# Decomposition and QE algorithms over the reals and over the integers

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- As well as collaborators: James H. Davenport (University of Bath), Matthew England (Coventry University), John May (Maplesoft), Bican Xia (Peking University).
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# Tentative Plan

1. Decomposition and QE algorithms over the reals
2. Decomposition and QE algorithms over the integers

# What does solving mean here?

- **Solving** over $\mathbb{C}$: that is, solving any system of multivariate polynomial equations (say, $f_1 = \cdots = f_m = 0$) and inequations $h \neq 0$:
  - computing **all** its solutions symbolically, or only the **generic** ones
  - providing tools to **extract information** (dimension, degree, etc.) about those solutions and,
  - **performing** (set or geometric) **operations** on solutions sets.
- **Solving** over $\mathbb{R}$: that is, solving any system of multivariate polynomial equations (say, $f_1 = \cdots = f_m = 0$, inequations $h \neq 0$ and inequalities $g_1 > 0, \ldots, g_p > 0$
  - doing the **same** as above, or
  - finding **sample** solutions, or
  - performing **cylindrical** algebraic decomposition (CAD) or **quantifier** elimination (QE).
- **Solving** over $\mathbb{Z}$: focusing on linear inequality systems, can mean:
  - **counting** the number of solutions, or
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# Outline

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# Solving for the real solutions of polynomial systems

## Classical tools as of 2010

- Cylindrical algebraic decomposition of polynomial systems:
  `SemiAlgebraicSetTools:-CylindricalAlgebraicDecompose` (James)
- Real root classification of parametric polynomial systems:
  `ParametricSystemTools:-RealRootClassification` (Bican)
- Decomposing polynomial systems over the algebraic closure of the base field:
  `RegularChains:-Triangularize` (ORCCA)

## New tools in the RegularChains library 2011

- Triangular decomposition of semi-algebraic systems: `RealTriangularize`
- Sampling all connected components of a semi-algebraic system: `SamplePoints`
- Set-theoretical operations on semi-algebraic sets:
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# Regular semi-algebraic system

## Notation

- Let $T \subset \mathbb{Q}[x_1 < \ldots < x_n]$ be a regular chain with $\mathbf{y} := \{\mathrm{mvar}(t) \mid t \in T\}$ and $\mathbf{u} := \mathbf{x} \setminus \mathbf{y} = u_1, \ldots, u_d$.
- Let $P$ be a finite set of polynomials, s.t. every $f \in P$ is regular modulo $\mathrm{sat}(T)$.
- Let $\mathcal{Q}$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{u}]$.

## Definition

We say that $R := [\mathcal{Q}, T, P_>]$ is a regular semi-algebraic system if:

$(i)$ $\mathcal{Q}$ defines a non-empty open semi-algebraic set $S$ in $\mathbb{R}^d$,

$(ii)$ the regular system $[T, P]$ specializes well at every point $u$ of $S$

$(iii)$ at each point $u$ of $S$, the specialized system $[T(u), P(u)_>]$ has at least one real solution.

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

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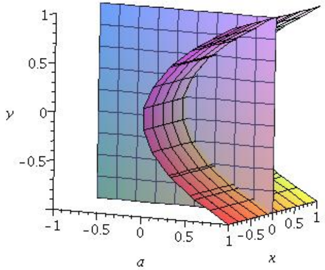
## Example

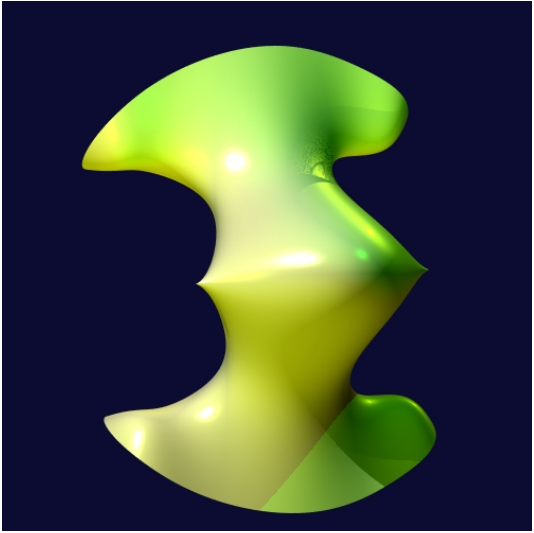
The system $[\mathcal{Q}, T, P_>]$, where

$$\mathcal{Q} := a > 0,\ T := \begin{cases} y^2 - a = 0 \\ x = 0 \end{cases},\ P_> := \{y > 0\}$$

is a regular semi-algebraic system.



