Efficient detection of redundancies in systems of linear inequalities

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July 18, 2024

Plan

$$
\begin{cases}
-x_3 \leq 1 \\
-x_1 - x_2 - x_3 \leq 2 \\
-x_1 + x_2 - x_3 \leq 2 \\
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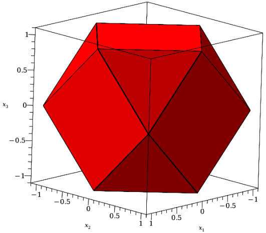

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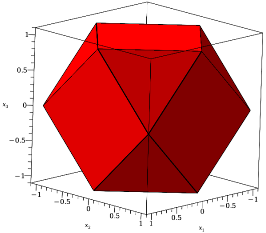

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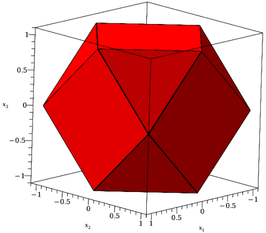

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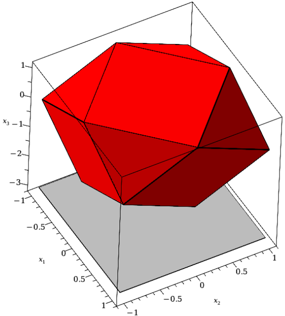

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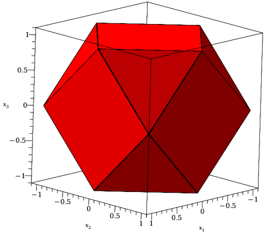

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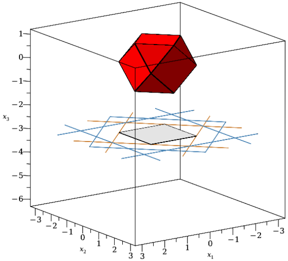

Application of FME: code generation

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for(i=0; i<=n; i++){
  c[i] = 0; c[i+n] = 0;
  for(j=0; j<=n; j++)
    c[i+j] += a[i]*b[j];
}
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skip slide

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FME reorders $p > t > i > j > n$ to $i > j > t > p > n$, thus eliminating i, j.

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The new representation allows us to generate the multithreaded code.

