Computing the Limit Points of Quasi-componets of Regular Chains in Diemnsion One

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Plan

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Specification of the problem

Input

- Let $R \subset \mathbb{C}[X_1, \ldots, X_s]$ be a regular chain.
- Let h_R be the product of initials of polynomials of R.
- Let $W(R)$ be the quasi-component of R, that is $V(R) \setminus V(h_R)$.

Output

The non-trivial limit points of $W(R)$, that is $\overline{W(R)}^Z \setminus W(R)$, denoted by $\lim(W(R))$.

Plan

Motivation (I): the Ritt problem

The Ritt problem

Given the characteristic sets of two prime differential ideals $\mathcal{I}_1$ and $\mathcal{I}_2$, determine whether $\mathcal{I}_1 \subseteq \mathcal{I}_2$ holds or not:

- No algorithm is known,
- Equivalent to other key problems, see (O. Golubitsky et al., 2009).

The algebraic counterpart of the Ritt problem

Given regular chains R_1 and R_2, determine whether $\mathrm{sat}(R_1) \subseteq \mathrm{sat}(R_2)$ holds or not, without computing a basis for $\mathrm{sat}(R_1)$ or $\mathrm{sat}(R_2)$:

- No algorithm is known,
- Such an algorithm could be used to solve the differential problem.

Our strategy for the algebraic version

- $\sqrt{\mathrm{sat}(R_1)} \subseteq \sqrt{\mathrm{sat}(R_2)} \Longleftrightarrow \overline{W(R_2)}^Z \subseteq \overline{W(R_1)}^Z$
- $\overline{W(R)}^Z = W(R) \cup \lim(W(R))$

Motivation (II): from Kalkbrener to Wu-Lazard decompositions

Specification (in the case of an irreducible variety)

Input: An irreducible algebraic set $V(F)$ and a regular chain R s.t. $V(F) = \overline{W(R)}^Z$

Output: Regular chains $R_1, \ldots, R_e$ s.t. $V(F) = W(R_1) \cup \cdots \cup W(R_e)$

Wu's trick

- Compute a triangular decomposition of $F \cup \{h_R\}$.
- The trick generalizes to the case where $V(F)$ is not irreducible.
- In practice, this process is very inefficient (many repeated calculations).

Our proposed strategy

- Compute $V(F) \setminus W(R)$ directly as the set $\lim(W(R))$.

Plan

Example one

The variable order is $x < y < z$. The regular chain is:

$$\begin{cases} xz - y^2 = 0 \\ y^5 - x^2 = 0 \end{cases}$$

What are the limits of y and z when x approaches 0?

Example one

The variable order is $x < y < z$. The regular chain is:

$$\begin{cases} xz - y^2 = 0 \\ y^5 - {\color{red}x^2} = 0 \end{cases}$$

What are the limits of y and z when x approaches 0?

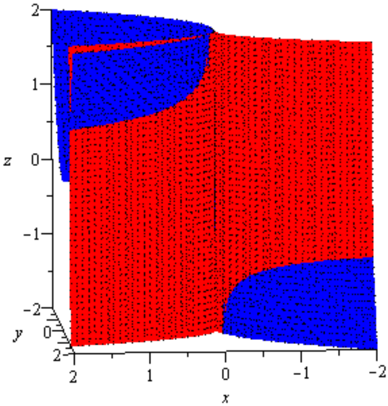

Figure: No limit points at $x = 0$

Example two

The variable order is $x < y < z$. The regular chain is:

$$\left\{ \begin{array}{l} xz - y^2 = 0 \\ y^5 - x^3 = 0 \end{array} \right.$$

What are the limits of y and z when x approaches 0?

Example two

The variable order is $x < y < z$. The regular chain is:

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What are the limits of y and z when x approaches 0?

