

When does $\langle T \rangle$ equal $\text{sat}(T)$?

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Abstract

Given a regular chain T , we aim at finding an efficient way for computing a system of generators of $\text{sat}(T)$, the saturated ideal of T . A natural idea is to test whether the equality $\langle T \rangle = \text{sat}(T)$ holds, that is, whether T generates its saturated ideal. By generalizing the notion of primitivity from univariate polynomials to regular chains, we establish a necessary and sufficient condition, together with a Gröbner basis free algorithm, for testing this equality. Our experimental results illustrate the efficiency of this approach in practice.

Key words: Regular chain, Saturated ideal, Primitivity of polynomials.

1. Introduction

Triangular decompositions are one of the most studied techniques for solving polynomial systems symbolically. Invented by J.F. Ritt in the early 30's for systems of differential polynomials, their stride started in the late 80's with the method of Wu (1986) dedicated to algebraic systems. Different concepts and algorithms extended the work of Wu. In the early 90's, the notion of a *regular chain*, introduced independently by Kalkbrener (1993) and by Yang and Zhang (1991), led to important algorithmic discoveries.

In Kalkbrener's vision, regular chains are used to represent the generic zeros of the irreducible components of an algebraic variety. In the original work of Yang and Zhang, they are used to decide whether a hypersurface intersects a quasi-variety (given by a

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regular chain). Regular chains have, in fact, several interesting properties and are the key notion in many algorithms for decomposing systems of algebraic or differential equations.

Regular chains have been investigated in many papers, among them are those of Aubry *et al.* (1999), Kalkbrener (1998) and Chou and Gao (1991). Several surveys (Boulier *et al.*, 2006; Hubert, 2001) are also available on this topic. The abundant literature on the subject can be explained by the many equivalent definitions of a regular chain. Actually, the original formulation of Kalkbrener is quite different from that of Yang and Zhang. In the papers by Chen *et al.* (2007) and Wang (2000), the authors provide bridges between the point of view of Kalkbrener and that of Yang and Zhang.

The key algebraic object associated with a regular chain is its *saturated ideal*. Let us review its definition. Let \mathbf{k} be a field and $x_1 \prec \dots \prec x_n$ be ordered variables. For a regular chain $T \subset \mathbf{k}[x_1, \dots, x_n]$, the saturated ideal of T , denoted by $\text{sat}(T)$ is defined by $\text{sat}(T) := \langle T \rangle : h^\infty$, where h is the product of the initial polynomials of T . (The next section contains a detailed review of these notions.) Given a polynomial $p \in \mathbf{k}[x_1, \dots, x_n]$, the memberships $p \in \text{sat}(T)$ and $p \in \sqrt{\text{sat}(T)}$ can be decided by means of pseudo-divisions and GCD computations, respectively. One should observe that these computations can be achieved without computing a system of generators of $\text{sat}(T)$. In some sense, the regular chain T is a “black box representation” of $\text{sat}(T)$ since the assertions $p \in \text{sat}(T)$ and $p \in \sqrt{\text{sat}(T)}$ can be evaluated without using an explicit representation of $\text{sat}(T)$.

Being able to compute a system of generators of $\text{sat}(T)$ remains, however, a fundamental question. For instance, given a second regular chain $U \subset \mathbf{k}[x_1, \dots, x_n]$, the only general method to decide the inclusion $\text{sat}(T) \subseteq \text{sat}(U)$ goes through the computation of a system of generators of $\text{sat}(T)$ by means of Gröbner bases. Unfortunately, such computations can be expensive (see Aubry and Moreno Maza, 1999) whereas one would like to obtain an inclusion test which could be used intensively in order to remove redundant components when computing the triangular decompositions of Kalkbrener’s algorithm or those arising in differential algebra. Note that for other kinds of triangular decompositions, such as those of Moreno Maza (1999) and Wang (2000), this question has been solved in Chen *et al.* (2007).

Therefore, testing the inclusion $\text{sat}(T) \subseteq \text{sat}(U)$ without Gröbner basis computation is a very important question in practice. Moreover, this can be regarded as an *algebraic version* of the Ritt problem in differential algebra. One case presents no difficulties: if $\text{sat}(T)$ is a zero-dimensional ideal, the product of the initial polynomials of T is invertible modulo $\langle T \rangle$ (see Moreno Maza and Rioboo, 1995, Proposition 5) and thus T generates $\text{sat}(T)$. In this case the inclusion test for saturated ideals reduces to the membership problem mentioned above.

In positive dimension, however, the ideal $\text{sat}(T)$ could be strictly larger than that generated by T . Consider for instance $n = 4$ and $T = \{x_1x_3 + x_2, x_2x_4 + x_1\}$, we have

$$\langle T \rangle = \langle x_1, x_2 \rangle \cap \langle x_1x_3 + x_2, -x_3x_4 + 1 \rangle.$$

Thus, we have

$$\text{sat}(T) = \langle T \rangle : (x_1x_2)^\infty = \langle x_1x_3 + x_2, -x_3x_4 + 1 \rangle.$$

In this article, we give a necessary and sufficient condition for the equality $\langle T \rangle = \text{sat}(T)$ to hold. Looking at the above example, one can feel that the ideal $\langle x_1, x_2 \rangle$ can be regarded as a “sort of content” of the ideal $\langle T \rangle$, which is discarded when computing $\text{sat}(T)$. We

observe also that the polynomials $x_1x_3 + x_2$ and $x_2x_4 + x_1$ are primitive in $(\mathbf{k}[x_1, x_2])[x_3]$ and $(\mathbf{k}[x_1, x_2])[x_4]$ respectively. Thus, the “usual notion” of primitivity (for a univariate polynomial over a UFD) is not sufficient to guarantee the equality $\langle T \rangle = \text{sat}(T)$. This leads us to the following two definitions.

Let R be a commutative ring with unity. We say that a non-constant polynomial $p = a_e x^e + \dots + a_0 \in R[x]$ is *weakly primitive* if for any $\beta \in R$ such that a_e divides $\beta a_{e-1}, \dots, \beta a_0$ then a_e divides β as well. This notion and its relations with similar concepts are discussed in Sections 3, 4, and 5.

We say that the regular chain $T = \{p_1, \dots, p_m\}$ is *primitive* if for all $1 \leq k \leq m$, the polynomial p_k is weakly primitive in $R[x_j]$, where x_j is the main variable of p_k and R is the residue class ring $\mathbf{k}[x_1, \dots, x_{j-1}]/\langle p_1, \dots, p_{k-1} \rangle$.

The first main result of this paper is the following: *the regular chain T generates its saturated ideal if and only if T is primitive*. This result, generalizing the concept of primitivity from univariate polynomials to regular chains, is established in Section 4.

Looking at regular chains from the point of view of regular sequences, we obtain our second main result: an algorithm to decide whether a regular chain generates its saturated ideal or not. The pseudo-code and its proof are presented in Section 6. This algorithm relies on a procedure for computing triangular decompositions. However, being applied to input systems which are regular sequences and “almost regular chains”, this procedure reduces simply to an iterated resultant computation. As a result, the proposed algorithm performs very well in practice and is Gröbner basis free. In Section 8 we report on experimentation, where we confirm the efficiency of this algorithm. Meanwhile, we observe that primitive regular chains are often present in the output of triangular decompositions.

Section 7, which is a new development w.r.t. our ISSAC paper (Lemaire *et al.*, 2008), proposes several criteria for testing the inclusion of saturated ideals. We point out that the notion of primitivity of regular chains provides a helpful tool for dealing with this question in practice. Section 9, which is also enhanced w.r.t. (Lemaire *et al.*, 2008), offers concluding remarks and open problems.

2. Preliminaries

2.1. Triangular set and regular chain

We denote by $\mathbf{k}[\mathbf{x}]$ the ring of multivariate polynomials with coefficients in a field \mathbf{k} and with ordered variables $\mathbf{x} = x_1 \prec \dots \prec x_n$. For a non-constant polynomial $p \in \mathbf{k}[\mathbf{x}]$, the greatest variable appearing in p is called *main variable*, denoted by $\text{mvar}(p)$. We regard p as a univariate polynomial in its main variable. The degree, the leading coefficient, the leading monomial and the reductum of p as a univariate polynomial in $\text{mvar}(p)$ are called *main degree*, *initial*, *rank* and *tail* of p ; they are denoted by $\text{mdeg}(p)$, $\text{init}(p)$, $\text{rank}(p)$ and $\text{tail}(p)$ respectively. Thus we have $p = \text{init}(p)\text{rank}(p) + \text{tail}(p)$.