# RealTriangularize applied to the Eve surface (1/2)



# RealTriangularize applied to the Eve surface (2/2)

$R := PolynomialRing([x, y, z]);\ F := [5*x^2 + 2*x*z^2 + 5*y^6 + 15*y^4 + 5*z^2 - 15*y^5 - 5*y^3];$

$$polynomial\_ring$$

$$[5x^2 + 2xz^2 + 5y^6 + 15y^4 + 5z^2 - 15y^5 - 5y^3]$$

$RealTriangularize(F, R, output = record);$

$$\left\{\begin{array}{l} 5x^2 + 2z^2x + 5y^6 + 15y^4 - 5y^3 - 15y^5 + 5z^2 = 0 \\ 25y^6 - 75y^5 + 75y^4 - z^4 - 25y^3 + 25z^2 < 0 \end{array}\right.,$$

$$\left\{\begin{array}{c} 5x + z^2 = 0 \\ 25y^6 - 75y^5 + 75y^4 - 25y^3 - z^4 + 25z^2 = 0 \\ 64z^4 - 1600z^2 + 25 > 0 \\ z \neq 0 \\ z - 5 \neq 0 \\ z + 5 \neq 0 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y - 1 = 0 \\ z = 0 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z = 0 \end{array}\right., \left\{\begin{array}{c} x + 5 = 0 \\ y - 1 = 0 \\ z - 5 = 0 \end{array}\right.,$$

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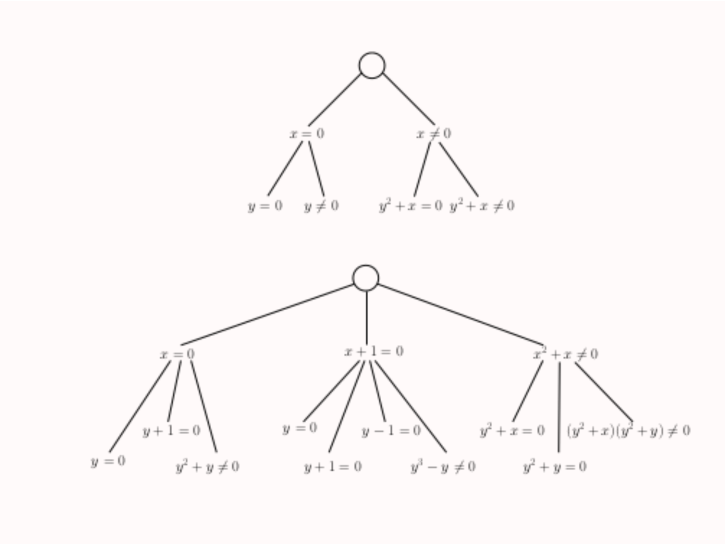


- A CAD of $\{y^2 + x, y^2 + y\}$ is computed **incrementally**: refining a CAD tree of $y^2 + x$ with $y^2 + y$.
- Experimental results in [5] (ASCM 2012) suggest that this approach outperforms the projection-and-lifting scheme of [7] (ISSAC 2009).

# Outline

```
> phi1 := ( ( 74 <= x ) &and ( x <= 76 ) &and ( v = 0 )
  &implies ( -v^2 - a * (x-75)^2 + b >= 0 ) ):

> phi2 := ( ( -v^2 - a * (x-75)^2 + b >= 0 )
  &implies (( 80 >= x ) &and ( x >= 70 )) ):

> phi3 := ( ( -v^2 - a * (x-75)^2 + b = 0 )
  &implies (( -2*v - a * 2 * (x-75)* v >= 0 ) &or ( 2*v - a
  * 2 * (x-75)* v >= 0 )) ):

> phi := phi1 &and phi2 &and phi3:
> t0 := time():
  psi := QuantifierElimination(&A([x,v]),phi,output=rootof);
  t1 := time() - t0;
```

$$\psi := ((0 < a \ \&\text{and}\ a \leq 1) \ \&\text{and}\ a \leq b) \ \&\text{and}\ b \leq \min\left(\frac{1}{a}, 25\, a\right)$$

$$t1 := 15.094$$

- **QE** based on regular chains and incremental CAD [6] (presented by James for us at ISSAC 2014) is illustrated above.
- This QE problem instance is related to a verification and synthesis of switched and hybrid dynamical systems (Sturm-Tiwari, ISSAC 2011).

# Outline

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# Dependence analysis

Cholesky's LU decomposition:

```
1:  for(i = 1; i <= n; i++){
        x = a[i][i];
        for(k = 1; k < i; k++)
2:          x = x - a[i][k] * a[i][k];
3:      p[i] = 1.0/sqrt(x);
        for(j = i+1; j <= n; j++){
4:          x = a[i][j];
            for(k = 1; k < i; k++)
5:              x = x - a[j][k] * a[i][k];
6:          a[j][i] = x * p[i];
        }
    }
```

system 1:

$$\begin{cases} 1 \le i \le n \\ i+1 \le j \le n \\ 1 \le k \le i-1 \\ 1 \le i' \le n \\ i'+1 \le j' \le n \\ j = j', k = i' \\ i < i' \end{cases}$$

system 2:

$$\begin{cases} 1 \le i \le n \\ i+1 \le j \le n \\ 1 \le k \le i-1 \\ 1 \le i' \le n \\ i'+1 \le j' \le n \\ j = j', k = i' \\ i = i', j < j' \end{cases}$$

system 3:

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# Delinearization

## Linearized multi-dimensional array

```
for (int i = 0; i < n; i ++)
    for (int j = i + 1; j < n; j ++)
        A[i * n + j] =
                A[n * n - n + j - 1];
```

$$\begin{cases} 0 \leq i_1 < n \\ i_1 + 1 \leq j_1 < n \\ 0 \leq i_2 < n \\ i_2 + 1 \leq j_2 < n \\ i_1 * n + j_1 = n^2 - n + j_2 - 1 \end{cases} \tag{1}$$

