

$$\mathcal{Q} := a > 0,\ T := \begin{cases} y^2 - a = 0 \\ x = 0 \end{cases},\ P_> := \{y > 0\}$$

is a regular semi-algebraic system.



Regular semi-algebraic system

Notations

Let $R := [\mathcal{Q}, T, P_>]$ be a regular semi-algebraic system. Recall that $\mathcal{Q}$ defines a non-empty open semi-algebraic set $\mathcal{O}$ in $\mathbb{R}^d$ and

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

Properties

- Each connected component C of $\mathcal{O}$ in $\mathbb{R}^d$ is a real analytic manifold, thus locally homeomorphic to the hyper-cube $(0, 1)^d$
- Above each C, the set $Z_{\mathbb{R}}(R)$ consists of disjoint graphs of semi-algebraic functions forming a real analytic covering of C.
- There is at least one such graph.

Consequences

- R can be understood as a parameterization of $Z_{\mathbb{R}}(R)$
- The Jacobian matrix $\left[\ \nabla t, t \in T\ \right]$ is full rank.

Triangular decomposition of semi-algebraic sets

Proposition

Let $S := [F_=, N_\geq, P_>, H_\neq]$ be a semi-algebraic system. Then, there exists a finite family of regular semi-algebraic systems $R_1, \ldots, R_e$ such that

$$Z_\mathbb{R}(S) = \cup_{i=1}^e Z_\mathbb{R}(R_i).$$

Triangular decomposition

- In the above decomposition, $R_1, \ldots, R_e$ is called a triangular decomposition of S and we denote by `RealTriangularize` an algorithm computing such a decomposition.
- Moreover, such a decomposition can be computed in an incremental manner with a function `RealIntersect`
  - taking as input a regular semi-algebraic system R and a semi-algebraic constraint $f = 0$ (resp. $f > 0$) for $f \in \mathbb{Q}[X_1, \ldots, X_n]$
  - returning regular semi-algebraic system $R_1, \ldots, R_e$ such that

$$Z_\mathbb{R}(f = 0) \cap Z_\mathbb{R}(R) \quad = \quad \cup_{i=1}^e Z_\mathbb{R}(R_i).$$

Outline

Generalization of concepts and basic lemmas (1/3)

Discriminant variety (Cadavid, Molina, and Vélez, 2013)

Let $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a rational function defined on a punctured ball D^*_δ. The discriminant variety $\chi(q)$ of q is the real zero-set of all 2-by-2 minors of

$$\begin{bmatrix} X_1 & \cdots & X_n \\ \frac{\partial q}{\partial X_1} & \cdots & \frac{\partial q}{\partial X_n} \end{bmatrix}$$

Limit along a semi-algebraic set

Let S be a semi-algebraic set of positive dimension (i. e. $\geqslant 1$) such that $\underline{o} \in \overline{S}$ in the Euclidean topology. Let $L \in \mathbb{R}$. We say

$$\lim_{\substack{(x_1,\ldots,x_n) \to (0,\ldots,0) \\ (x_1,\ldots,x_n) \in S}} q(x_1,\ldots,x_n) = L$$

whenever

$(\forall \varepsilon > 0)\,(\exists 0 < \delta)\,(\forall (x_1,\ldots,x_n) \in S \cap D^*_\delta)\ \ |q(x_1,\ldots,x_n) - L| < \varepsilon$

Generalization of concepts and basic lemmas (2/3)

Lemma 1

For all $L \in \mathbb{R}$ the following assertions are equivalent:

- $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1,\ldots,x_n)$ exists and equals L,
- $\lim_{\substack{(x_1,\ldots,x_n)\to(0,\ldots,0)\\(x_1,\ldots,x_n)\in\chi(q)}} q(x_1,\ldots,x_n)$ exists and equals L.

Lemma 2

Let $R_1,\ldots,R_e$ be regular semi-algebraic systems forming a triangular decomposition of $\chi(q)$. Then, for all $L \in \mathbb{R}$ the following are equivalent:

- $\lim_{\substack{(x_1,\ldots,x_n)\to(0,\ldots,0)\\(x_1,\ldots,x_n)\in\chi(q)}} q$ exists and equals L.
- for all $i \in \{1,\ldots,e\}$ such that $Z_{\mathbb{R}}(R_i)$ has dimension at least 1 and the origin belongs to $\overline{Z_{\mathbb{R}}(R_i)}$, we have $\lim_{\substack{(x_1,\ldots,x_n)\to(0,\ldots,0)\\(x_1,\ldots,x_n)\in Z_{\mathbb{R}}(R_i)}} q$ exists and equals L.