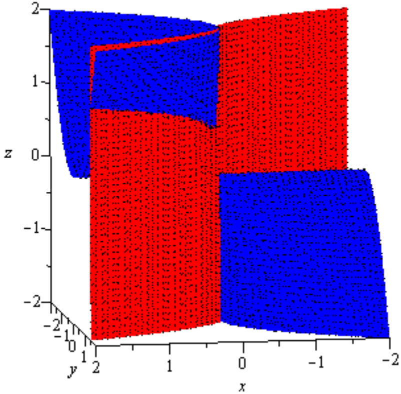


Figure: One limit point at $x = 0$.

How to compute the limit points

The variable order is $x < y < z$.

$$R_1 := \begin{cases} xz - y^2 = 0 \\ y^5 - x^2 = 0 \end{cases}$$

(1) solve $y^5 - x^2 = 0$, we get $y = x^{\frac{2}{5}}$

(2) substitute $y = x^{\frac{2}{5}}$ into $xz - y^2 = 0$, we get $xz - x^{\frac{4}{5}} = 0$

(3) since $x \neq 0$, we have $z = x^{-\frac{1}{5}}$

(4) so there are no limit points

$$R_2 := \begin{cases} xz - y^2 = 0 \\ y^5 - x^3 = 0 \end{cases}$$

(1) $y = x^{\frac{3}{5}}$

(2) $z = x^{\frac{1}{5}}$

(3) the limit point is $(x = 0, y = 0, z = 0)$

Plan

The problem

- Input: the regular chain R below with $X_1 < X_2 < X_3$

$$R := \begin{cases} r_2 &= (X_1+2)X_1X_3^2 + (X_2+1)(X_3+1) \\ r_1 &= X_1X_2^2 + X_2 + 1 \end{cases}$$

  The product of the initials of its polynomials is $h_R := X_1(X_1+2)$.
- Output: Limit points of $W(R)$ at $h_R = 0$.

Puiseux series expansions of r_1 at $X_1 = 0$

- The two Puiseux expansions of r_1 at $X_1 = 0$ are:

$$[X_1 = T, X_2 = -1 - T + O(T^2)],$$

$$[X_1 = T, X_2 = -T^{-1} + 1 + T + O(T^2)].$$

- The second expansion cannot result in a limit point while the first one might.

Limit points of $W(R)$ at $X_1 = 0$

- After substituting the first expansion into r_2, we have:

$$r_2' = (T+2)TX_3^2 + (-T + O(T^2))(X_3 + 1)$$

- Now, we compute Puiseux series expansions of r_2' which are

$$[T = T, X_3 = 1 - 1/3T + O(T^2)],$$
$$[T = T, X_3 = -1/2 + 1/12\,T + O(T^2)].$$

- So the regular chains

$$\begin{cases} X_3 - 1 = 0 \\ X_2 + 1 = 0 \\ X_1 = 0 \end{cases}, \quad \begin{cases} X_3 + 1/2 = 0 \\ X_2 + 1 = 0 \\ X_1 = 0 \end{cases}$$

give the limit points of $W(R)$ at $X_1 = 0$.

Limit points of $W(R)$ at $X_1 = -2$

- Puiseux series expansions of r_1 at the point $X_1 = -2$:

$$[X_1 = T-2, X_2 = 1 + 1/3\,T + O(T^2)],$$
$$[X_1 = T-2, X_2 = -1/2 - 1/12\,T + O(T^2)].$$

- After substitution into r_2, we obtain:

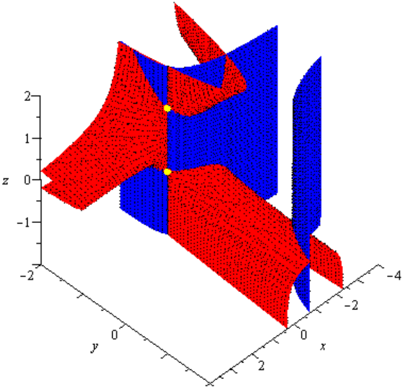
$$\begin{aligned} r'_{12} &= (T-2)\,T{X_3}^2 + \left(2 + 1/3\,T + O(T^2)\right)(X_3+1) \\ r'_{22} &= (T-2)\,T{X_3}^2 + \left(1/2 - 1/12\,T + O(T^2)\right)(X_3+1)\,. \end{aligned}$$