## Delinearized multi-dimensional array

```
for (int i = 0; i < n; i ++)
    for (int j = i + 1; j < n; j ++)
        A[i][j] = A[n - 1][j - 1];
```

$$\begin{cases} 0 \leq i_1 < n \\ i_1 + 1 \leq j_1 < n \\ 0 \leq i_2 < n \\ i_2 + 1 \leq j_2 < n \\ i_1 = n - 1 \\ j_1 = j_2 - 1 \end{cases} \tag{2}$$

# Problem definition

**Input**:

$(\mathbf{i}_1 \cdots ; \ \cdots ; \ i_1{+}{+})$
$\ldots (i_d \cdots ; \cdots ; \ i_d{+}{+})$
$A[R(i_1, \ldots, i_d, m_1, \ldots, m_\delta)] \leftarrow \cdots \ \ldots$

- $i_1, \ldots, i_d$ take non-negative integer values such that
$$L \begin{pmatrix} i_1 \\ \vdots \\ i_d \end{pmatrix} \leq \begin{pmatrix} r_1 \\ \vdots \\ r_d \end{pmatrix},$$
- L is a lower-triangular full-rank matrix over $\mathbb{Z}$ (known at compile time) defining the iteration domain
- $m_1, \ldots, m_\delta, r_1, \ldots, r_d$: data parameters (known only at execution time)
- $R(i_1, \ldots, i_d, m_1, \ldots, m_\delta)$ is a polynomial, the coefficients of which are known at compile time.

**Output**:

$(i_1 \cdots ; \ \cdots ; \ i_1{+}{+})$
$\ldots (i_d \cdots ; \cdots ; \ i_d{+}{+})$
$\widetilde{A}[f_1] \cdots [f_\delta] \leftarrow \cdots \ \ldots$

- $f_1, \ldots, f_\delta$ are affine forms in $i_1, \ldots, i_d$ the coefficients of which are integers to-be-determined,
- $\widetilde{A}$ is an $M_1 \times \cdots \times M_\delta$-array,
- $M_1, \ldots, M_\delta$ are affine forms in $m_1, \ldots, m_\delta$ the coefficients of which are integers TBD,

such that:
$$R = f_1 M_2 \cdots M_\delta \ + \ \cdots \ + \ f_{\delta-1} M_2 + f_\delta$$
holds and for each $(i_1, \ldots, i_d)$ in the iteration domain we have:
$$0 \leq f_1 < M_1, \ \ \ldots, 0 \leq f_\delta < M_\delta.$$

# Problem definition

**Input**:

$(\mathbf{i}_1 \cdots ;\ \cdots ;\ i_1{+}{+})$
$\ldots(i_d \cdots ; \cdots ;\ i_d{+}{+})$
$A[R(i_1, \ldots, i_d, m_1, \ldots, m_\delta)] \leftarrow \cdots\ \ldots$

- $i_1, \ldots, i_d$ take non-negative integer values such that
$$L \begin{pmatrix} i_1 \\ \vdots \\ i_d \end{pmatrix} \leq \begin{pmatrix} r_1 \\ \vdots \\ r_d \end{pmatrix},$$
- L is a lower-triangular full-rank matrix over $\mathbb{Z}$ (known at compile time) defining the iteration domain
- $m_1, \ldots, m_\delta, r_1, \ldots, r_d$: data parameters (known only at execution time)
- $R(i_1, \ldots, i_d, m_1, \ldots, m_\delta)$ is a polynomial, the coefficients of which are known at compile time.

**Output**:

$(i_1 \cdots ;\ \cdots ;\ i_1{+}{+})$
$\ldots(i_d \cdots ; \cdots ;\ i_d{+}{+})$
$\widetilde{A}[f_1] \cdots [f_\delta] \leftarrow \cdots\ \ldots$

- $f_1, \ldots, f_\delta$ are affine forms in $i_1, \ldots, i_d$ the coefficients of which are integers to-be-determined,
- $\widetilde{A}$ is an $M_1 \times \cdots \times M_\delta$-array,
- $M_1, \ldots, M_\delta$ are affine forms in $m_1, \ldots, m_\delta$ the coefficients of which are integers TBD,

such that:
$$\boxed{R = f_1 M_2 \cdots M_\delta\ +\ \cdots\ +\ f_{\delta-1} M_2 + f_\delta}$$
holds and for each $(i_1, \ldots, i_d)$ in the iteration domain we have:
$$\boxed{0 \leq f_1 < M_1,\ \ \ldots, 0 \leq f_\delta < M_\delta.}$$

# Two problems to solve

## Polynomial system solving

Find $f_1, \ldots, f_\delta$ so that

$$R = f_1 M_2 \cdots M_\delta + \cdots + f_{\delta-1} M_2 + f_\delta$$

holds.

- This part can be done off-line.

## Quantifier elimination

$\forall(i_1, \ldots, i_d)$ in the iteration domain, we have:

$$0 \leq f_1 < M_1, \ldots, 0 \leq f_\delta < M_\delta$$

- At run-time, all the parameters are known, we can solve this problem in the integer domain.
- But we would rather do it off-line (thus parametrically).

# Integer QE problem

**For each $f_k$ and $M_k$, we need to ensure $\max f_k < M_k$**

$$\begin{array}{ll} \text{maximize} & f_k \\ \text{subject to} & i_1, \ldots, i_d \in \mathbb{Z} \\ & \forall (i_1, \ldots, i_d) \ \in \mathbf{D} \end{array}$$

- At compile time, $f_k$ and $M_k$ cannot be determined numerically because of the parameters.
- Thus, the above problem becomes a **parametric** integer linear programming problem (PILP) which is very similar to a **parametric** integer hull problem.
- This has motivated what follows.