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}
```

```
parallel_for (p=0; p<=2*n; p++){
  c[p] = 0;
  for (t=max(0,n-p);
       t<=min(n,2*n-p);t++)
    c[p] += A[t+p-n] * B[n-t];
}
```

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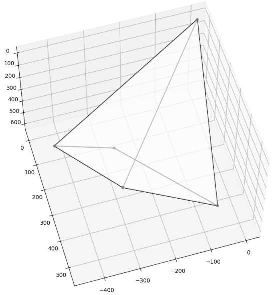
Application of FME: computing integer hulls (1/3)

The input polyhedral set:

$$\left\{\begin{array}{ccc} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -5405x_1 + 4965x_2 + 3870x_3 & \leq & 4303504 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{array}\right.$$



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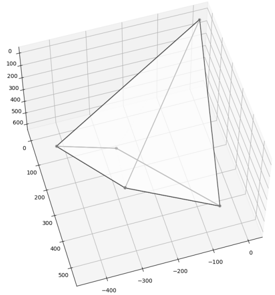
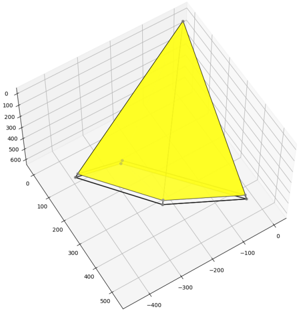
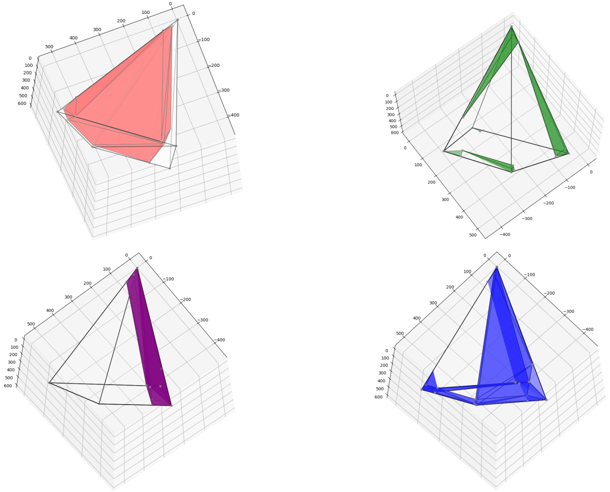


Normalization (leaves the integer hull unchanged):

$$\left\{ \begin{array}{rcl} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -1081x_1 + 993x_2 + 774x_3 & \leq & 860700 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{array} \right.$$



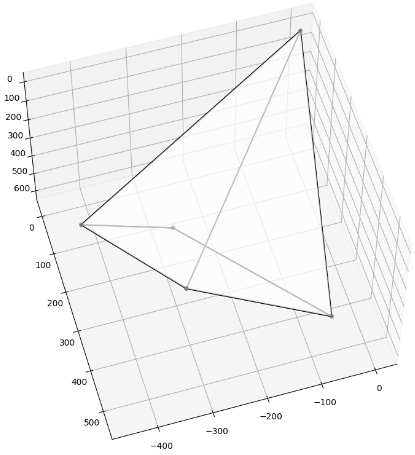
Application of FME: computing integer hulls (2/3)



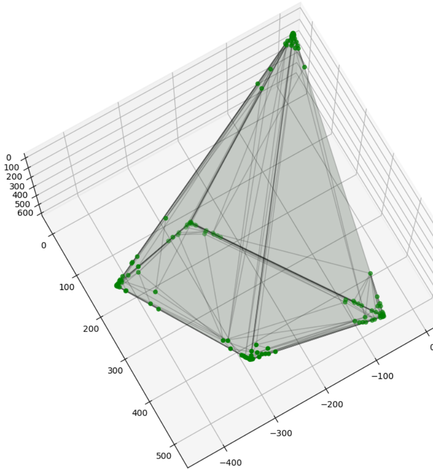
1. The **red** is an approximation of the integer hull of the input.
2. The integer hulls of border regions (green, blue, purple) are brute-force computed via FME.
3. Then `QuickHull` is applied to obtain the integer hull of the input.

Application of FME: computing integer hulls (3/3)

The input has only 5 vertices.



Its integer hull has 139 vertices.



All details are in https://ir.lib.uwo.ca/etd/8985/ and in https://doi.org/10.1007/978-3-031-14788-3_14

Plan

Polyhedral sets

1. A *polyhedral set* P is any $\{\mathbf{x} \mid A\mathbf{x} \le \mathbf{b}\}$, where $A \in \mathbb{Q}^{m\times n}$ and $\mathbf{b} \in \mathbb{Q}^m$. Such a linear system is called an *H-representation* of P.