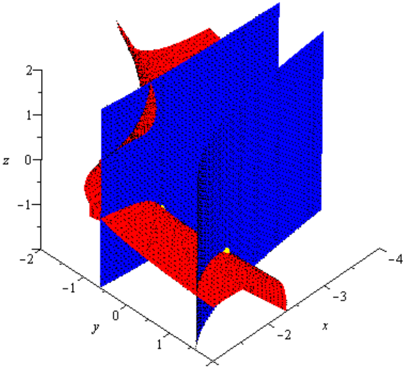
- Puiseux expansions of r'_{12} and r'_{22} at $T = 0$ resulting in limit points:
  - i) for r'_{12}: $[T = T, X_3 = -1 + T + O(T^2)]$
  - ii) for r'_{22}: $[T = T, X_3 = -1 + 4T + O(T^2)]$
- The limit points of $W(R)$ at $X_1 = -2$ are represented by the regular chains $\{X_1 + 2, X_2 - 1, X_3 + 1\}$ and $\{X_1 + 2, X_2 + 1/2, X_3 + 1\}$.

Visualizing the limit points of $W(R)$

The limit points are:

$$\left\{\begin{array}{l} X_3 - 1 = 0 \\ X_2 + 1 = 0 \\ X_1 = 0 \end{array}\right., \quad \left\{\begin{array}{l} X_3 + 1/2 = 0 \\ X_2 + 1 = 0 \\ X_1 = 0 \end{array}\right., \quad \left\{\begin{array}{l} X_3 + 1 = 0 \\ X_2 - 1 = 0 \\ X_1 + 2 = 0 \end{array}\right., \quad \left\{\begin{array}{l} X_3 + 1 = 0 \\ X_2 + 1/2 = 0 \\ X_1 + 2 = 0 \end{array}\right.$$





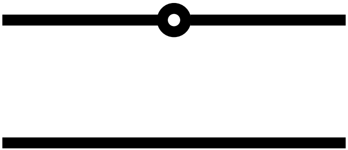
Plan

Zariski topology

Zariski closure

- Let $\mathbf{k}$ be an algebraically closed field, like $\mathbb{C}$.
- We denote by $\mathbb{A}^s$ the *affine s-space* over $\mathbf{k}$.
- An *affine algebraic variety* of $\mathbb{A}^s$ is the set of common zeroes of a collection $F \subseteq \mathbf{k}[X_1, \ldots, X_s]$ of polynomials.
- The Zariski topology on $\mathbb{A}^s$ is the topology whose closed sets are the affine algebraic varieties of $\mathbb{A}^s$.
- The Zariski closure of a subset $W \subseteq \mathbb{A}^s$ is the intersection of all affine algebraic varieties containing W.

The set $\{y = 0, x \neq 0\}$ and its Zariski closure $\{y = 0\}$.



Zariski topology and the Euclidean topology

The relation between the two topologies

- With $\mathbf{k} = \mathbb{C}$, the affine space $\mathbb{A}^s$ is endowed with both topologies.
- The basic open sets of the Euclidean topology are the open balls.
- The basic open sets of Zariski topology are the complements of hypersurfaces.
- Thus, a Zariski closed (resp. open) set is closed (resp. open) in the Euclidean topology on $\mathbb{A}^s$.
- That is, Zariski topology is coarser than the Euclidean topology.

Theorem (The relation between two closures (D. Mumford))

- *Let $V \subseteq \mathbb{A}^s$ be an irreducible affine variety.*
- *Let $U \subseteq V$ be nonempty and open in the Zariski topology induced on V.*

Then, U has the same closure in both topologies. In fact, we have

$$V = \overline{U}^Z = \overline{U}^E.$$

Limit points

- Let (X, τ) be a topological space and $S \subseteq X$ be a subset.
- A point $p \in X$ is a limit point of S if every neighborhood of p contains at least one point of S different from p itself.
- If X is a metric space, the point p is a limit point of S if and only if there exists a sequence $(x_n, n \in \mathbb{N})$ of points of $S \setminus \{p\}$ such that $\lim_{n\to\infty} x_n = p$.
- The limit points of S which do not belong to S are called non-trivial, denoted by $\lim(S)$.