# Outline

# A for-loop nest and its associated parametric polyhedral set

```
for(i = 0; i ≤ n; i ++)
for(j = i; j ≤ n; j ++)
A[i][j]...
```

$$\begin{cases} 0 \leq i \leq n \\ i \leq j \leq n \end{cases}$$

- Loop counters can only be integers
- This leads to the problem of finding the integer points of a polyhedral set, called the iteration space
- Often this space is parametric (e.g. the variable $n$)

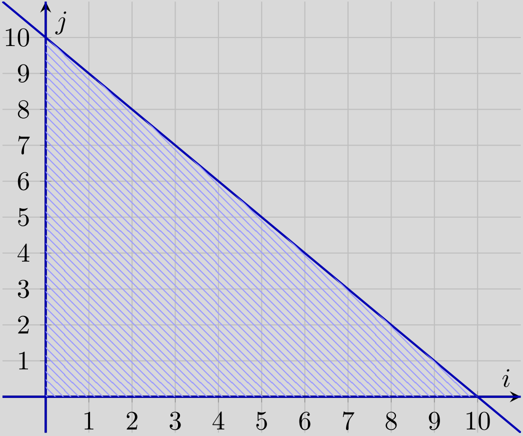
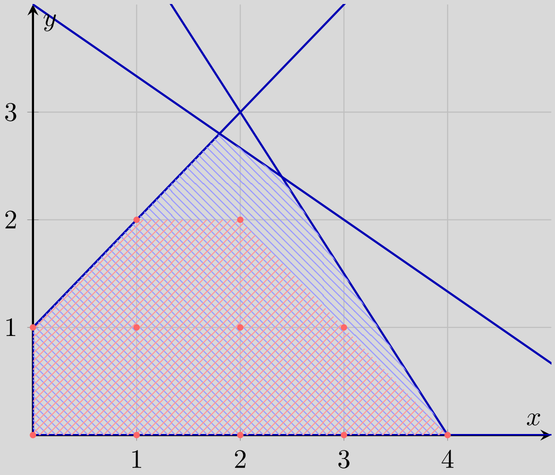



Figure: Iteration space when $n = 10$

# Integer hull: simple non-parametric example

$$\begin{cases} 0 & \leq & x \\ 0 & \leq & y \\ 3\,x + 2\,y & \leq & 12 \\ 2\,x + 3\,y & \leq & 12 \\ -x + y & \leq & 1 \end{cases}$$

$$\begin{cases} 0 & \leq & x \\ 0 & \leq & y \\ y & \leq & 2 \\ x + y & \leq & 4 \\ -x + y & \leq & 1 \end{cases}$$




Figure

▸▸ skip slide

# Outline

# Decomposing the integer points of a polyhedron

## Example

Input: $K_1 : \begin{cases} 3x_1 - 2x_2 + x_3 \leq 7 \\ -2x_1 + 2x_2 - x_3 \leq 12 \\ -4x_1 + x_2 + 3x_3 \leq 15 \\ -x_2 \leq -25 \end{cases}$, assume $x_1 > x_2 > x_3$.

Output: $K_1^1, K_1^2, K_1^3, K_1^4, K_1^5$ given by:

$$\begin{cases} 3x_1 - 2x_2 + x_3 \leq 7 \\ -2x_1 + 2x_2 - x_3 \leq 12 \\ -4x_1 + x_2 + 3x_3 \leq 15 \\ 2x_2 - x_3 \leq 48 \\ -5x_2 + 13x_3 \leq 67 \\ -x_2 \leq -25 \\ 2 \leq x_3 \leq 17 \end{cases}, \begin{cases} x_1 = 15 \\ x_2 = 27 \\ x_3 = 16 \end{cases}, \begin{cases} x_1 = 18 \\ x_2 = 33 \\ x_3 = 18 \end{cases}, \begin{cases} x_1 = 14 \\ x_2 = 25 \\ x_3 = 15 \end{cases}, \begin{cases} x_1 = 19 \\ x_2 = 50 + t \\ x_3 = 50 + 2t \\ -25 \leq t \leq -16. \end{cases}$$

# Decomposing the integer points of a polyhedron

Output: $K_1^1, K_1^2, K_1^3, K_1^4, K_1^5$ given by:

$$\begin{cases} 3x_1 - 2x_2 + x_3 \le 7 \\ -2x_1 + 2x_2 - x_3 \le 12 \\ -4x_1 + x_2 + 3x_3 \le 15 \\ 2x_2 - x_3 \le 48 \\ -5x_2 + 13x_3 \le 67 \\ -x_2 \le -25 \\ 2 \le x_3 \le 17 \end{cases}, \begin{cases} x_1 = 15 \\ x_2 = 27 \\ x_3 = 16 \end{cases}, \begin{cases} x_1 = 18 \\ x_2 = 33 \\ x_3 = 18 \end{cases}, \begin{cases} x_1 = 14 \\ x_2 = 25 \\ x_3 = 15 \end{cases}, \begin{cases} x_1 = 19 \\ x_2 = 50 + t \\ x_3 = 50 + 2t \\ -25 \le t \le -16. \end{cases}$$

- An integer point solves $K_1$ iff it solves either $K_1^1$, $K_1^2$, $K_1^3$, $K_1^4$ or $K_1^5$.
- Each of $K_1^1, K_1^2, K_1^3, K_1^4, K_1^5$ has at least one integer point.
- For each $K_1^i$, each integer point in any (standard) projection of $K_1^i$ can be lifted to an integer point in the polyhedron.