Lemma 3

- Assume $n \geq 3$. Let $R = [\mathcal{Q}, \{t_n\}, P_>]$ be a regular semi-algebraic system of $\mathbb{Q}[X_1, \ldots, X_n]$ such that $Z_{\mathbb{R}}(R)$ has dimension $d := n-1$, and $\underline{o} \in \overline{Z_{\mathbb{R}}(R)}$. W.l.o.g. we assume that $\mathrm{mvar}(t_n) = X_n$ holds.
- Let $\mathcal{M} := \left[\begin{array}{ccc} X_1 & \cdots & X_n \\ \frac{\partial t_n}{\partial X_1} & \ldots & \frac{\partial t_n}{\partial X_n} \end{array} \right]$

Then, there exists a non-empty set $\mathcal{U} \subset D_\rho^* \cap Z_{\mathbb{R}}(R)$, which is open relatively to $Z_{\mathbb{R}}(R)$, such that $\mathcal{M}$ is full rank at any point of $\mathcal{U}$, and $\underline{o} \in \overline{\mathcal{U}}$.

Outline

Overview of RationalFunctionLimit

Input: a rational function $q \in \mathbb{Q}(X_1, \ldots, X_n)$ such that origin is an isolated zero of the denominator.

Output: $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1, \ldots, x_n)$

1. Apply RealTriangularize on $\chi(q)$, obtaining rsas $R_1, \ldots, R_e$
2. Discard R_i if either $\dim(R_i) = 0$ or $\underline{o} \notin \overline{Z_{\mathbb{R}}(R_i)}$
   - QuantifierElimination checks whether $\underline{o} \in \overline{Z_{\mathbb{R}}(R_i)}$ or not.
3. Apply LimitInner(R) on each regular semi algebraic system of dimension higher than one.
   - main task: solving constrained optimization problems
4. Apply LimitAlongCurve on each one-dimensional regular semi algebraic system resulting from Step 3
   - main task: Puiseux series expansions

Principles of LimitInner

Input: a rational function q and a regular semi algebraic system $R := [Q, T, P_>]$ with $\dim(Z_{\mathbb{R}}(R)) \geq 1$ and $\underline{o} \in \overline{Z_{\mathbb{R}}(R)}$

Output: limit of q at the origin along $Z_{\mathbb{R}}(R)$

1. if $\dim(Z_{\mathbb{R}}(R)) = 1$ then return LimitAlongCurve(q, R)
2. otherwise build $\mathcal{M} := \begin{bmatrix} X_1 & \cdots & X_n \\ \multicolumn{3}{c}{\nabla t, t \in T} \end{bmatrix}$
3. For all $m \in$ Minors$(\mathcal{M})$ such that $Z_{\mathbb{R}}(R) \not\subseteq Z_{\mathbb{R}}(m)$ build

   $$\mathcal{M}' := \begin{bmatrix} \frac{\partial E_r}{\partial X_1} & \cdots & \frac{\partial E_r}{\partial X_n} \\ X_1 & \cdots & X_n \\ \multicolumn{3}{c}{\nabla t, t \in T} \end{bmatrix} \text{ with } E_r := \sum_{i=1}^{n} A_i X_i^2 - r^2$$

   For all $m' \in$ Minors$(\mathcal{M}')$ $\mathcal{C} :=$ `RealIntersect`$(R, m' = 0, m \neq 0)$

   For all $C \in \mathcal{C}$ such that $\dim(Z_{\mathbb{R}}(C)) > 0$ and $\underline{o} \in \overline{Z_{\mathbb{R}}(C)}$

   1. compute $L =$ LimitInner(q, C);
   2. if L is `no_finite_limit` or L is finite but different from a previously found finite L then return `no_finite_limit`
4. If the search completes then a unique finite was found and is returned.

Principles of LimitAlongCurve

Input: a rational function q and a curve C given by $[Q, T, P_>]$

Output: limit of q at the origin along C

1. Let f, g be the numerator and denominator of q
2. Let $T' := \{gX_{n+1} - f\} \cup T$ with X_{n+1} a new variable
3. Compute the real branches of $W_{\mathbb{R}}(T') := Z_{\mathbb{R}}(T') \setminus Z_{\mathbb{R}}(h_{T'})$ in $\mathbb{R}^n$ about the origin via Puiseux series expansions
4. If no branches escape to infinity and if $W_{\mathbb{R}}(T')$ has only one limit point $(x_1, \ldots, x_n, x_{n+1})$ with $x_1 = \cdots = x_n = 0$, then x_{n+1} is the desired limit of q
5. Otherwise return `no_finite_limit`

Example

Let $q(x, y, z, w) = \frac{z\,w + x^2 + y^2}{x^2 + y^2 + z^2 + w^2}$.