Example

Consider the interval $S := [1, 2) \subset \mathbb{R}$. The point 2 is a non-trivial limit point of S.

Limit points of the quasi-component of a regular chain

Recall Mumford's Theorem

- Let $V \subseteq \mathbb{A}^s$ be an irreducible affine variety.
- Let $U \subseteq V$ be nonempty and open in the Zariski topology induced on V.

Then $V = \overline{U}^Z = \overline{U}^E$.

Corollary

Let R be a regular chain. Recall that $\mathrm{sat}(R) := \langle R \rangle : \mathrm{init}(R)^\infty$ is its saturated ideal and $W(R) = V(R) \setminus V(\mathrm{init}(R))$ is its quasi-component. Then, we have

$$V(\mathrm{sat}(R)) = \overline{W(R)}^Z = \overline{W(R)}^E.$$

- We use $\overline{W(R)}$ to denote this common closure.
- $\lim(W(R)) := \overline{W(R)} \setminus W(R)$ denotes the limit points of $W(R)$.

Field of Puiseux series

- Let T be a symbol.
- $\mathbb{C}[[T]]$: ring of formal power series.
- $\mathbb{C}\langle T\rangle$: ring of convergent power series.
- $\mathbb{C}[[T^*]] = \cup_{n=1}^{\infty}\mathbb{C}[[T^{\frac{1}{n}}]]$: ring of formal Puiseux series.
- $\mathbb{C}\langle T^*\rangle = \cup_{n=1}^{\infty}\mathbb{C}\langle T^{\frac{1}{n}}\rangle$: ring of convergent Puiseux series.
- $\mathbb{C}((T^*))$: quotient field of $\mathbb{C}[[T^*]]$, or the field of Puiseux series.
- $\mathbb{C}(\langle T^*\rangle)$: quotient field of $\mathbb{C}\langle T^*\rangle$, or the field of convergent Puiseux series.

We have

- $\mathbb{C}[[T]] \subset \mathbb{C}[[T^*]] \subset \mathbb{C}((T^*))$; $\mathbb{C}\langle T\rangle \subset \mathbb{C}\langle T^*\rangle \subset \mathbb{C}(\langle T^*\rangle)$
- $\mathbb{C}\langle T\rangle \subset \mathbb{C}[[T^*]]$; $\mathbb{C}\langle T^*\rangle \subset \mathbb{C}[[T^*]]$; $\mathbb{C}(\langle T^*\rangle) \subset \mathbb{C}((T^*))$

Example

We have $\sum_{i=0}^{\infty} T^i \in \mathbb{C}\langle T\rangle$, $\sum_{i=0}^{\infty} T^{\frac{i}{2}} \in \mathbb{C}\langle T^*\rangle$ and $\sum_{i=-3}^{\infty} T^{\frac{i}{2}} \in \mathbb{C}(\langle T^*\rangle)$.

Theorem (Puiseux)

Both $\mathbb{C}((T^*))$ *and* $\mathbb{C}(\langle T^* \rangle)$ *are algebraically closed fields.*

Puiseux expansions

- Let $\mathbf{k} = \mathbb{C}((X^*))$ or $\mathbb{C}(\langle X^* \rangle)$.
- Let $f \in \mathbf{k}[Y]$, where $d := \deg(f, Y) > 0$.
- There exist $\varphi_i \in \mathbf{k}$, $i = 1, \ldots, d$, such that

$$\frac{f}{\mathrm{lc}(f, Y)} = (Y - \varphi_1) \cdots (Y - \varphi_d).$$

- We call $\varphi_1, \ldots, \varphi_d$ the *Puiseux expansions* of f at the origin.

Example

- $(Y^2 - X) = (Y - X^{\frac{1}{2}})(Y + X^{\frac{1}{2}})$.
- Puiseux expansions of $Y^2 - XY - X$:
  $Y - (X^{\frac{1}{2}} + \frac{1}{2}X + \frac{1}{8}X^{\frac{3}{2}} + O(X^2)), Y - (-X^{\frac{1}{2}} + \frac{1}{2}X - \frac{1}{8}X^{\frac{3}{2}} + O(X^2))$.