# Outline

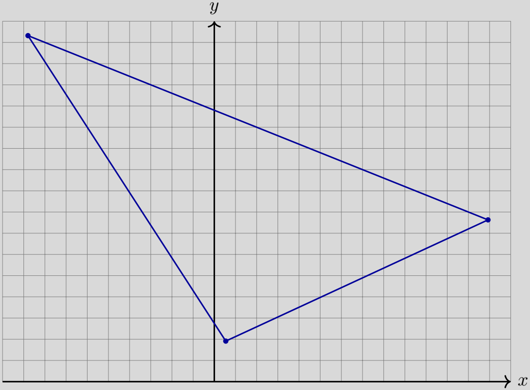
# Example (0/3)

## Input

Let's look at a simple example first.

Vertices: $(-44/5, 408/25), (349/27, 206/27), (85/57, 109/57)$

$$\begin{cases} 2x + 5y & \leq & 64 \\ 7x + 5y & \geq & 20 \\ 3x - 6y & \leq & -7 \end{cases}$$



# Example (1/3)

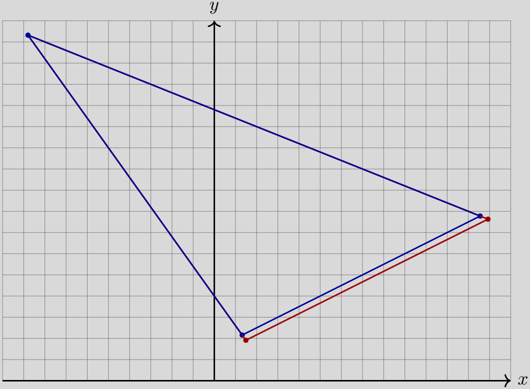
## Normalization

Replace the facets that could not have integer point

Vertices: $(-44/5, 408/25)$,~~$(349/27, 206/27)$,$(85/57, 109/57)$,~~
$(113/9, 70/9)$,$(25/19, 41/19)$

~~$3x - 6y \leq -7$~~

$$\begin{cases} 2x + 5y & \leq & 64 \\ 7x + 5y & \geq & 20 \\ 3x - 6y & \leq & -9 \end{cases}$$

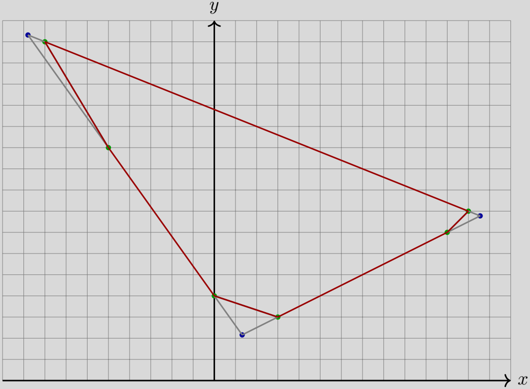


# Example (2/3)

## Partition

Vertices: $(-44/5, 408/25), (113/9, 70/9), (25/19, 41/19)$
Find the triangles with vertices: $[(-8, 16), (-44/5, 408/25), (-5, 11)]$, $[(3, 3), (25/19, 41/19), (0, 4)]$, $[(12, 8), (113/9, 70/9), (11, 7)]$

$$\begin{cases} 5y & \leq & -2x + 64 \\ 5y & \geq & -7x + 20 \\ 2y & \geq & x + 3 \end{cases}$$



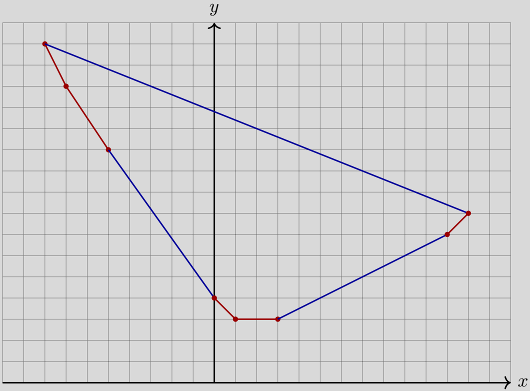
# Example (3/3)

## Merging

Vertices: $(-8,16),(-7,14),(-5,11),(0,4),(1,3),(3,3),(11,7),(12,8)$

$$\begin{cases} 5y & \leq & -2x+64 \\ 5y & \geq & -7x+20 \\ 2y & \geq & x+3 \end{cases}$$

$$\begin{cases} y & \geq & -2x \\ 2y & \geq & 3x+7 \\ y & \geq & -x+4 \\ y & \geq & 3 \\ y & \geq & x-4 \end{cases}$$



# Main steps of our algorithm

Our algorithm has 3 main steps:

- **Normalization**: construct a new polyhedral set $Q$ from $P$ as follows. Consider in turn each facet $F$ of $P$:
  1. if the hyperplane $H$ supporting $F$ contains an integer point, then $H$ is a hyperplane supporting a facet of $Q$,
  2. otherwise we slide $H$ towards the center of $P$ along the normal vector of $F$, stopping as soon as we hit a hyperplane $H'$ containing an integer point, then making $H'$ a hyperplane supporting a facet of $Q$.