Let R be a commutative ring with unity and F be a subset of R . Denote by $\langle F \rangle$ the ideal it generates, by $\sqrt{\langle F \rangle}$ the radical of $\langle F \rangle$, and by $R/\langle F \rangle$ the *residue class ring* of R with respect to $\langle F \rangle$. For an element p in R , we say that p is zero modulo $\langle F \rangle$ if p belongs to $\langle F \rangle$, that is, p is zero as an element in $R/\langle F \rangle$. An element $p \in R$ is a *zerodivisor* modulo $\langle F \rangle$, if there exists $q \in R$ such that $p \notin \langle F \rangle$ and $q \notin \langle F \rangle$ but $pq \in \langle F \rangle$. We say that p is *regular* modulo $\langle F \rangle$ if it is neither zero, nor a zerodivisor modulo $\langle F \rangle$. Furthermore, p is *invertible* in R if there exists a $q \in R$ such that $pq = 1$.

Example 1. Consider the polynomials in $\mathbf{k}[x_1, x_2, x_3]$

$$p_1 = x_2^2 - x_1^2, p_2 = (x_2 - x_1)x_3 \text{ and } p_3 = x_2x_3^3 - x_1.$$

The above notions are illustrated in the following table.

	mvar	init	mdeg	rank	tail
p_1	x_2	1	2	x_2^2	$-x_1^2$
p_2	x_3	$x_2 - x_1$	1	x_3	0
p_3	x_3	x_2	3	x_3^3	$-x_1$

The initial $x_2 - x_1$ of p_2 is a zerodivisor modulo $\langle p_1 \rangle$, since $(x_2 + x_1)(x_2 - x_1)$ is in $\langle p_1 \rangle$, while neither $x_2 + x_1$ nor $x_2 - x_1$ belongs to $\langle p_1 \rangle$. However, the initial x_2 of p_3 is regular modulo $\langle p_1 \rangle$.

In what follows, we recall the notions of regular chain and saturated ideal, which are the main objects in our study.

A set T of non-constant polynomials in $\mathbf{k}[\mathbf{x}]$ is called a *triangular set*, if for all $p, q \in T$ with $p \neq q$ we have $\text{mvar}(p) \neq \text{mvar}(q)$. For a nonempty triangular set T , we define the *saturated ideal* $\text{sat}(T)$ of T to be the ideal $\langle T \rangle : h^\infty$, that is,

$$\text{sat}(T) := \langle T \rangle : h^\infty = \{q \in \mathbf{k}[\mathbf{x}] \mid \exists e \in \mathbb{Z}_{\geq 0} \text{ s.t. } h^e q \in \langle T \rangle\},$$

where h is the product of the initials of the polynomials in T . The empty set is also regarded as a triangular set, whose saturated ideal is the trivial ideal $\langle 0 \rangle$.

One way of solving (or decomposing) a polynomial set $F \subseteq \mathbf{k}[\mathbf{x}]$ is to compute triangular sets $T_1, \dots, T_e \subseteq \mathbf{k}[\mathbf{x}]$ such that

$$\sqrt{\langle F \rangle} = \sqrt{\text{sat}(T_1)} \cap \dots \cap \sqrt{\text{sat}(T_e)}.$$

It is thus desirable to require $\text{sat}(T_1), \dots, \text{sat}(T_e)$ to be proper ideals. This observation has led to the notion of a regular chain which was introduced independently in Kalkbrener (1993); Yang and Zhang (1991).

Definition 1 (Regular chain). Let T be a triangular set in $\mathbf{k}[\mathbf{x}]$. If T is empty, then it is a regular chain. Otherwise, let p be the polynomial of T with the greatest main variable and let C be the set of other polynomials in T . We say that T is a *regular chain*, if C is a regular chain and $\text{init}(p)$ is regular modulo $\text{sat}(C)$.

In commutative algebra (see Eisenbud, 1994) there is a closely related concept called *regular sequence* which is a sequence r_1, \dots, r_s of nonzero elements in the ring $\mathbf{k}[\mathbf{x}]$ satisfying

- (1) $\langle r_1, \dots, r_s \rangle$ is a proper ideal of $\mathbf{k}[\mathbf{x}]$;
- (2) r_i is regular modulo $\langle r_1, \dots, r_{i-1} \rangle$, for each $2 \leq i \leq s$.

When we sort polynomials in a regular chain by increasing main variable, the following example says that the resulting sequence may not be a regular sequence of $\mathbf{k}[\mathbf{x}]$.

Example 2. Let $T = \{t_1, t_2\}$ be a triangular set in $\mathbf{k}[x_1, x_2, x_3]$ with $t_1 = x_1x_2$ and $t_2 = x_1x_3$. Then $\{t_1\}$ is a regular chain with $\text{sat}(\{t_1\}) = \langle x_1x_2 \rangle : x_1^\infty = \langle x_2 \rangle$. Since $\text{init}(t_2) = x_1$ is regular modulo $\text{sat}(\{t_1\})$, the triangular set T is a regular chain with

$$\text{sat}(T) = \langle x_1x_2, x_1x_3 \rangle : x_1^\infty = \langle x_2, x_3 \rangle.$$

However, t_1, t_2 is not a regular sequence since $t_2 = x_1x_3$ is not regular modulo $\langle x_1x_2 \rangle$. Here, the saturation operation *discards* the content introduced by the initials.

2.2. Properties of regular chains

We recall several important results on regular chains and saturated ideals, which will be used throughout this paper. Pseudo-division and iterated resultant are fundamental tools in this context.

Let p and q be polynomials of $\mathbf{k}[\mathbf{x}]$, with $q \notin \mathbf{k}$. Denote by $\text{prem}(p, q)$ and $\text{pquo}(p, q)$ the *pseudo-remainder* and the *pseudo-quotient* of p by q , regarding p and q as univariate polynomials in $x = \text{mvar}(q)$. Using these notations, we have

$$\text{init}(q)^e p = \text{pquo}(p, q)q + \text{prem}(p, q), \quad (1)$$

where $e = \max\{\deg(p, x) - \deg(q, x) + 1, 0\}$; moreover either $r := \text{prem}(p, q)$ is null or $\deg(r, x) < \deg(q, x)$. Pseudo-division generalizes as follows given a polynomial p and a regular chain T :

$$\text{prem}(p, T) = \begin{cases} p & \text{if } T = \emptyset, \\ \text{prem}(\text{prem}(p, t), T') & \text{if } T = T' \cup \{t\}, \end{cases}$$

where t is the polynomial in T with greatest main variable. We have the *pseudo-division formula* (Wu, 1986): there exist non-negative integers e_1, \dots, e_s and polynomials q_1, \dots, q_s in $\mathbf{k}[\mathbf{x}]$ such that

$$h_1^{e_1} \dots h_s^{e_s} p = \sum_{i=1}^s q_i t_i + \text{prem}(p, T), \quad (2)$$

where $T = \{t_1, \dots, t_s\}$ and $h_i = \text{init}(t_i)$, for $1 \leq i \leq s$.

We denote by $\text{res}(p, q)$ the *resultant* of p and q regarding them as univariate polynomials in $\text{mvar}(q)$. Note that $\text{res}(p, q)$ may be different from $\text{res}(q, p)$, if they have different main variables. For a polynomial p and a regular chain T , we define the *iterated resultant* of p w.r.t. T , denoted by $\text{ires}(p, T)$, as follows:

$$\text{ires}(p, T) = \begin{cases} p & \text{if } T = \emptyset, \\ \text{ires}(\text{res}(p, t), T') & \text{if } T = T' \cup \{t\}, \end{cases}$$

where t is the polynomial in T with greatest main variable.

Theorem 1. For a regular chain T and a polynomial p we have:

- (1) p belongs to $\text{sat}(T)$ if and only if $\text{prem}(p, T) = 0$,
- (2) p is regular modulo $\text{sat}(T)$ if and only if $\text{ires}(p, T) \neq 0$,
- (3) p is a zerodivisor modulo $\text{sat}(T)$ if and only if $\text{ires}(p, T) = 0$ and $\text{prem}(p, T) \neq 0$.

For the proofs, we refer to Aubry *et al.* (1999) for item (1), and to Wang (2000); Chen *et al.* (2007) for item (2). Item (3) is a direct consequence of (1) and (2).

Remark 1. Theorems 1 and 2 highlight the structure of the associated primes of $\text{sat}(T)$ which makes regularity test easier than with an arbitrary polynomial ideal. In general, deciding if a polynomial p is regular modulo an ideal I is equivalent to checking if p does not belong to any associated primes of I .