Puiseux parametrizations

Let $f \in \mathbb{C}\langle X\rangle[Y]$. A Puiseux parametrization of f is a pair $(\psi(T), \varphi(T))$ of elements of $\mathbb{C}\langle T\rangle$ for some new variable T, such that

- $\psi(T) = T^{\varsigma}$, for some $\varsigma \in \mathbb{N}_{>0}$.
- $f(X = \psi(T), Y = \varphi(T)) = 0$ holds in $\mathbb{C}\langle T\rangle$,
- there is no integer $k > 1$ such that both $\psi(T)$ and $\varphi(T)$ are in $\mathbb{C}\langle T^k\rangle$.

The index ς is the *ramification index* of the parametrization $(T^{\varsigma}, \varphi(T))$.

Relation to Puiseux expansions

- Let $z_1, \ldots, z_{\varsigma}$ denote the primitive roots of unity of order ς in $\mathbb{C}$. Then $\varphi(z_i X^{1/\varsigma})$, for $i = 1, \ldots, \varsigma$, are ς Puiseux expansions of f.
- For a Puiseux expansion φ of f, let c minimum s.t. $\varphi = g(T^{1/c})$ and $g \in \mathbb{C}\langle T\rangle$. Then $(T^c, g(T))$ is a Puiseux parametrization of f.

Example

Puiseux parametrization of $Y^2 - XY - X$:

$$\left(X = T^2, Y = T + \tfrac{1}{2}T^2 + \tfrac{1}{8}T^3 + O(T^4)\right)$$

Plan

Puiseux expansions of a regular chain

Notation

- Let $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\} \subset \mathbb{C}[X_1 < \cdots < X_s]$ be a 1-dim regular chain.
- Assume R is strongly normalized, that is, $\mathrm{init}(R) \in \mathbb{C}[X_1]$.
- Let $\mathbf{k} = \mathbb{C}(\langle X_1^* \rangle)$.
- Then R generates a zero-dimensional ideal in $\mathbf{k}[X_2, \ldots, X_s]$.
- Let $V^*(R)$ be the zero set of R in $\mathbf{k}^{s-1}$.

Definition

We call *Puiseux expansions* of R the elements of $V^*(R)$.

Remarks

- The *strongly normalized assumption* is only for presentation ease.
- Generically, The 1-*dim* assumption extends to d-*dim* $d \leq 2$.
- Higher dimension requires the Jung-Abhyankar theorem.

An example

A regular chain R

$$R := \begin{cases} X_1 X_3^2 + X_2 \\ X_1 X_2^2 + X_2 + X_1 \end{cases}$$

Puiseux expansions of R

$$\begin{cases} X_3 &= & 1 + O(X_1^2) \\ X_2 &= & -X_1 + O(X_1^2) \end{cases} \qquad \begin{cases} X_3 &= & -1 + O(X_1^2) \\ X_2 &= & -X_1 + O(X_1^2) \end{cases}$$

$$\begin{cases} X_3 &= & {X_1}^{-1} - \frac{1}{2} X_1 + O(X_1^2) \\ X_2 &= & -{X_1}^{-1} + X_1 + O(X_1^2) \end{cases} \qquad \begin{cases} X_3 &= & -{X_1}^{-1} + \frac{1}{2} X_1 + O(X_1^2) \\ X_2 &= & -{X_1}^{-1} + X_1 + O(X_1^2) \end{cases}$$

Relation between $\lim_0(W(R))$ and Puiseux expansions of R

Theorem

For $W \subseteq \mathbb{C}^s$, denote

$$\lim_0(W) := \{x = (x_1, \ldots, x_s) \in \mathbb{C}^s \mid x \in \lim(W) \text{ and } x_1 = 0\},$$

and define

$$V^*_{\geq 0}(R) := \{\Phi = (\Phi^1, \ldots, \Phi^{s-1}) \in V^*(R) \mid \operatorname{ord}(\Phi^j) \geq 0, j = 1, \ldots, s-1\}.$$