`RealTriangularize` $(\chi(q))$:

$$Z_{\mathbb{R}}(\chi(q)) = Z_{\mathbb{R}}(R_1) \cup Z_{\mathbb{R}}(R_2) \cup Z_{\mathbb{R}}(R_3) \cup Z_{\mathbb{R}}(R_4),$$

where

$$R_1 := \left\{ \begin{array}{l} x = 0 \\ y = 0 \\ z = 0 \\ w = 0 \end{array} \right. , R_2 := \left\{ \begin{array}{l} x = 0 \\ y = 0 \\ z + w = 0 \end{array} \right. ,$$

$$R_3 := \left\{ \begin{array}{l} x = 0 \\ y = 0 \\ z - w = 0 \end{array} \right. , R_4 := \left\{ \begin{array}{l} z = 0 \\ w = 0 \end{array} \right. .$$

Example

- $\dim(Z_{\mathbb{R}}(R_1)) = 0$
- $\dim(Z_{\mathbb{R}}(R_2)) = 1 \Longrightarrow$ LimitAlongCurve $(q, R_2) = \frac{-1}{2}$
- $\dim(Z_{\mathbb{R}}(R_3)) = 1 \Longrightarrow$ LimitAlongCurve $(q, R_3) = \frac{1}{2}$
- $\dim(Z_{\mathbb{R}}(R_4)) = 2 \Longrightarrow$ LimitInner (q, R_4)
  -

$$R_5 := \begin{cases} z = 0 \\ w = 0 \\ x = 0 \\ y \neq 0 \end{cases}, R_6 := \begin{cases} z = 0 \\ w = 0 \\ y = 0 \\ x \neq 0 \end{cases}$$

  - $\dim(Z_{\mathbb{R}}(R_5)) = 1 \Longrightarrow$ LimitAlongCurve $(q, R_5) = 1$
  - $\dim(Z_{\mathbb{R}}(R_6)) = 1 \Longrightarrow$ LimitAlongCurve $(q, R_6) = 1$

$\Longrightarrow$ the limit does not exists.

Outline

Experimentation

Ex	NV	TD	LM	TL	RFL	LV
1	2	4	0.061	0.097	0.312	-1
2	2	4	0.056	wrong answer	0.309	-1
3	2	2	0.015	0.002	0.121	undefined
4	2	4	0.096	0.001	0.814	undefined
5	2	4	0.064	0.089	0.313	-1
6	3	5	N/A	0.508	4.952	0
7	3	8	N/A	$> 2GB$	$> 2GB$	0
8	3	18	N/A	10.422	0.185	0
9	3	18	N/A	0.502	0.164	0
10	4	4	N/A	0.002	1.411	undefined
11	4	2	N/A	0.003	0.241	undefined
12	4	4	N/A	0.002	1.414	undefined
13	4	5	N/A	$> 2GB$	2.727	0
14	4	21	N/A	$> 2GB$	4.502	0
15	4	6	N/A	$> 2GB$	1.986	0

- NV : number of variables
- TD : total degree
- LV : limit value
- LM : limit/multi
- TL : TestLimit
- RFL : RationalFunctionLimit

Ex	NV	TD	LM	TL	RFL	LV
16	5	19	N/A	$> 2GB$	0.400	0
17	5	4	N/A	2.705	1.053	0
18	6	6	N/A	Error	1.140	0
19	6	6	N/A	Error	1.274	undefined
20	6	18	N/A	Error	0.269	0
21	6	10	N/A	$> 2GB$	5.395	0
22	6	10	N/A	$> 2GB$	2.474	0
23	6	6	N/A	Error	4.372	0
24	7	6	N/A	0.002	0.012	undefined
25	8	5	N/A	$> 2GB$	7.895	0
26	8	9	N/A	Error	20.132	undefined
27	9	4	N/A	0.003	3.058	undefined
28	9	10	N/A	Error	72	0
29	9	5	N/A	Error	0.526	0
30	10	10	N/A	Error	15.198	0

- NV : number of variables
- TD : total degree
- LV : limit value
- LM : limit/multi
- TL : TestLimit
- RFL : RationalFunctionLimit

Outline

Concluding remarks

- We have presented a procedure for determining the existence and possible value of finite limits of n-variate rational function over $\mathbb{Q}$
- We rely on the theory of regular chains, which allows us to avoid computing singular loci and decompositions into irreducible components
- Our main tool is the `RealTriangularize` algorithm.
- We have implemented our procedure within the `RegularChains` library.
- Our code is available at `www.regularchains.org`
- Experimental results show that our code solves more test cases than the implementation of S.J. Xiao and G.X. Zeng (2014), in particular as variable number or total degree increases.

Current works

- Extending our algorithm to the case where the origin is not an isolated zero of the denominator is work in progress.
- Currently, our algorithm returns either a finite limit, when it exists, or `no_finite_limit`. Handling infinite is also work in progress.
- Currently, `RealTriangularize` decomposes an arbitrary semi-algebraic set S whereas what we really need here are the connected components of S which have the origin in their closure. This is also work in progress.