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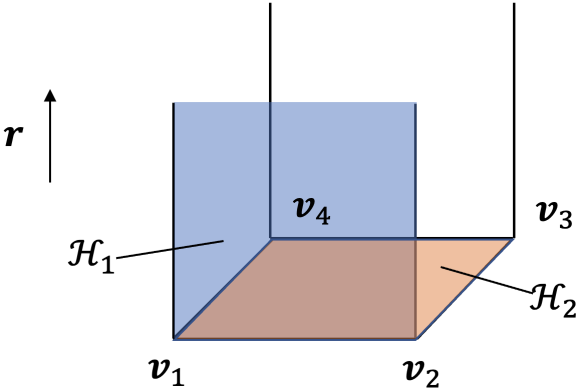
11. Since P is pointed, an extreme ray of P is a one-dimensional face of $\mathrm{CharCone}(P)$.
12. Let V and R denote the set of vertices and extreme rays of P. Then, the pair $\mathcal{VR}(F) := (V, R)$ is called a *V-representation* of P.

An unbounded polyhedral set and its representations



The open cube $P := \{(x, y, z) \mid -z \leq 1, 0 \leq x \leq 1, 0 \leq y \leq 1\}$ shown above has 4 vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and extreme ray $\mathbf{r}$.

Redundant inequalities, the saturation matrix

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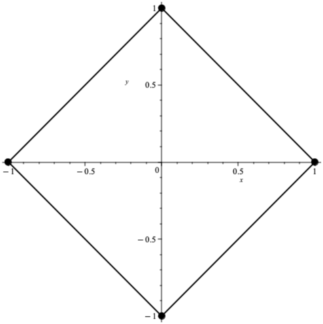
The *saturation matrix* of F is the $0-1$ matrix $S \in \mathbb{Q}^{m\times k}$, where $S_{i,j} = 1$ iff the j-th element of $\mathcal{VR}(F)$ saturates the i-th inequality of F.

A bounded polyhedral set and its the saturation matrix

F	$\mathcal{VR}(F)$
$\ell_1 : x + y \leq 1$	$\mathbf{v}_1 : (0, 1)$
$\ell_2 : -x - y \leq 1$	$\mathbf{v}_2 : (1, 0)$
$\ell_3 : x - y \leq 1$	$\mathbf{v}_3 : (-1, 0)$
$\ell_4 : -x + y \leq 1$	$\mathbf{v}_4 : (0, -1)$

satM(F)

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
ℓ_1	1	1	0	0
ℓ_2	0	0	1	1
ℓ_3	0	1	0	1
ℓ_4	1	0	1	0



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Theorem

We have:

$$\mathcal{S}^{\mathcal{VR}}(\mathrm{proj}(\{\ell_{pos}, \ell_{neg}\}, \{x\})) = \mathrm{proj}(\mathcal{S}^{\mathcal{VR}}(\ell_{pos}) \cap \mathcal{S}^{\mathcal{VR}}(\ell_{neg}), \{x\}).$$

Plan

Algorithm 1: CheckRedundancy

Input: 1. the inequality system F with m inequalities;
2. the saturation matrix satM.

Output: the minimal system F_{irred} and the corresponding saturation matrix $\mathrm{satM}_{\mathrm{irred}}$.