An ideal in $\mathbf{k}[\mathbf{x}]$ is *unmixed*, if all its associated primes have the same dimension. In particular, an unmixed ideal has no embedded associated primes.

Theorem 2. Let $T = C \cup \{t\}$ be a regular chain in $\mathbf{k}[\mathbf{x}]$ with t having greatest main variable in T . The following properties hold:

- (1) $\text{sat}(T)$ is an unmixed ideal with dimension $n - |T|$,
- (2) $\text{sat}(T \cap \mathbf{k}[x_1, \dots, x_i]) = \text{sat}(T) \cap \mathbf{k}[x_1, \dots, x_i]$,
- (3) $\text{sat}(T) = \langle \text{sat}(C) \cup \{t\} \rangle : \text{init}(t)^\infty$.

For the proofs, we refer to Boulier *et al.* (2006); Chou and Gao (1991) for item (1), to Aubry *et al.* (1999) for item (2), and to Kalkbrener (1998) for item (3). From (1), we deduce that the saturated ideal of a regular chain T consisting of n polynomials has dimension 0.

3. Primitivity of Polynomials

In this section, we introduce the notion of weak primitivity of a polynomial in a general univariate polynomial ring, and then present several of its properties.

The following Lemma 1 may be seen as a generalization of Gauss lemma over an arbitrary commutative ring with unity. It will be used in the proof of our main theorem. We found that this lemma is not new and can be deduced from the Dedekind-Mertens Lemma (See Arnold and Gilmer (1970); Corso *et al.* (1998); Coquand *et al.* (2003) and the references therein). For the sake of reference, we include our direct proof here. In the sequel, the ring R is a commutative Noetherian ring with unity. We say that p divides q , denoted by $p \mid q$, if there exists r such that $q = pr$ holds.

Lemma 1. Let $p = \sum_{i=0}^m a_i y^i$ and $q = \sum_{i=0}^n b_i y^i$ be polynomials in $R[y]$ with $\deg(p) = m \geq 0$ and $\deg(q) = n \geq 0$. Then for each $h \in R$,

- (i) $h \mid pq$ implies $h \mid b_0 a_i^{n+1}$ for $0 \leq i \leq m$,
- (ii) $h \mid pq$ implies $h \mid b_n a_i^{n+1}$ for $0 \leq i \leq m$.

Proof. First, we prove (i). Considering first the special case $m = 0$, we observe that $h \mid pq$ implies $h \mid a_0 b_0$ and the conclusion follows. Now we assume that $m > 0$ holds.

For $i = 0$, the claim is also clear, for the same reason as the case $m = 0$. For $1 \leq i \leq m$, we introduce the polynomials A_i and B_i below in order to simplify our expressions:

$$A_i = \sum_{j=0}^{i-1} a_j y^j, \text{ and } B_i = - \sum_{j=i}^m a_j y^j. \quad (3)$$

Clearly, we have $p = A_i - B_i$. The key observation is to consider the polynomial $\tilde{p} = A_i^{n+1} - B_i^{n+1}$, as suggested by the forms of our claims. To avoid talking about the degree of a zero polynomial, we assume that both A_i^{n+1} and B_i^{n+1} are nonzero polynomials.

According to the construction of A_i and B_i in (3), we have the following degree estimates:

$$\deg(A_i^{n+1}) \leq \deg(A_i)(n+1) \leq (i-1)(n+1), \quad (4)$$

$$\text{trdeg}(B_i^{n+1}) \geq \text{trdeg}(B_i)(n+1) \geq i(n+1), \quad (5)$$

where $\text{trdeg}(\cdot)$ denotes the trailing degree, that is, the degree of the term with lowest degree in a polynomial. Therefore there is no term cancellation between A_i^{n+1} and B_i^{n+1} . With the assumption that A_i and B_i nonzero, the polynomial \tilde{p} is nonzero too. Now we write \tilde{p} in the form

$$\tilde{p} = (A_i - B_i)(A_i^n + \cdots + B_i^n) = p(A_i^n + \cdots + B_i^n).$$

It follows that $p \mid \tilde{p}$ holds. Therefore $h \mid \tilde{p}q$ holds since we have $h \mid pq$. Observe now that if qA_i^{n+1} is nonzero, then

$$\deg(qA_i^{n+1}) \leq (i-1)(n+1) + n < i(n+1). \quad (6)$$

Similarly, if qB_i^{n+1} is nonzero, then its trailing degree is bounded

$$\text{trdeg}(qB_i^{n+1}) \geq i(n+1). \quad (7)$$

Combining (6) with (7), we know that in $q\tilde{p} = qA_i^{n+1} - qB_i^{n+1}$, the polynomial qA_i^{n+1} only contributes to terms with degree smaller than $i(n+1)$. Thus we have

$$\text{coeff}(q\tilde{p}, y^{i(n+1)}) = \text{coeff}(-qB_i^{n+1}, y^{i(n+1)}) = b_0 a_i^{n+1} \quad (8)$$

which implies $h \mid b_0 a_i^{n+1}$, as desired.

Now we handle the special cases where $A_i^{n+1} = 0$ and $B_i^{n+1} = 0$. It is easy to see that $A_i^{n+1} = 0$ does not affect the proof above. When $B_i^{n+1} = 0$, simply we have $a_i^{n+1} = 0$, and then the claim is also clear.

Finally, we prove (ii). Let $P = y^m p(1/y)$ and $Q = y^n q(1/y)$. Since $h \mid pq$, h will also divide $PQ = y^{m+n}(pq)(1/y)$. Assume that

$$\begin{aligned} a_0 = \cdots = a_{r-1} = 0, a_r \neq 0, \\ b_0 = \cdots = b_{s-1} = 0, b_s \neq 0. \end{aligned}$$

Then $r \leq m$ and $s \leq n$ hold. According to (i), for any $r \leq i \leq m$, $h \mid b_n a_i^{s+1}$. It follows that $h \mid b_n a_i^{n+1}$ for any $0 \leq i \leq m$. \square

Definition 2. Let $p = a_0 + \cdots + a_e x^e \in R[x]$ with $e \geq 1$. The polynomial p is *strongly primitive* if the ideal generated by $\{a_0, \dots, a_e\}$ is the whole ring R . The polynomial p is *weakly primitive* if for any $\beta \in R$ such that $a_e \mid \beta a_i$ holds for all $0 \leq i \leq e-1$, we have $a_e \mid \beta$ as well.

Proposition 1. Strong primitivity implies weak primitivity.

Proof. We use the same notation as in Definition 2. Let p be strongly primitive. Then there exist $c_e, \dots, c_0 \in R$ such that $c_e a_e + \cdots + c_0 a_0 = 1$. Let $\beta \in R$ such that for $0 \leq j \leq e-1$, we have $a_e \mid \beta a_j$. Then there exist $d_0, \dots, d_{e-1} \in R$ such that $a_e d_j = \beta a_j$. Since $\beta c_e a_e + \cdots + \beta c_0 a_0 = \beta$, we have $a_e(\beta c_e + d_{e-1} c_{e-1} \cdots + d_0 c_0) = \beta$. Thus, we have $a_e \mid \beta$, and therefore p is weakly primitive. \square

Remark 2.

- (1) If any a_i is invertible, then p is strongly primitive and then is weakly primitive. As a particular case, p is weakly primitive if one of its coefficients is a nonzero constant of a field.

- (2) Weak primitivity does not imply strong primitivity. For example, let $R = \mathbb{Z}[t]$ and $p = tx + 2 \in \mathbb{Z}[t][x]$. Then p is not strongly primitive, since $\langle t, 2 \rangle \neq \langle 1 \rangle_R$. In $R[x]$, the polynomial p is weakly primitive. If $t \mid 2\beta$, then $t \mid \beta$ must hold.
- (3) The definition of strongly primitive does not depend on the *order* of the coefficients in p . However, the definition of weakly primitive relies on it. Indeed, let $R = \mathbb{Z}_4[t]$, $p = \bar{2}x + t$ and $q = tx + \bar{2}$. Then we have
- (a) p is weakly primitive in $R[x]$. For any $\beta \in R[x]$, if $\bar{2} \mid t\beta$ then $\bar{2} \mid \beta$.
 - (b) q is not weakly primitive in $R[x]$. Let $\beta = t + \bar{2} \in R[x]$. Then we have $t \mid \bar{2}(t + \bar{2}) = \bar{2}t$, and $t \nmid (t + \bar{2})$.
- (4) Weak primitivity may not be extended. That is to say, if p is weakly primitive, assuming that $\deg(p) = e > 0$, then $\bar{p} = p + qx^{e+1}$ may not be weakly primitive. For example, let $R = \mathbb{Z}_4[t]$, $p = \bar{2}x + t$ and $\bar{p} = p + tx^2 = tx^2 + \bar{2}x + t$. Then p is weakly primitive, and \bar{p} is not weakly primitive. Indeed taking $\beta = t + \bar{2}$, we have $t \mid t\beta$ and $t \mid \bar{2}\beta$, but $t \nmid \beta$.