  Clearly $Q_I = P_I$.
- **Partitioning**: make each part of the partition a polyhedron $R$ which:
  1. either has integer points as vertices so that $R_I = R$,
  2. or has a small volume so that any algorithm (including exhaustive search) can be applied to compute $R_I$.
- **Merging**: Once the integer hull of each part of the partition is computed and given by the list of its vertices, an algorithm for computing the convex hull of a set points, such as `QuickHull`, can be applied to deduce $P_I$.

skip slide

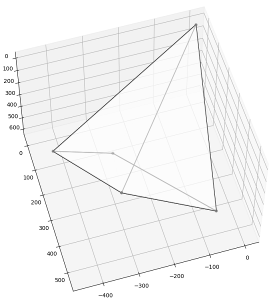
# The general algorithm on a 3D example

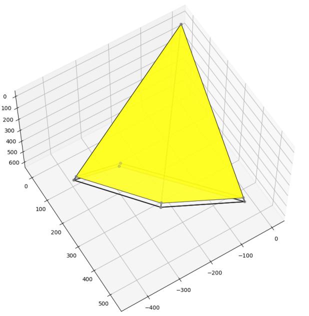
## Normalization

The integer hull of the normalized polyhedral set should be the same as that of the input

$$\left\{\begin{array}{rcl} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -5405x_1 + 4965x_2 + 3870x_3 & \leq & 4303504 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{array}\right.$$

$$\left\{\begin{array}{rcl} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -1081x_1 + 993x_2 + 774x_3 & \leq & 860700 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{array}\right.$$



# The general algorithm: building the partition

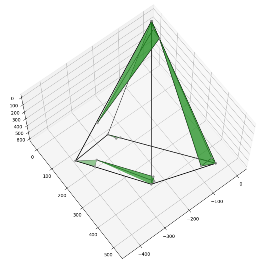
## Partition

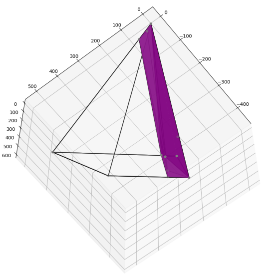
For each face $f$ of $P$:

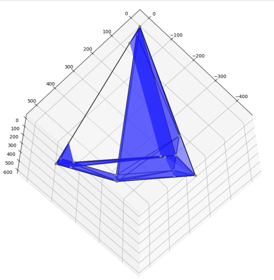
- let $\mathcal{F}$ be the set of all facets that intersect at $f$
- if there exist integer points on $f$ (which implies that the closest integer points on f to each of its vertices do exist as well), then for each vertex $v$ of $f$, a “corner” polyhedral is built as the convex hull of:
  - $v$,
  - the closest integer point to $v$ on $f$,
  - all the closest integer points to $v$ on $F$, for $F \in \mathcal{F}$.
- if there is no integer point on $f$, a single “corner” polyhedral set is built for $f$ as the convex hull of:
  - the vertex set of $f$,
  - all the closest integer points to $v$ on $F$, for $F \in \mathcal{F}$.

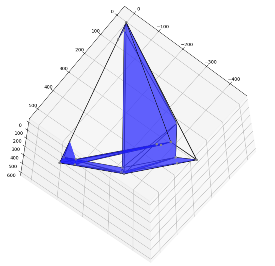
# The general algorithm on a 3D example

## Partition



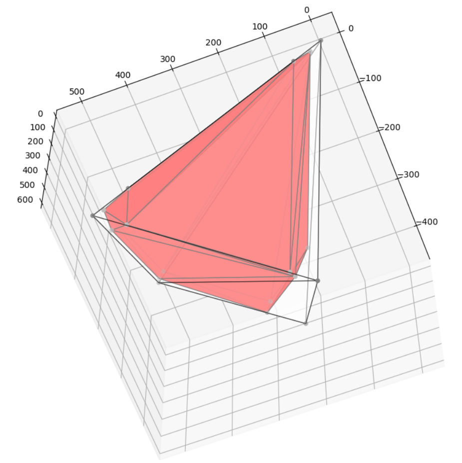



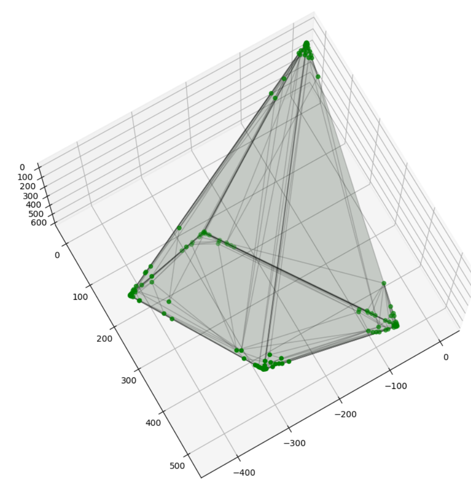



# The general algorithm on a 3D example

## Merging

The integer hull has 139 vertices



# "Closest integer points" on a facet to each of its vertices

## Projection and recursive call

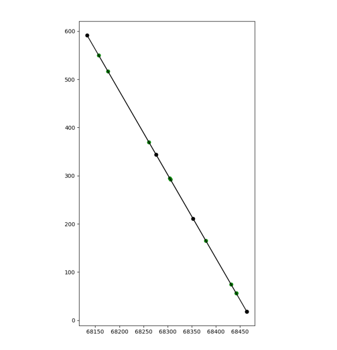
In $\mathbb{Q}^d$, for a facet $F$ of dimension $d-1 < d$, and its vertex set $V$:

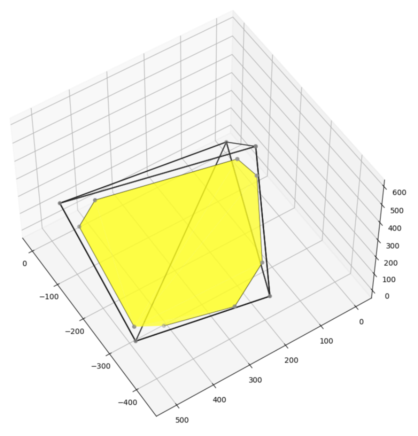
1. make a projection on a full-dimensional polyhedron $G$ using Hermite normal form $\vec{c}^t U = [\mathbf{0} H]$ (where $U = [U_L U_R]$ and $\vec{c}^t \mathbf{x} = s$ is the hyperplane supporting $F$)
2. we obtain a parametrization $R_F$ of $F$ of the form:
$$R_F : \begin{cases} \mathbb{Q}^{d-1} & \rightarrow & \mathbb{Q}^d \\ \mathbf{z} & \longmapsto & \mathbf{x} = \mathbf{v} + U_L \mathbf{z}. \end{cases} \quad (3)$$
3. thus $R_F(G) = F$. Moreover, we have
$$R_F(G_I) = F_I.$$
4. q recursive call to our integer hull algorithm computes the vertices $V_I'$ of the integer hull of $G$
5. we deduce the vertices $V_I$ of $F_I$ by $R_F(V_I') = V_I$
6. finally, we find in $V_I$ the "closest integer points" to each $v$ of $V$.

# Closest integer points on a face to one of its vertices

## Projection and recursive call

$$R_F : \begin{cases} x_1 = 993x'_1 + 573x'_2 - 67995300 \\ x_2 = 1081x'_1 + 623x'_2 - 74020200 \\ x_3 = x'_2 \end{cases}$$



# The PolyhedralSets:-IntegerHull command in Maple

```
> with(PolyhedralSets) :
> ineqs := [2 x + 5 y ≤ 64, 7 x + 5 y ≥ 20 , 3 x − 6 y ≤ -7] :
> poly := PolyhedralSet(ineqs, [x, y]);
```

$$poly := \begin{cases} Coordinates & : \quad [x, y] \\ Relations & : \quad \left[ -x - \frac{5\,y}{7} \leq -\frac{20}{7},\, x - 2\,y \leq -\frac{7}{3},\, x + \frac{5\,y}{2} \leq 32 \right] \end{cases}$$

```
> IntegerHull(poly);
```

$$[[[12, 8], [-8, 16], [-7, 14], [-5, 11], [0, 4], [1, 3], [3, 3], [11, 7]], [\ ]]$$

```
> IntegerHull(poly, returntype=polyhedralset);
```

$$\begin{cases} Coordinates & : \quad [x, y] \\ Relations & : \quad \left[ -y \leq -3,\, -x - y \leq -4,\, -x - \frac{5\,y}{7} \leq -\frac{20}{7},\, -x - \frac{2\,y}{3} \leq -\frac{7}{3},\, -x - \frac{y}{2} \leq 0,\, x - 2\,y \leq -3,\, x \right. \end{cases}$$

```
>
```

# The PolyhedralSets:-IntegerHull command in Maple

```
> restart; with(PolyhedralSets) :
> vertices := [[10, 10, 10, 10/3], [-140/8, -220/12, -10, -10/3], [60/8, 20, -100/12, -70/3], [-10
     /4, -100/12, 70/2, 35/3], [0, 0, 0, 50/3]] :
  vars := [x1, x2, x3, x4] :
  poly := PolyhedralSet(vertices, [ ], vars);
```

$$poly := \begin{cases} Coordinates & : [x1, x2, x3, x4] \\ Relations & : \left[ -x1 + \frac{503\,x2}{694} + \frac{85\,x3}{694} + \frac{311\,x4}{2082} \leq \frac{7775}{3123}, -x1 + \frac{2715\,x2}{2234} + \frac{603\,x3}{2234} + \cdots \right. \end{cases}$$

```
> IntegerHull(poly);
```

[[[-15, -16, -6, -2], [-15, -15, -9, -4], [-14, -15, -4, -1], [-13, -13, -8, -1], (1
[-13, -12, -9, -5], [-12, -13, -4, 1], [-12, -13, -4, 2], [-12, -12, -3, -3], [-11, -12, -3, -1], [-11, -11, -6, -3], [-11, -11, -1, -3], [-10, -8, -8, -5], [-9, -6, -8, -8], [-7, -7, -4, 7], [-7, -6, -5, 3], [-7, -3, -8, -10], [-7, -3, -7, -10], [-6, -4, -5, 0], [-5, -9, 23, 8], [-5, -4, -4, 5], [-5, -4, -3, -2], [-4, -5, 3, 10], [-4, -5, 3, 11], [-4, -3, -2, -3], [-4, 0, -6, -9], [-3, -8, 30, 10], [-3, -7, 23, 10], [-3, -7, 23, 11], [-3, -6, 24, 6], [-3, 3, -8, -12], [-3, 3, -7, -13], [-2, -7, 31, 11], [-2, -6, 24, 8], [-2, -6, 24, 12], [-2, -6, 26, 11], [-2, -5, 25, 7], [-2, -5, 26, 6],