```
Irredundant := {seq(i, i = 1..m)}.
for i from 1 to m do
    if the number of nonzero elements in satM[i] is less than n then
        Irredundant := Irredundant ∖ {i}.
        next.
    for j in Irredundant ∖ {i} do
        if satM[i] = satM[i]&satM[j] then
            Irredundant := Irredundant ∖ {i}.
            break.
F_irred := [seq(F[i], i in Irredundant)] and
 satM_irred := [seq(satM[i], i in Irredundant)].
return F_irred and satM_irred.
```

Algorithm 2: Minimal projected representation

Input: 1. an inequality system F;

2. a variable order $x_1 > x_2 > \ldots > x_n$.

Output: the minimal projected representation *res* of F.

Compute the V-representation V of F by DD method;
Set $res := \text{table}()$.
Sort the elements in V w.r.t. the reverse lexico order.
Compute the saturation matrix satM.
$F, \text{satM} := \mathrm{CheckRedundancy}(F, \text{satM}(F))$.
$res[x_1] := F^{x_1}$.
for *i from 1 to* $n-1$ **do**
    $(F^p, F^n, F^0) := \text{partition}(F)$.
    $V_{new} := \text{proj}(V, \{x_i\})$.
    Merging: $\text{satM} := \text{Merge}(\text{satM})$.
    Let $F_{new} := F^0$ and $\text{satM}_{new} := \text{satM}[F^0]$.
    foreach $f_p \in F^p$ *and* $f_n \in F^n$ **do**
        Append $\text{proj}((f_p, f_n), \{x_i\})$ to F_{new},
        Append $\text{satM}[f_p]\&\text{satM}[f_n]$ to satM_{new}.
    $F, \text{satM} := \mathrm{CheckRedundancy}(F_{new}, \text{satM}_{new})$.
    $V := V_{new}$, $res[x_{i+1}] := F^{x_{i+1}}$.
return *res*.

Plan

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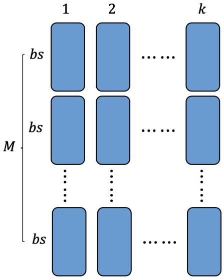

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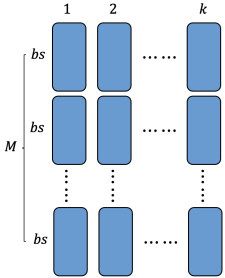

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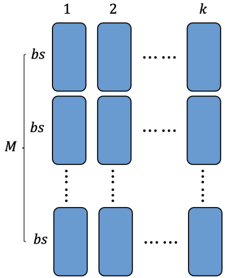

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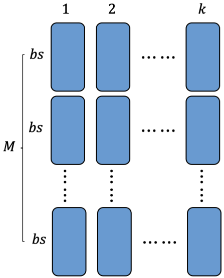
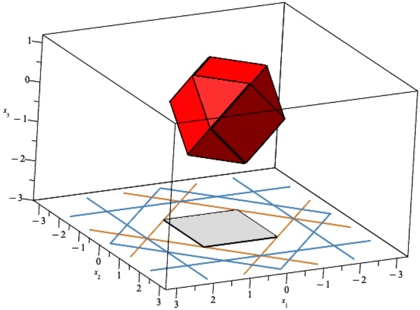


Plan

Cuboctahedron



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2. **weakly redundant inequalities eliminated by cardinality**
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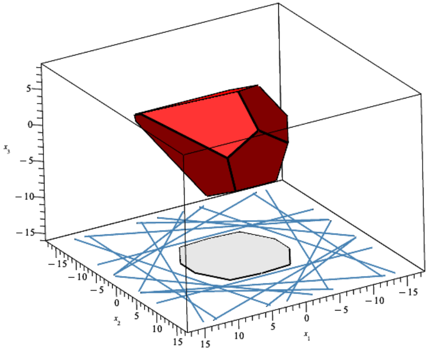
Snub disphenoid (triangular dodecahedron)



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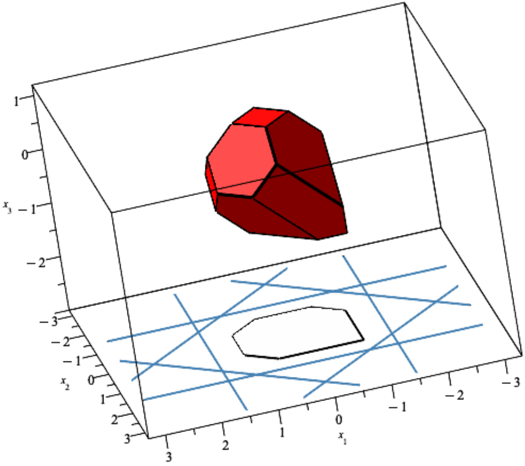