According to Proposition 2 the notion of weak primitivity turns out to be a generalization of the ordinary notion of primitivity (the gcd of the coefficients of a univariate polynomial is 1).

Proposition 2. Let R be a UFD and $p = \sum_{i=0}^e a_i x^i \in R[x]$ with $a_e \neq 0$ and $e \geq 1$. Then, the following statements are equivalent

- (i) p is weakly primitive in $R[x]$.
- (ii) $\text{content}(p) := \gcd(a_0, \dots, a_e) = 1$.

Proof. We prove (i) \Rightarrow (ii). Assume that $\gcd(a_0, \dots, a_e) \neq 1$. Then there is a prime factor f of $\gcd(a_0, \dots, a_e)$. Let $\beta = a_e/f$. Then $a_e \mid \beta a_i$, for $0 \leq i \leq e-1$. Since $a_e \nmid \beta$, p is not weakly primitive, a contradiction.

We prove (ii) \Rightarrow (i). Assume that there exists $\beta \in R$ such that

$$(\forall 0 \leq j \leq e-1) \quad a_e \mid \beta a_j \quad \text{and} \quad a_e \nmid \beta.$$

Then $a_e \mid \text{content}(\beta p) = \beta \text{content}(p)$. Since $a_e \nmid \beta$, some prime factor f of a_e divides $\text{content}(p)$, a contradiction. \square

The following property on weak primitivity will be used in the next section. It states the following fact: if one raises each coefficient of a weakly primitive polynomial p to some power, then the resulting polynomial is still weakly primitive. To avoid the cancellation of the leading coefficient of p , we assume that this coefficient is a regular element of the ground ring.

Proposition 3. Let $p = \sum_{i=0}^e a_i x^i \in R[x]$ with a_e being regular in R , and $\{n_i \mid 0 \leq i \leq e\}$ be a set of non-negative integers. Define $q = \sum_{i=0}^e a_i^{n_i} x^i$. Then if p is weakly primitive, q is also weakly primitive.

The proof directly follows from the following two lemmas.

Lemma 2. Let $p = a_0 + \dots + a_e x^e \in R[x]$ with a_e being regular in R and n be a non-negative integer. If p is weakly primitive, then $p_n = a_0 + \dots + a_{e-1} x^{e-1} + a_e^n x^e$ is also weakly primitive.

Proof. By induction on $n \geq 0$. The case $n = 0$ follows from Remark 2. So we assume that the claim is true for $n - 1$, that is, p_{n-1} is weakly primitive, with $n \geq 1$. Let $\beta \in R$ such that $a_e^n \mid a_i\beta$, for $0 \leq i \leq e - 1$. There exist $h_0, \dots, h_{e-1} \in R$ such that we have

$$a_e^n h_i = a_i \beta, \quad 0 \leq i \leq e - 1. \quad (9)$$

Since p_{n-1} is weakly primitive and since we have $a_e^{n-1} \mid a_i\beta$, we deduce $a_e^{n-1} \mid \beta$, that is, there exists $h' \in R$ such that

$$a_e^{n-1} h' = \beta. \quad (10)$$

With (9) and (10) we have $a_e^n h_i = a_i a_e^{n-1} h'$, and then $a_e h_i = a_i h'$, since a_e is regular. Hence $a_e \mid a_i h'$. By the weak primitivity of p , $a_e \mid h'$ holds, that is, there exists $h'' \in R$ such that

$$a_e h'' = h'. \quad (11)$$

By (10) and (11) we have $a_e^n h'' = \beta$. So $a_e^n \mid \beta$ and p_n is weakly primitive. \square

Lemma 3. Let $p = a_0 + \dots + a_e x^e \in R[x]$ with $a_e \neq 0$ and n be a non-negative integer. Let j be an index such that $0 \leq j \leq e - 1$. Define $q = a_0 + \dots + a_j^n x^j + \dots + a_e x^e = p + (a_j^n - a_j)x^j$. If p is weakly primitive, then q is also weakly primitive.

Proof. The claim is clear if $n = 0$, so we assume $n \geq 1$. Let $\beta \in R$ such that, for $0 \leq i \leq e - 1$ and $i \neq j$

$$a_e \mid a_i \beta, \text{ and } a_e \mid a_j^n \beta. \quad (12)$$

We prove that $a_e \mid \beta$ holds. We have, for $0 \leq i \leq e - 1$ and $i \neq j$

$$a_e \mid a_i(a_j^{n-1}\beta), \text{ and } a_e \mid a_j(a_j^{n-1}\beta).$$

Define $\beta' = a_j^{n-1}\beta$. Hence $a_e \mid \beta'$ holds, since p is weakly primitive. With (12), for $0 \leq i \leq e - 1$ and $i \neq j$ we have

$$a_e \mid a_i \beta, \text{ and } a_e \mid a_j^{n-1} \beta. \quad (13)$$

We deduce that $a_e \mid a_j^{n-2}\beta$ holds. Continuing in this manner, we reach $a_e \mid \beta$. Thus q is also weakly primitive. \square

4. Primitive Regular Chain

In this section, we generalize the notion of primitivity to any regular chain T . Then we prove that $\text{sat}(T) = \langle T \rangle$ holds if and only if T is primitive.

Definition 3. Let $T = \{p_1, \dots, p_m\} \subset \mathbf{k}[\mathbf{x}] = \mathbf{k}[x_1, \dots, x_n]$ be a regular chain with $\text{mvar}(p_1) \prec \dots \prec \text{mvar}(p_m)$. We say that T is *primitive* if for all $1 \leq k \leq m$, p_k is weakly primitive in $R[x_j]$ where $x_j = \text{mvar}(p_k)$ and

$$R = \mathbf{k}[x_1, \dots, x_{j-1}] / \langle p_1, \dots, p_{k-1} \rangle.$$

Proposition 4 (Base case of Theorem 3). Let $p = a_e x^e + \dots + a_0 \in \mathbf{k}[\mathbf{y}][x]$ and $c = \text{gcd}_{\mathbf{k}[\mathbf{y}]}(a_0, \dots, a_e)$, where $e \geq 1$ and \mathbf{y} is a finite set of variables. Then we have $\langle p \rangle = \langle p \rangle : a_e^\infty \iff c = 1$.

Proof. First we prove that $\langle p \rangle \subsetneq \text{sat}(p) := \langle p \rangle : a_e^\infty$ if $c \neq 1$. Denote $\bar{p} = p/c$. Then $a_e \bar{p} = a_e p/c \in \langle p \rangle$, hence $\bar{p} \in \text{sat}(p)$. Assume that \bar{p} is in $\langle p \rangle$. Then there exists $q \in \mathbf{k}[\mathbf{y}][x]$ such that $p/c = \bar{p} = pq$. It follows that $qc = 1$ which is a contradiction since $c \notin \mathbf{k}$. Therefore \bar{p} is in $\text{sat}(p)$ but not in $\langle p \rangle$.

Conversely, we prove that if $c = 1$ then $\text{sat}(p) \subseteq \langle p \rangle$. For any $q \in \text{sat}(p)$, there exist $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbf{k}[\mathbf{y}][x]$ such that $a_e^n q = \beta p$. Taking the content w.r.t. x , we have

$$\begin{aligned} a_e^n \text{content}(q, x) &= \text{content}(\beta, x) \text{content}(p, x) \\ &= \text{content}(\beta, x) \end{aligned}$$

Thus $a_e^n \mid \beta$. There exists $\beta' \in \mathbf{k}[\mathbf{y}][x]$ such that $\beta = a_e^n \beta'$. So we have $a_e^n q = \beta p = a_e^n \beta' p$, and then $q = \beta' p$, that is, $q \in \langle p \rangle$. \square

Remark 3. Let $T = \{p_1\}$ be a regular chain consisting of a single polynomial. By definition, T is primitive if and only if p_1 is weakly primitive in $R = \mathbf{k}[x_1, \dots, x_{j-1}]$, where $x_j = \text{mvar}(p_1)$. Since R is a UFD, it follows from Proposition 2, that T is primitive if and only if p_1 is primitive in ordinary sense, that is, whenever the gcd of the coefficients of p_1 (as a univariate polynomial in $R[x_j]$) is 1. Therefore, the notion of primitivity for a regular chain extends that of primitivity for a polynomial.