Then we have

$$\lim_0(W(R)) = \cup_{\Phi \in V^*_{\geq 0}(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}.$$

$$V^*_{\geq 0}(R) := \left\{ \begin{array}{lcl} X_3 & = & 1 + O(X_1^2) \\ X_2 & = & -X_1 + O(X_1^2) \end{array} \right. \cup \left\{ \begin{array}{lcl} X_3 & = & -1 + O(X_1^2) \\ X_2 & = & -X_1 + O(X_1^2) \end{array} \right.$$

Thus the limit ponts are $\lim_0(W(R)) = \{(0, 0, 1), (0, 0, -1)\}$.

Puiseux parametrizations of a regular chain

Idea

- Let $\Phi_i = (\Phi_i^1, \ldots, \Phi_i^{s-1}) \in V^*_{\geq 0}(R)$ be a Puiseux expansion, $1 \leq i \leq M := |V^*_{\geq 0}(R)|$. Recall that $\Phi_i^1, \ldots, \Phi_i^{s-1} \in \mathbb{C}(\langle X_1^* \rangle)$.
- Φ_i can be associated with a Puiseux parametrization $(X_1 = T^{\varsigma_i}, X_2 = g_i^1(T), \ldots, X_s = g_i^{s-1}(T))$ with $g_i^j \in \mathbb{C}\langle T \rangle$.

Details

- Note: Φ_i^j is an expansion of $r_j(X_1, X_2 = \Phi_i^1, \ldots, X_j = \Phi_i^{j-1}, X_{j+1})$.
- Let $(T^{\varsigma_{i,j}}, X_j = \varphi_i^j(T))$ be the corresponding Puiseux parametrization of Φ_i^j, where $\varsigma_{i,j}$ is the ramification index of Φ_i^j.
- Let ς_i be the l.c.m. of $\{\varsigma_{i,1}, \ldots, \varsigma_{i,s-1}\}$ and $g_i^j := \varphi_i^j(T = T^{\varsigma_i/\varsigma_{i,j}})$.

Definition

$\mathfrak{G}_R := \{(X_1 = T^{\varsigma_i}, X_2 = g_i^1(T), \ldots, X_s = g_i^{s-1}(T)), i = 1, \ldots, M\}$ is a *system of Puiseux parametrizations* of R.

Relation between $\lim_0(W(R))$ and Puiseux parametrizations of R

Notation (recall)

Let $\mathfrak{G}_R := \{(X_1 = T^{\varsigma_i}, X_2 = g_i^1(T), \ldots, X_s = g_i^{s-1}(T)), i = 1, \ldots, M\}$ be a system of Puiseux parametrizations of R.

Theorem

We have

$$\lim_0(W(R)) = \mathfrak{G}_R(T = 0).$$

Plan

Limit points of a plane curve (without Puiseux parametrizations)

Theorem (Lemaire-MorenoMaza-Pan-Xie 08, $\langle T \rangle \stackrel{?}{=} \mathrm{sat}(T)$)

Let $f \in \mathbb{C}[X][Y]$. Assume that f is primitive in Y. Then $\lim_0(W(f)) = \{(0, y) \mid f(0, y) = 0\}$.

Theorem (R.J. Walker, 50)

Let $f \in \mathbb{C}[X][Y]$. Assume that f is general in Y, that is $f(0, Y) \neq 0$. Then, $\lim_0(W(f)) = \{(0, y) \mid f(0, y) = 0\}$.

Theorem

- *Let $f \in \mathbb{C}\langle X \rangle[Y]$.*
- *Assume that f is general in Y.*
- *Let $\rho > 0$ be small enough such that f converges in $|X| < \rho$.*
- *Let $V_\rho(f) := \{(x, y) \mid 0 < |x| < \rho, f(x, y) = 0\}$.*

Then, we have $\lim_0(V_\rho(f)) = \{(0, y) \mid f(0, y) = 0\}$.

From algebra to computer: what is the challenge?