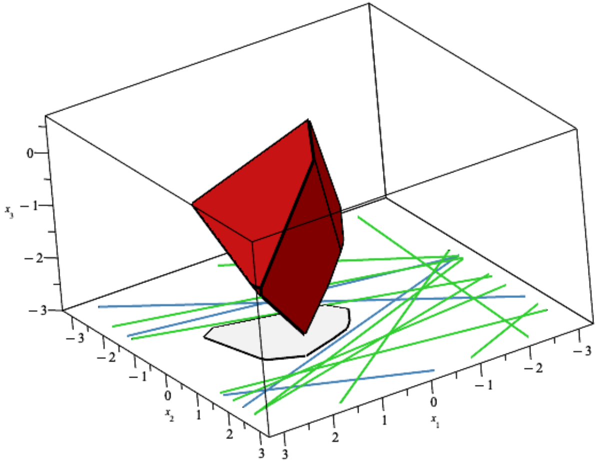
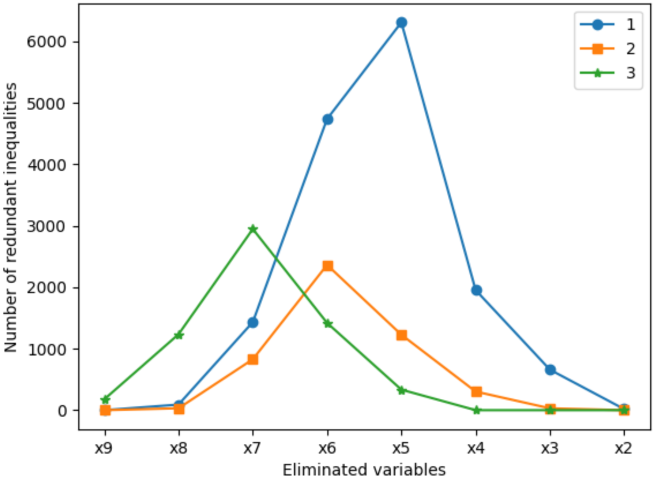
Truncated octahedron



Random 3D polyhedron



Random 10D polyhedron



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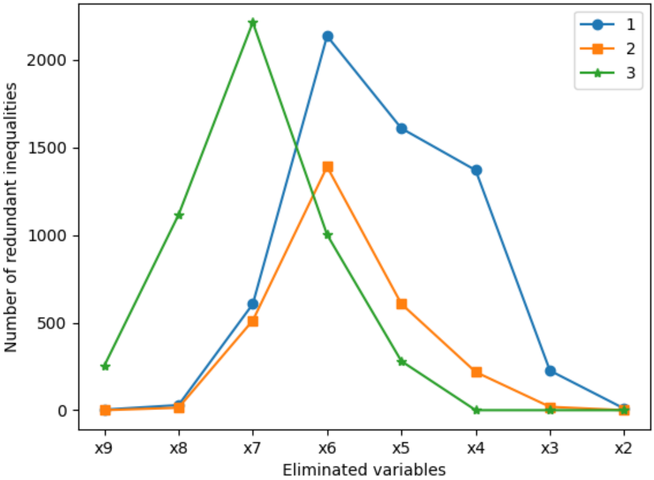

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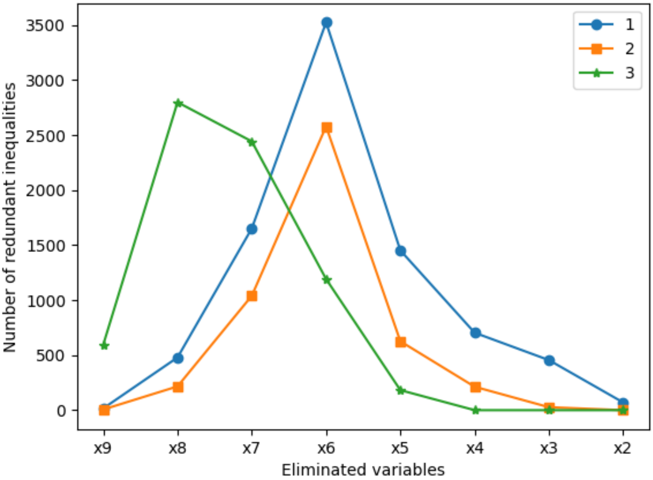

Comparative experimentation (1/3)

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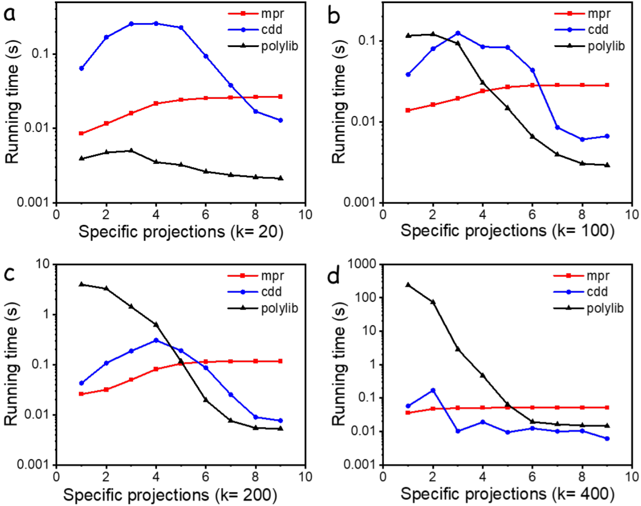
All the experimental results were collected on a PC (Intel(R) Xeon(R) Gold 6258R CPU 2.70GHz, 503G RAM, Ubuntu 20.04.3).

Comparative experimentation (2/3)



1. Four different random polyhedra with $m = 15$ and $n = 10$.
2. For $1 \le i \le 9$, in the hor. axiss, the first i variables are eliminated.
3. The vert. axis in each figure shows the running time (in seconds).

test case	(n, m, k)	mpr	BPAS	cdd	polylib
32hedron	(6, 32, 11)	6.54	16.80	4183.08	**1.92**
64hedron	(7,64,13)	13.05	52.42	>5min	**1.67**
francois	(13,27,2304)	499.92	**253.66**	388.36	> 5min
francois2	(13,31,384)	**41.80**	140.34	55.17	80.63
herve.in	(14,25,262)	34.42	140.34	294.01	**30.08**
c6.in	(11,17,31)	**9.85**	12.72	84.11	5.56
c9.in	(16,18,140)	**25.08**	65.54	151.17	131.53
c10.in	(18,20,142)	22.10	98.68	249.02	**16.06**
S24	(24, 25,25)	23.50	58.80	748.67	**17.47**
S35	(35, 36,36)	46.55	182.14	3575.00	**46.007**
cube	(10, 20,1024)	**81.33**	201.92	125.900	161.06
C56	(5, 6,6)	3.67	4.09	11.81	**0.79**
C1011	(10, 11,11)	24.99	115.68	1716.25	**9.99**
C510	(5, 42,10)	12.00	40.01	>5min	**4.42**
T1	(5, 10,38)	**5.61**	16.44	27.42	8.81
T3	(10,12,29)	21.29	141.64	288.07	**12.07**
T5	(5, 10,36)	8.12	15.62	22.92	**4.76**
T6	(10,20,390)	**1142.9**	23800.11	14937.61	>5min
T7	(5, 8,26)	5.81	10.79	13.96	**4.00**
T9	(10,12,36)	**36.56**	414.53	479.18	100.34
T10	(6, 8,24)	**4.58**	13.65	18.39	5.27
T12	(5, 11,42)	**8.52**	19.03	38.65	8.60
R_15_20	(15, 20,1328)	**28430.40**	336035.00	38037.21	>5min

Plan

Complexity estimates (1/2)

Recall the notations

1. m is the number of inequalities and n is the dimension of the ambient space. If the input H-representation is irredundant, the m is also the number of facets of P.
2. Let $h := \text{height}([A, \mathbf{b}])$, let θ be the coefficient of linear algebra and ω the bit-size of a machine word.