Theorem 3. Let $T \subset \mathbf{k}[x_1, \dots, x_n]$ be a regular chain. Then T is primitive if and only if $\langle T \rangle = \text{sat}(T)$.

Proof. We prove the theorem by induction on the number of polynomials in T . The base case is Proposition 4, where $|T| = 1$. Now assume that $T = \{p_1, \dots, p_m\}$ consists of $m \geq 2$ polynomials with $\text{mvar}(p_1) \prec \dots \prec \text{mvar}(p_m)$. We denote by T_k the regular chain consisting of the first k polynomials in T .

First, assume indirectly that T is not primitive. We need to prove that $\langle T \rangle$ is a proper subset of $\text{sat}(T)$. Let k be the smallest integer such that p_k is not weakly primitive in $R[y]$, where $y = x_j = \text{mvar}(p_k)$ and $R = \mathbf{k}[x_1, \dots, x_{j-1}]/\langle T_{k-1} \rangle$. By Proposition 4, we know $k \geq 2$.

Let $p_k = a_e y^e + \dots + a_0$. By induction, $\text{sat}(T_{k-1}) = \langle T_{k-1} \rangle$ holds and thus a_e is regular in R . Since p_k is not weakly primitive over R , there exists $\beta \in \mathbf{k}[x_1, \dots, x_{j-1}]$ such that, in R , we have

$$(\forall 0 \leq r \leq e-1) \quad a_e \mid \beta a_r \quad \text{and} \quad a_e \nmid \beta.$$

Define $q_k = \beta p_k / a_e$. Then $q_k \in R[y]$, since

$$\frac{\beta}{a_e} p_k = \beta y^e + \sum_{0 \leq r < e} \frac{\beta a_r}{a_e} y^r.$$

We claim that $q_k \in \langle p_k \rangle : a_e^\infty$ and $q_k \notin \langle p_k \rangle$ in $R[y]$, which leads to $\text{sat}(T_k) \neq \langle T_k \rangle$.

Indeed, we have $a_e q_k = \beta p_k \in \langle p_k \rangle$ in $R[y]$. Thus, $q_k \in \langle p_k \rangle : a_e^\infty$. Now if $q_k \in \langle p_k \rangle$, there exists $\alpha \in R[y]$ such that $q_k = \alpha p_k$ in $R[y]$. By the construction of q_k , $\deg(q_k, y)$ equals $\deg(p_k, y)$. Hence $\alpha \in R$ and $\beta - \alpha a_e = 0$ in R . This contradicts $a_e \nmid \beta$.

Secondly, we assume that T is primitive and show $\langle T \rangle = \text{sat}(T)$. By induction, $\text{sat}(T_{k-1}) = \langle T_{k-1} \rangle$ holds. We shall prove that $\text{sat}(T_k) = \langle T_k \rangle$ holds, too. To do so, we consider $p \in \text{sat}(T_k)$ and show that we have $p \in \langle T_k \rangle$. Let $\text{mvar}(p) = x_i$ and $\text{mvar}(p_k) = x_j$. If $i > j$, then $p \in \text{sat}(T_k)$ if and only if all coefficients of p w.r.t x_i are in $\text{sat}(T_k)$, since T_k is a regular chain. So we can concentrate on the case $p \in \mathbf{k}[x_1, \dots, x_j]$.

Let h_{p_k} be the leading coefficient of p_k w.r.t. $y = x_j$, that is, w.r.t. the main variable of p_k . By virtue of Theorem 2 we have

$$\begin{aligned}\text{sat}(T_k) &= \langle \text{sat}(T_{k-1}), p_k \rangle : h_{p_k}^\infty \\ &= \langle \langle T_{k-1} \rangle, p_k \rangle : h_{p_k}^\infty.\end{aligned}$$

By virtue of Theorem 1 we have $\text{prem}(p, T_k) = 0$, since $p \in \text{sat}(T_k)$. Consequently, $\text{prem}(p, p_k)$ is in $\text{sat}(T_{k-1}) = \langle T_{k-1} \rangle$. Now the pseudo-division formula (1) in Section 2 leads to

$$h_{p_k}^\alpha p = \text{pquo}(p, p_k)p_k + \text{prem}(p, p_k), \quad (14)$$

where $\alpha = \max\{0, \deg(p, y) - \deg(p_k, y) + 1\}$. If $\deg(p, y) < \deg(p_k, y)$, then $p = \text{prem}(p, p_k) \in \langle T_{k-1} \rangle \subset \langle T_k \rangle$ holds and we are done. From now on, we assume $\deg(p, y) \geq \deg(p_k, y)$ and we write $\alpha = \deg(p, y) - \deg(p_k, y) + 1$. With (14) we observe that we have the following equation in $R[y]$

$$h_{p_k}^\alpha p = q p_k. \quad (15)$$

We consider a more general situation: let $s \in \text{sat}(T_k)$, let δ be a non-negative integer and let $u \in \mathbf{k}[x_1, \dots, x_n]$ such that

$$h_{p_k}^\delta s = u p_k \quad (16)$$

holds in $R[y]$. In order to prove that $p \in \langle T_k \rangle$ holds, we prove that $s \in \langle T_k \rangle$ by induction on the number of terms in u . For simplicity, we denote

$$p_k = \sum_{i=0}^e a_i y^i \quad \text{and} \quad u = \sum_{i=0}^f b_i y^i,$$

with $a_e \neq 0$ and $b_f \neq 0$. Note that $a_e = h_{p_k}$.

If $u = 0$ in $R[y]$, then $a_e^\delta s = 0$ in $R[y]$. Since a_e is regular in R , we deduce $s = 0$ in $R[y]$, that is, $s \in \langle T_{k-1} \rangle$ and thus $s \in \langle T_k \rangle$. Assume $u \neq 0$ in $R[y]$. Let f' be the largest integer such that $b_{f'} \notin \langle T_{k-1} \rangle$ and write $u' = \sum_{i=0}^{f'} b_i y^i$. We have

$$a_e^\delta s = u' p_k \text{ in } R[y]. \quad (17)$$

By Lemma 1, for any $0 \leq i \leq e$, we have $a_e^\delta \mid b_{f'} a_i^{f'+1}$ in R . Since p_k is weakly primitive in $R[y]$, by Proposition 3 we have $a_e^\delta \mid b_{f'}$ in R . Thus there exists $\gamma \in \mathbf{k}[x_1, \dots, x_{j-1}]$, $\gamma \neq 0$ in R , such that

$$a_e^\delta \gamma = b_{f'} \quad \text{in } R. \quad (18)$$

We define

$$s' = s - \gamma y^{f'} p_k. \quad (19)$$

Since $s \in \text{sat}(T_k)$ we have $s' \in \text{sat}(T_k)$. Moreover we have

$$u' = a_e^\delta \gamma y^{f'} + \text{tail}(u').$$

Therefore, the following holds in $R[y]$:

$$a_e^\delta s' = \text{tail}(u') p_k. \quad (20)$$

By induction hypothesis we have $s' \in \langle T_k \rangle$. With (19) we conclude $s \in \langle T_k \rangle$, as desired. \square

5. Weak Primitivity Test

In this section, we point out the componentwise nature of weak primitivity. That is, if R can be written as a direct product of rings, then checking weak primitivity over R reduces to checking weak primitivity over each of its “components”.

Lemma 4. Let R_1, \dots, R_n be commutative rings with 1. Let $R = \prod_{i=1}^n R_i$ be their direct product and let π_k be the canonical projection from R to R_k . Let $a, b \in R$. Then $a \mid b$ in R if and only if $\pi_k(a) \mid \pi_k(b)$ for each $1 \leq k \leq n$.

The proof of this lemma is straightforward, and thus is omitted.

Proposition 5. Let $R = \prod_{i=1}^n R_i$ be a direct product of rings and let π_k be the canonical projection from R to R_k and τ_k be the canonical injection from R_k to R . Let $p = \sum_{i=0}^e a_i x^i \in R[x]$ be a polynomial with a_e being regular in R . Then p is weakly primitive in $R[x]$ if and only if $\pi_k(p) = \sum_{i=0}^e \pi_k(a_i) x^i$ is weakly primitive in $R_k[x]$ for each $1 \leq k \leq n$.

Proof. For any $1 \leq k \leq n$, denote $p_k = \pi_k(p)$. Since a_e is regular in R , $\pi_k(a_e) \neq 0$ for each k , and then each p_k is a polynomial of degree e .