# The PolyhedralSets:-IntegerHull command in Maple

```
> ineqs := [-x1-(132*x2)/205-(62*x3)/205 <= -1358/123, -x1 + (34*x2)/34 + (4*x3)/4
      <= 1405/17, x1-(12*x2)/118 + (83*x3)/177 <= 3500/59]:
  poly := PolyhedralSet(ineqs, [x1, x2, x3]);
  IsBounded(poly);
```

$$poly := \begin{cases} \text{Coordinates} & : [x1, x2, x3] \\ \text{Relations} & : \left[ -x1 - \frac{132\,x2}{205} - \frac{62\,x3}{205} \leq -\frac{1358}{123},\ -x1 + x2 + x3 \leq \frac{1405}{17},\ x1 - \frac{6\,x2}{59} + \frac{8}{} \right. \end{cases}$$

$$false \quad \textbf{(14}$$

```
> IntegerHull(poly);
```

$$\Big[ [[-20, 36, 26], [-4, -25, 103], [-2, -30, 107], [-1, -36, 117], [0, -38, 118], [0, -36, 118], [1, -37, 112], [1, -34, 117], [2, -39, 113], [2, -38, 114], [10, -43, 95], [26, -51, 60], [399, -238, -776], [403, -240, -785], [453, -265, -897], [1544, -811, -3342]], \left[\left[\frac{101}{260}, 1, -\frac{159}{260}\right], \left[\frac{2012}{4509}, -\frac{6041}{27054}, -1\right], \left[-\frac{70}{337}, \frac{267}{337}, -1\right]\right]\Big] \quad \textbf{(15}$$

# Benchmarks 2D

E&C represents “enumeration and convex hull”, which in Maple is done by `ZPolyhedralSets:-EnumerateIntegerPoints` and `ConvexHull`. `Normaliz` is an open source tool for computations in affine monoids, vector configurations, lattice polytopes, and rational cones.

| Volume | 27.95 | | 111.79 | | 11179.32 | |
|---|---|---|---|---|---|---|
| Algorithm | IntegerHull | E&C | IntegerHull | E&C | IntegerHull | E&C |
| Maple (ms) | 172 | 410 | 244 | 890 | 159 | 58083 |
| C/C++ (ms) | 0.284 | 0.768 | 0.339 | 1.676 | 0.286 | 6.883 |
| Normaliz (ms) | 835.730 | | 462.116 | | 1559.401 | |

Table: Integer hulls of triangles

| Volume | 58.21 | | 5820.95 | | 23283.82 | |
|---|---|---|---|---|---|---|
| Algorithm | IntegerHull | E&C | IntegerHull | E&C | IntegerHull | E&C |
| Maple (ms) | 303 | 752 | 275 | 31357 | 304 | 123159 |
| C/C++ (ms) | 0.451 | 0.565 | 0.478 | 0.657 | 0.396 | 0.682 |
| Normaliz (ms) | 2.837 | | 1216.238 | | 740.559 | |

Table: Integer hulls of hexagons

# Benchmarks 3D

| Volume | 447.48 | | 6991.89 | | 55935.2 | |
|---|---|---|---|---|---|---|
| Algorithm | IntegerHull | E&C | IntegerHull | E&C | IntegerHull | E&C |
| Maple (ms) | 977 | 7289 | 1223 | 74804 | 1378 | 531904 |
| C/C++ (ms) | 4.488 | 0.826 | 4.615 | 0.923 | 4.624 | 1.527 |
| Normaliz (ms) | 851.495 | | 956.666 | | 793.192 | |

Table: Integer hulls of tetrahedrons (4 vertices, 4 facets and 6 edges)

| Volume | 412.58 | | 7050.81 | | 60417.63 | |
|---|---|---|---|---|---|---|
| Algorithm | IntegerHull | E&C | IntegerHull | E&C | IntegerHull | E&C |
| Maple (ms) | 1476 | 5711 | 1573 | 60233 | 1728 | 512101 |
| C/C++ (ms) | 11.049 | 21.235 | 16.001 | 145.068 | 23.822 | 2082.559 |
| Normaliz (ms) | 7862.109 | | N/A | | N/A | |

Table: Integer hulls of triangular bipyramids (5 vertices, 6 facets and 9 edges)

# Outline

# Conclusions and remarks

Over the reals:

- The notion of regular semi-algebraic system is a natural generalization of that of a regular chain for isolating real solutions.
- The incremental flavor of `RealTriangularize` is experimentally more effective than its elimination approach.
- `RealTriangularize` has inspired follow-up works (CAD and QE based on regular chains and proceeding incrementally).
- The implementation of `RealTriangularize` relies on CAD but this could be relaxed. (The complexity analysis uses Renagar's work.)
- Can the notion of a regular semi-algebraic system be weakened so as to reduce the cost of the decomposition while remaining useful?

Over the integers:

- The `IntegerPointDecomposition` is also inspired by the theory of regular chains.
- It often produces more information than needed and this has a cost.
- Our `IntegerHull` solves that issue and is currently adapted to support parameters and thus QE problems.

# Thank You!

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