Algebra

Let $\mathfrak{G}_R$ be a system of Puiseux parametrizations of R. Recall that we have

$$\lim_0(W(R)) = \mathfrak{G}_R(T = 0).$$

When Walker's theorem applies or when the T is a primitive regular chain, we do not need to compute $\mathfrak{G}_R(T = 0)$. However, those are criteria only!

How to compute $\mathfrak{G}_R$ when the previous criteria do not apply?

- We shall not compute $\mathfrak{G}_R$.
- We need to compute $\mathfrak{G}_R(T = 0)$.
- In fact, we compute a truncation (approximation) of $\mathfrak{G}_R$.

The back-substitution process for computing $\mathfrak{G}_R$

Specifications

Input: $R := \{r_1(X_1, X_2), \ldots, r_{s-1}(X_1, \ldots, X_s)\}$ a 1-dim strongly normalized regular chain.

Output: $\mathfrak{G}_R$: a system of Puiseux parametrizations of R.

Algorithm

Polynomial	Substitution	Puiseux parametrsation
$r_1(X_1, X_2)$	N/A	$(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1))$
$r_2(X_1, X_2, X_3)$	$r_2(T_1^{\varsigma_1}, \varphi_1(T_1), X_3)$	$(T_1 = T_2^{\varsigma_2}, X_3 = \varphi_2(T_2))$
$r_3(X_1, X_2, X_3, X_4)$	$r_3(T_2^{\varsigma_1\varsigma_2}, \varphi_1(T_2^{\varsigma_2}), \varphi_2(T_2), X_4)$	$(T_2 = T_3^{\varsigma_3}, X_4 = \varphi_3(T_3))$

More generally, for $i = 2, \ldots, s-1$, we define:

- $f_i := r_i(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1), \ldots, X_i = \varphi_{i-1}(T_{i-1}), X_{i+1}) \in \mathbb{C}\langle T_{i-1}\rangle[X_{i+1}]$,
- $(T_i := T_{i-1}^{\varsigma_i}, X_{i+1} := \varphi_i(T_i))$.

New problem: compute Puiseux parametrizations of f_i of given accuracy.

Puiseux parametrizations of $f \in \mathbb{C}\langle X\rangle[Y]$ of finite accuracy

Definition

- Let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathbb{C}[[X]]$.
- For any $\tau \in \mathbb{N}$, let $f^{(\tau)} := \sum_{i=0}^{\tau} a_i X^i$.
- We call $f^{(\tau)}$ the polynomial part of f *of accuracy* $\tau + 1$.

Definition

- Let $f \in \mathbb{C}\langle X\rangle[Y]$, $\deg(f, Y) > 0$.
- Let $\sigma, \tau \in \mathbb{N}_{>0}$ and $g(T) = \sum_{k=0}^{\tau-1} b_k T^k$.
- Let $\{T^{k_1}, \ldots, T^{k_m}\}$ be the support of $g(T)$.
- The pair $(T^\sigma, g(T))$ is called a Puiseux parametrization of f of accuracy τ if there exists a Puiseux parametrization $(T^\varsigma, \varphi(T))$ of f such that
  - (i) σ divides ς.
  - (ii) $\gcd(\sigma, k_1, \ldots, k_m) = 1$.
  - (iii) $g(T^{\varsigma/\sigma})$ is the polynomial part of $\varphi(T)$ of accuracy $(\varsigma/\sigma)(\tau - 1) + 1$.

Computing Puiseux parametrizations of $f \in \mathbb{C}\langle X\rangle[Y]$ of finite accuracy

Theorem

- *Let* $f = \sum_{i=0}^{d} \sum_{j=0}^{\infty} a_{i,j} Y^i \in \mathbb{C}\langle X\rangle[Y]$.
- *Then we can compute* $m \in \mathbb{N}$ *such that the Puiseux parametrizations of* f *of accuracy* τ *are exactly the Puiseux parametrizations of* $\sum_{i=0}^{d} \sum_{j=0}^{m-1} a_{i,j} Y^i$ *of accuracy* τ.