First we prove that if all p_k are weakly primitive then p is also weakly primitive. Let $\beta \in R$ satisfying $a_e \mid a_i \beta$ for $0 \leq i \leq e-1$. By definition, we need to prove that $a_e \mid \beta$ in R .

Applying π_k to $a_e \mid a_i \beta$, we have $\pi_k(a_e) \mid \pi_k(a_i) \pi_k(\beta)$, for $0 \leq i \leq e-1$. By the weak primitivity of p_k , we have $\pi_k(a_e) \mid \pi_k(\beta)$. So there exists $u_k \in R_k$ such that $\pi_k(a_e) u_k = \pi_k(\beta)$. Define $u = (u_1, \dots, u_n) \in \prod_{i=1}^n R_i$. Then $\pi_k(u) = u_k$, and hence $\pi_k(a_e) \pi_k(u) = \pi_k(\beta)$, for each $1 \leq k \leq n$. By Lemma 4, $a_e \mid \beta$ in R . We proved that p is weakly primitive in $R[x]$.

On the other hand, we prove that, if p_k is not weakly primitive over R_k for *some* $1 \leq k \leq n$ then p is not weakly primitive over R . For simplicity, we assume $k = 1$. So, there exists $\beta_1 \in R_1$ such that $\pi_1(a_e) \mid \pi_1(a_i) \beta_1$ for $0 \leq i \leq e-1$, but $\pi_1(a_e) \nmid \beta_1$. Define $\beta = \tau_1(\beta_1) = (\beta_1, 0, \dots, 0) \in R$. Then we claim that $a_e \nmid \beta$ and $a_e \mid a_i \beta$ for $0 \leq i \leq e-1$. This implies that p is not weakly primitive over R , as desired.

Indeed, first we have $a_e \nmid \beta$, since $\pi_1(a_e) \nmid \pi_1(\beta) = \beta_1$. Second, to prove $a_e \mid a_i \beta$ for $0 \leq i \leq e-1$, by Lemma 4, we need to prove that $\pi_k(a_e) \mid \pi_k(a_i \beta)$ for $1 \leq k \leq n$ and $0 \leq i \leq e-1$. If $k = 1$, it follows from the choice of β_1 . If $2 \leq k \leq n$, we have

$$\pi_k(a_i \beta) = \pi_k(a_i) \pi_k(\beta) = \pi_k(a_i) \cdot 0 = 0$$

for $1 \leq i \leq e-1$. Thus $\pi_k(a_e) \mid \pi_k(a_i \beta)$ holds for $1 \leq i \leq e-1$. \square

Example 3. Let $T = \{p_1, p_2\}$ be a regular chain in $\mathbb{Q}[t \prec x \prec y]$ with $p_1 = x(x-t)$, $p_2 = (x+t)y+t$. Since $p_1 = x^2 - tx$ is strongly primitive in $(\mathbb{Q}[t])[x]$, p_1 is weakly primitive in $(\mathbb{Q}[t])[x]$. Let $R = \mathbb{Q}[t, x]/\langle x(x-t) \rangle$. Then we have

$$R = R_1 \times R_2 = \mathbb{Q}[x, t]/\langle x \rangle \times \mathbb{Q}[x, t]/\langle x-t \rangle \simeq \mathbb{Q}[t] \times \mathbb{Q}[t].$$

Over R_1 , $p_2 = ty+t$ is not weakly primitive, since t is not invertible over R_1 and according to the definition we can choose $\beta = 1$. Hence T is not a primitive regular chain.

In order to generalize the construction of the above example into an algorithm, one would need to use algebraic factorization. In the next section, we propose a primitivity test for regular chains which avoids algebraic factorization, relying instead on polynomial GCDs modulo regular chains. Based on the algorithms and software tools available today we view it as a practical solution, as confirmed in Section 8.

6. A Primitivity Test Algorithm

In Section 4, we define the notion of primitive regular chain which generalizes that of primitive polynomial over a UFD. In this section, we present another characterization on primitivity in terms of regularity of a polynomial. As a consequence, we obtain an algorithm to test whether a regular chain is primitive or not.

Lemma 5, 6, 7 and 8 are well-known facts. The proofs of Lemma 5 and 8 are straightforward. Lemma 6 can be found as Lemma 9.2.3 in Ischebeck and Rao (2005) whereas Lemma 7 is in Coquand *et al.* (2003, Lemma 7).

Lemma 5. Let I be a proper ideal of R and let h be an element of R . Then h is regular modulo I if and only if $I = I : h^\infty$ holds.

Lemma 6. Let a and b be two regular elements of R . Assume that a and b are not invertible. If a is regular modulo $\langle b \rangle$ then b is also regular modulo $\langle a \rangle$.

Lemma 7 (Mc Coy Lemma). A non-zero polynomial $f \in R[x]$ is a zerodivisor if and only if there exists a non-zero element $a \in R$ such that $af = 0$ holds.

Lemma 8. Let $f \in R[x]$ be a non-constant polynomial. If its leading coefficient is a regular element in R , then f is not a unit.

Proposition 6. Let R be a Noetherian commutative ring with 1. Consider a polynomial $f = \sum_{i=0}^n a_i x^i \in R[x]$. Assume that n is at least 1 and a_n is regular in R . Then $\langle f \rangle = \langle f \rangle : a_n^\infty$ holds if and only if a_n is invertible in R , or $\text{tail}(f)$ is regular modulo $\langle a_n \rangle$.

Proof. If a_n is invertible in R , then clearly $\langle f \rangle : a_n^\infty = \langle f \rangle$ holds. So we assume that a_n is not invertible in R . Note that both a_n and f are regular in $R[x]$; this follows from Lemma 7. Since a_n is not invertible in R , a_n is not invertible in $R[x]$ either. Since a_n is regular in R , it follows from Lemma 8 that f is not invertible in $R[x]$. Then, applying Lemma 5 and 6, we deduce

$$\begin{aligned} \langle f \rangle = \langle f \rangle : a_n^\infty &\iff a_n \text{ is regular modulo } \langle f \rangle \\ &\iff f \text{ is regular modulo } \langle a_n \rangle \\ &\iff \text{tail}(f) \text{ is regular modulo } \langle a_n \rangle. \end{aligned}$$

This completes the proof. \square

The following corollary may be seen as another characterization of the primitivity of a regular chain. This also provides an algorithm for checking whether a regular chain is primitive or not.

Corollary 1 (Primitivity test of a regular chain). Let $T \subset \mathbf{k}[x_1, \dots, x_{s-1}]$ be a primitive regular chain. Let $p = \sum_{i=0}^e a_i x_s^i \in \mathbf{k}[x_1, \dots, x_s]$ with a_e being regular modulo $\text{sat}(T)$. Denote $\text{tail}(p) = \sum_{i=0}^{e-1} a_i x_s^i$. Then $T \cup \{p\}$ is a primitive regular chain if and only if a_e is invertible modulo $\text{sat}(T)$, or $\text{tail}(p)$ is a regular polynomial modulo $\langle T \cup \{a_e\} \rangle$.

Proof. This is a direct consequence of Proposition 6, Theorem 3 and the definition of a regular chain. \square

Thus the problem of checking whether a regular chain $T \cup \{p\}$ is primitive or not, reduces to checking whether the polynomial $\text{tail}(p)$ is regular or not modulo $\langle T, a_e \rangle$. We next show that $\langle T, a_e \rangle$ in Corollary 1 generates an unmixed ideal; this result is crucial in view of Algorithm 1 below. Indeed, it allows us to deal with the following subtle point: a polynomial p regular modulo the radical \sqrt{I} of an ideal I may not be regular modulo I . For example, consider $p = y$ and $I = \langle xy, x^2 \rangle$. Then y is a zerodivisor modulo I but y is regular modulo $\sqrt{I} = \langle x \rangle$. If I is unmixed, then p is regular modulo I if and only if p is regular modulo \sqrt{I} .

Lemma 9. Let $R = \mathbf{k}[x_1, \dots, x_n]$ and T be a primitive regular chain of R . If $t \in R$ is regular but not invertible modulo $\text{sat}(T)$, then (T, t) is a regular sequence of R and the ideal $\langle T, t \rangle$ is unmixed with dimension $n - |T| - 1$.

Proof. Denote $T_i = T \cap \mathbf{k}[x_1, \dots, x_i]$. Since T is primitive, $\text{sat}(T_i) = \langle T_i \rangle$ holds for each i . Thus T is already a regular sequence of R . Now since t is regular but not invertible modulo $\text{sat}(T) = \langle T \rangle$, by definition (T, t) is a regular sequence.