Well-known bounds

1. The size k of the V-representation (V, R) is at most $\binom{m}{n} + \binom{m}{n-1} \leq \frac{m^n}{n!}$.
2. From [2], for $1 \leq i < n$, after eliminating i variables during the process of FME, the number of irredundant inequalities defining the projection is at most $\binom{m}{n-i-1} \leq m^n$.

Theorem

The costs for computing all the inequalities (redundant and irredundant) and generating the initial saturation matrix are within $O(m^{2n} n^{\theta+\varepsilon} h^{1+\varepsilon})$ *bit operations, while the costs for updating and operating on the saturation matrices are bounded over by* $\frac{3m^{3n-4}}{\omega}$ *word operations.*

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Bounds for FME

1. FME based on LP: $O(n^2\, m^{2n}\, \mathrm{LP}(n, 2^n h n^2 m^n))$ bit operations, where $\mathrm{LP}(d, H)$ is an upper bound for the number of bit operations required for solving a linear program in d variables and with total bit size H. For instance, in the case of Karmarkar's algorithm [4], we have $\mathrm{LP}(d, H) \in O(d^{3.5} H^2 \cdot \log H \cdot \log\log H)$.
2. FME based on redundancy test cone: $O(m^{\frac{5n}{2}}\, n^{\theta+1+\epsilon}\, h^{1+\epsilon})$ bit operations, for any $\epsilon > 0$.
3. This paper: $O(m^{2n} n^{\theta+\varepsilon} h^{1+\varepsilon})$ bit operations and $\frac{3m^{3n-4}}{\omega}$ word operations.

Plan

Concluding remarks

Summary and notes

1. We proposed a technique for removing redundant inequalities in linear systems.
2. It relies on the analysis of 3 different types of redundancies
3. Our redundancy tests allow for efficient implementation based on bit-vector arithmetic.
4. From the experimental results, our method works best on hard problems.
5. This is promising to solve large scale problems in areas like information theory, SMT and optimizing compilers.

Work in progress

1. Our implementation has room for improvements.
2. Indeed, our algorithms have opportunities for both multithreaded parallelism and instruction-level parallelism.
3. The third criterion (redundancy test based on containment) needs further study to discover the container.

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