Lemma

- *Let* $f = a_d(X)Y^d + \cdots + a_0(X) \in \mathbb{C}\langle X\rangle[Y]$.
- *Let* $\delta := \mathrm{ord}(a_d(X))$.
- *Then "generically", we can choose* $m = \tau + \delta$.

Recall the back-substitution process for computing $\mathfrak{G}_R$

Algorithm

Polynomial	Substitution	Puiseux parametrsation
$r_1(X_1, X_2)$	N/A	$(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1))$
$r_2(X_1, X_2, X_3)$	$r_2(T_1^{\varsigma_1}, \varphi_1(T_1), X_3)$	$(T_1 = T_2^{\varsigma_2}, X_3 = \varphi_2(T_2))$
$r_3(X_1, X_2, X_3, X_4)$	$r_3(T_2^{\varsigma_1\varsigma_2}, \varphi_1(T_2^{\varsigma_2}), \varphi_2(T_2), X_4)$	$(T_2 = T_3^{\varsigma_3}, X_4 = \varphi_3(T_3))$
$\vdots$	$\vdots$	$\vdots$

More generally, for $i = 2, \ldots, s-1$, we define:

- $f_i := r_i(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1), \ldots, X_i = \varphi_{i-1}(T_{i-1}), X_{i+1}) \in \mathbb{C}\langle T_{i-1}\rangle[X_{i+1}]$,
- $(T_i := T_{i-1}^{\varsigma_i}, X_{i+1} := \varphi_i(T_i))$.

Putting everything together

Let $R := \{r_1(X_1, X_2), \dots, r_{s-1}(X_1, \dots, X_s)\} \subset \mathbb{C}[X_1 < \cdots < X_s]$. For $1 \leq i \leq s-1$, let

- $h_i := \mathrm{init}(r_i)$
- $d_i := \deg(r_i, X_{i+1})$
- $\delta_i := \mathrm{ord}(h_i)$.

Theorem

One can compute positive integer numbers $\tau_1, \dots, \tau_{s-1}$ such that, in order to compute $\lim_0(W(R))$, it sufficies to compute Puiseux parametrizations of f_i of accuracy τ_i, for $i = 1, \dots, s-1$. Moreover, generically, we can choose τ_i, $i = 1, \dots, s-1$, as follows

- $\tau_{s-1} := 1$
- $\tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k)\delta_{s-1} + 1$
- $\tau_i = (\prod_{k=1}^{s-2} \varsigma_k)(\sum_{k=2}^{s-1} \delta_i) + 1$, $i = 1, \dots, s-3$.

Moreover, the indices ς_k can be replaced with d_k, $k = 1, \dots, s-2$.

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Maple packages used: RegularChains and algcurves:-puiseux.

- T: timings of Triangularize
- $\#(T)$: number of regular chains returned by Triangularize
- d-1, d-0: number of one and zero dimensional components
- R: timings spent on removing redundant components
- $\#(R)$: number of irredundant components

Table: Removing redundant components in Kalkbrener decompositions.

Sys	T	#(T)	d-1	d-0	R	#(R)
f-744	14.360	4	1	3	432.567	1
Liu-Lorenz	0.412	3	3	0	216.125	3
MontesS3	0.072	2	2	0	0.064	2
Neural	0.296	5	5	0	1.660	5
Solotareff-4a	0.632	7	7	0	32.362	7
Vermeer	1.172	2	2	0	75.332	2
Wang-1991c	3.084	13	13	0	6.280	13

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Concluding remarks

- We proposed an algorithm for computing the limit points of the quasi-component of a regular chain in dimension one.
- To this end, we make use of the *Puiseux series expansions* of a regular chain.
- In addition, we have sharp bounds on the degree of truncations that are required to compute *approximate Puiseux series expansions* from which the desired limit points can be obtained.
- Our experimental results show that this is a useful tool for dealing with triangular decompositions of polynomial systems.
- For instance, for testing inclusion between saturated ideals of regular chains in a direct manner (i.e. without computing a basis).
- Computing limit points in higher dimension may require the help of the Abhyankar-Jung theorem. This is work in progress.