Let $I = \langle T, t \rangle$ and $d = |T|$. According to the Principal Ideal Theorem (see Eisenbud, 1994, Theorem 10.2) the dimension $\dim(I)$ of I is at least $n - (d + 1)$. On the other hand, since (T, t) is a regular sequence of length $d + 1$, the dimension of I is at most $n - (d + 1)$. Hence, $\dim(I) = n - (d + 1)$ and then I is unmixed, by Macaulay Unmixedness Theorem (see Sturmfels, 2002, Theorem 5.7). \square

Remark 4. Before proving the above algorithm, we comment on its subprocedures and possible optimization.

- (1) The function **Triangularize** decomposes a polynomial system F into a finite set of regular chains U_i such that $\sqrt{\langle F \rangle} = \bigcap_i \sqrt{\text{sat}(U_i)}$ holds; this is called a triangular decomposition of F in the sense of Kalkbrener (Aubry and Moreno Maza, 1999). According to the above specification, the set of the associated primes of $\sqrt{\langle F \rangle}$ are “implicitly” represented by U_i ’s.

Triangularize is one of the core functions in the REGULARCHAINS library in MAPLE (Lemaire *et al.*, 2005); it implements the triangular decomposition algorithm of Moreno Maza (1999). While computing in Kalkbrener’s sense, it has the same specification as the function **solve_n** in Kalkbrener (1993), although the algorithms of Moreno Maza (1999) and Kalkbrener (1993) are quite different.

Apart from Kalkbrener’s sense, **Triangularize** can also work in the Lazard sense (see Aubry and Moreno Maza, 1999), where all solutions of the input systems will be explicitly represented by means of regular chains. In general, this function runs faster in Kalkbrener’s sense, since only *generic* solutions will be represented explicitly.

Algorithm 1 IsPrimitive

Input: T , a regular chain of $\mathbf{k}[x_1, \dots, x_n]$.

Output: true if T is primitive, false otherwise.

```
1: if  $|T| = 1$  then
2:    $t \leftarrow$  the defining polynomial of  $T$ 
3:   if  $\text{content}(t, \text{mvar}(t)) \in \mathbf{k}$  then return true else return false
4: else
5:   write  $T$  as  $T' \cup \{t\}$ , where  $t$  has the greatest main variable
6:   if not IsPrimitive( $T'$ ) then
7:     return false
8:   else
9:      $h \leftarrow \text{init}(t)$ ,  $r \leftarrow \text{tail}(t)$ 
10:    for  $U \in \text{Triangularize}(T' \cup \{h\})$  do
11:      if  $\text{ires}(r, U) = 0$  then return false
12:    end for
13:    return true
14:   end if
15: end if
```

- (2) The use of **Triangularize** seems hard to avoid. The purpose is to represent all associated primes of the ideal $\langle T \cup \{h\} \rangle$ by means of regular chains. Geometrically, it is the intersection of the zero set of T with the hypersurface defined by h .
- (3) Algorithm 1 can be optimized using Item (1) of Remark 2: if a coefficient a_i of $t = a_e x^e + \dots + a_0$ is an invertible constant, then lines 10-12 can be skipped since t is strongly primitive.

Proof. We prove the above algorithm **IsPrimitive**. Its termination of the algorithm follows from the fact that in each recursive call the number of polynomials in the input regular chain decreases by 1.

For the correctness, we proceed by induction on the number of polynomials in the regular chain T . When $|T| = 1$, the specification follows from Remark 3. So we assume $|T| > 1$. Definition 3 and Theorem 3 imply that if T is primitive then T' is also primitive. So we assume that T' is primitive and branch to line 9.

Let \mathcal{U} be the output of **Triangularize** in line 10 and let $I = \langle T' \cup \{h\} \rangle$. From the specification of **Triangularize**, we have

$$\bigcap_{U \in \mathcal{U}} \sqrt{\text{sat}(U)} = \sqrt{I}.$$

By Corollary 1, we need to distinguish two cases: h is invertible (resp. not invertible) modulo $\langle T' \rangle = \text{sat}(T')$.

If h is invertible modulo $\langle T' \rangle$ then \mathcal{U} is empty, and the algorithm correctly returns true. Assume from now on that h is not invertible modulo $\langle T' \rangle$. In this case by Lemma 9, the triangular decomposition \mathcal{U} is not empty. So T is primitive if and only if r is regular modulo I . By Lemma 9 again, the ideal I is unmixed and therefore T is primitive if and only if r is regular modulo \sqrt{I} . This holds if and only if r is regular modulo $\text{sat}(U)$ for each $U \in \mathcal{U}$. Finally, the correctness of Algorithm 1 follows from Theorem 1. \square

Example 4. Let $R = \mathbf{k}[z \prec y \prec x]$ be a polynomial ring and $T = \{t_1, t_2\}$ be a regular chain of R with $t_1 = y^5 - z^4, t_2 = zx - y^2$. Clearly, $\{t_1\}$ is a primitive regular chain. Let $I = \langle t_1, \text{lc}(t_2) \rangle = \langle t_1, z \rangle = \langle z, y^5 \rangle$. In Algorithm 1 the call to **Triangularize** will produce $\sqrt{I} = \sqrt{\text{sat}(U)}$ where $U = \{z, y\}$ is a regular chain. Thus, the computation

$$\text{ires}(\text{tail}(t_2), U) = \text{ires}(-y^2, U) = 0$$

implies that $\text{tail}(t_2) = -y^2$ is not regular modulo I . Thus T is not primitive. In fact, the prime ideal $\text{sat}(T) = \langle x^3 - yz, xz - y^2, z^2 - x^2y \rangle$ can not be generated by only two polynomials (see Şahin, 2002, page 43). Hence, in *any* variable ordering, one cannot find a primitive regular chain C such that $\langle C \rangle = \text{sat}(T)$.

7. An Application to Inclusion Test

A fundamental problem in the theory of regular chains is the inclusion test for saturated ideals, that is, deciding if $\text{sat}(T) \subseteq \text{sat}(U)$ holds for two regular chains T and U . For a regular chain T , denote by $\text{mvar}(T)$ the set of main variables of polynomials in T , which is also called the set of algebraic variables of T . In this section, we first show that when T and U share the same set of algebraic variables the inclusion test is simple. Then we point out that the notion of primitivity presented in this paper solves the inclusion test problem partially.

Lemma 10. Let T and U be two regular chains. If $\text{sat}(T) \subseteq \text{sat}(U)$ and $|T| = |U|$ hold, then each associated prime of $\text{sat}(U)$ is also an associated prime of $\text{sat}(T)$.

Proof. Let \mathcal{T} and \mathcal{U} be the set of associated primes of $\text{sat}(T)$ and $\text{sat}(U)$ respectively. Then we have

$$\sqrt{\text{sat}(T)} = \bigcap_{P \in \mathcal{T}} P \quad \text{and} \quad \sqrt{\text{sat}(U)} = \bigcap_{Q \in \mathcal{U}} Q.$$

Since $\text{sat}(T) \subseteq \text{sat}(U)$ implies $\sqrt{\text{sat}(T)} \subseteq \sqrt{\text{sat}(U)}$, for each $Q \in \mathcal{U}$ there exists $P \in \mathcal{T}$ such that $P \subseteq Q$. Since T and U are unmixed with same height, $\dim(P)$ equals $\dim(Q)$, which implies $Q = P$. Hence \mathcal{U} is a subset of \mathcal{T} . \square

Proposition 7. Let T and U be two regular chains with the same set of algebraic variables. Write T as $T = T' \cup \{t\}$ with t having largest main variable. Then $\text{sat}(T) \subseteq \text{sat}(U)$ if and only if $\text{sat}(T') \subseteq \text{sat}(U)$ and $\text{prem}(t, U) = 0$.

Proof. Clearly, we only need to show that $\text{sat}(T) \subseteq \text{sat}(U)$ holds if $\text{sat}(T') \subseteq \text{sat}(U)$ and $\text{prem}(t, U) = 0$.

Denote by h the initial of t . We first prove that h is regular modulo $\text{sat}(U)$. Since h is regular modulo $\text{sat}(T')$, h is not contained in any associated prime of $\text{sat}(T')$. Let u be the polynomial in U such that $\text{mvar}(t) = \text{mvar}(u)$ and define $U' = U \setminus \{u\}$. Then we have $\text{sat}(T') \subseteq \text{sat}(U')$. By Lemma 10, h is not contained in any associated prime of $\text{sat}(U')$. Hence h is regular modulo $\text{sat}(U')$. It follows that h is regular modulo $\text{sat}(U)$ since the main variable of h is smaller than that of u .

For arbitrary $f \in \text{sat}(T)$, we have $\text{prem}(f, t) \in \text{sat}(T') \subseteq \text{sat}(U)$. By the pseudo-division formula, $h^e f = \text{prem}(f, t) + qt$ for some $e \geq 0$ and some q . Since $\text{prem}(t, U) = 0$, we have $t \in \text{sat}(U)$. Therefore $h^e f$ belongs to $\text{sat}(U)$, which implies $f \in \text{sat}(U)$ since h is regular modulo $\text{sat}(U)$. \square

The above proposition handles the case in which two regular chains have the same set of algebraic variables.

Example 5. Let $R = \mathbf{k}[x \prec y \prec z]$ and let $T = \{xz+y\}$ and $U = \{x, y\}$ be regular chains of R . Then $\text{sat}(T) = \langle xz+y \rangle \subsetneq \langle x, y \rangle = \text{sat}(U)$ holds, although we have $\text{mvar}(T) = \{z\}$ and $\text{mvar}(U) = \{x, y\}$.

In practice, the inclusion $\text{sat}(T) \subseteq \text{sat}(U)$ is often established by proving that $\langle T \rangle \subseteq \text{sat}(U)$ holds and that all initials in T are regular modulo $\text{sat}(U)$. This simple criterion follows immediately from the definition of a saturated ideal and Lemma 5.

Now with the notion of primitivity for a regular chain, we have another useful way to detect if $\text{sat}(T) \subseteq \text{sat}(U)$ holds. That is, $\text{sat}(T) \subseteq \text{sat}(U)$ holds whenever $\langle T \rangle \subseteq \text{sat}(U)$ holds and T is primitive. In the above example, the initial of $zx+y$ is not regular modulo $\text{sat}(U)$. However, we know that $\text{sat}(T)$ is contained in $\text{sat}(U)$, since T is primitive and $\langle T \rangle \subseteq \text{sat}(U)$ holds. In the following Section 8, we shall see that algorithm **IsPrimitive** is efficient and primitive regular chains appear quite often in practice.

Corollary 2 below is a direct consequence of Proposition 7, which shows that it is an easy task to check whether two regular chains have the same saturated ideal. Actually, testing $\text{sat}(T) = \text{sat}(U)$ can be done “directly” without testing the inclusions $\text{sat}(T) \subseteq \text{sat}(U)$ and $\text{sat}(U) \subseteq \text{sat}(T)$. The algorithm concluding this section combines together the different criteria reported above for testing the inclusion of saturated ideals. Observe that this algorithm is not always able to check whether the inclusion holds or not.

Corollary 2. Let $T = T \cup \{t\}$ and $U = U' \cup \{u\}$ be two regular chains with t and u having the greatest main variable in T and U respectively. The equality $\text{sat}(T) = \text{sat}(U)$ holds if and only if the following conditions hold

- (1) $\text{sat}(T') = \text{sat}(U')$,
- (2) $\text{mvar}(t) = \text{mvar}(u)$,
- (3) $t \in \text{sat}(U)$ and $u \in \text{sat}(T)$.

Algorithm 2 IsIncluded

Input: T and U , regular chains of $\mathbf{k}[x_1, \dots, x_n]$.

Output: If true (resp. false) is returned then $\text{sat}(T) \subseteq \text{sat}(U)$ holds (resp. does not hold). If failed is returned then the inclusion could not be proved nor disproved.

```

1: if  $T = \emptyset$  then return true end if
2: if  $U = \emptyset$  then return false end if
3: if  $\text{mvar}(T) = \text{mvar}(U)$  then
4:    $v := \max \text{mvar}(T)$ 
5:    $T' := T \setminus \{T_v\}$ 
6:   if IsIncluded( $T', U$ ) and  $\text{prem}(T_v, U) = 0$  then return true end if
7: end if
8: if  $T \subseteq \text{sat}(U)$  then
9:   if  $\text{ires}(\prod_{t \in T} \text{init}(t), U) \neq 0$  then return true end if
10:  if IsPrimitive( $T$ ) then return true end if
11: end if
12: return failed

```

8. Experimentation

We have implemented the algorithm **IsPrimitive** on top of the REGULARCHAINS library in MAPLE (Lemaire *et al.*, 2005). The experimentation, described hereafter, was conducted on well-known problems used in Chen *et al.* (2007)¹, and the tests were performed in MAPLE 11 on an Intel Pentium 4 machine (3.20GHz CPU, 2.0GB memory).

First, we computed their triangular decompositions using the **Triangularize** command in the sense of Kalkbrener. Then, we applied the **IsPrimitive** algorithm to each regular chain in the output.

In Table 1, we summarize the features of the problems and our experimental results. The name of the problems are listed in the first column. The second column gives the number n of variables and the maximal total degree d . For each triangular decomposition (which is a list of regular chains) we record the total running time (in seconds) of **IsPrimitive** in the third column. The last column is the result of mapping **IsPrimitive** to each triangular decomposition: in each of these patterns Y stands for *true* and N for *false*.

These data show that the procedure **IsPrimitive** is efficient in practice. This agrees with the fact that, in Algorithm 1, the input polynomial set in each call to **Triangularize** is rather structured. We also observe that primitive regular chains appear quite often in the output of triangular decompositions.

Table 1. Tests for **IsPrimitive** on 14 examples

System	(n, d)	Time	Pattern
KdV575	(26, 3)	3.525	[Y, Y, Y, Y, Y, Y, Y]
MontesS11	(6, 4)	.001	[Y]
MontesS16	(15, 2)	.103	[Y, Y, Y, N, Y, Y, Y]
Wu-Wang2	(13, 3)	0.099	[Y, N, Y, Y, Y]
MontesS10	(7, 3)	.145	[N]
Lazard2001	(7, 4)	2.314	[Y, Y, Y, N, Y, N]
Lanconelli	(11, 3)	.062	[N, Y]
Wang93	(5, 3)	.142	[N]
Leykin-1	(8, 4)	.228	[Y, Y, Y, Y, Y, Y, Y, Y, N, Y, Y, Y, N, N]
MontesS14	(5, 4)	1.171	[Y, N, N]
MontesS15	(12, 2)	.312	[N]
Maclane	(10, 2)	.157	[Y, Y, N, Y, N]
MontesS12	(8, 2)	.042	[N]
Liu-Lorenz	(5, 2)	1.117	[N, Y]

¹ The defining polynomial systems can be found at <http://www.orcca.on.ca/~panwei/issac08/>

9. Discussion

We have generalized the notion of primitivity from univariate polynomials to regular chains. This has allowed us to establish a necessary and sufficient condition for a regular chain T to generate its saturated ideal $\text{sat}(T)$. Assume that T is not empty and write $T = T' \cup \{p\}$ where p is the polynomial of T with largest main variable. Theorem 3 states that the equality $\langle T \rangle = \text{sat}(T)$ holds whenever $\langle T' \rangle = \text{sat}(T')$ holds and the polynomial p is *weakly primitive* over $\mathbf{k}[\mathbf{x}]/\langle T' \rangle$. This latter property is a generalization of the usual notion of primitivity for polynomials over a UFD.

Examining the proof of Theorem 3, we make the following observation. When p is not weakly primitive over $\mathbf{k}[\mathbf{x}]/\langle T' \rangle$, the proof exhibits a polynomial q which belongs to $\text{sat}(T)$ but not to $\langle T \rangle$. When p is weakly primitive over $\mathbf{k}[\mathbf{x}]/\langle T' \rangle$, the proof shows that every polynomial q of $\text{sat}(T)$ belongs to $\langle T \rangle$. The argument is constructive providing that one has at hand an algorithm for dividing a by b modulo $\langle T' \rangle$, where b is a polynomial regular modulo $\langle T' \rangle$ and is a multiple of the polynomial a modulo $\langle T' \rangle$. This can be done via Gröbner basis computations (see Monagan and Pearce, 2006). An algorithmic solution based on the algorithms of the REGULARCHAINS library is an ongoing research work.

Theorem 3 and its proof do not lead directly to an algorithm for testing the equality $\langle T \rangle = \text{sat}(T)$. Algorithm 1 provides such a decision procedure. This algorithm reduces to testing whether a polynomial is regular modulo an ideal. Fortunately the involved ideal is unmixed which allows us to rely on the algorithms of the REGULARCHAINS library avoiding Gröbner basis computations. Our experimentation illustrates the practical efficiency of Algorithm 1.

Algorithm 1 does not generalize easily in the differential setting. Indeed, consider the polynomial $p = u_x^2 - 4u$ as in (Ritt, 1950, example 1 page 120). We recall hereafter that we have $[u_x^2 - 4u] \subsetneq [u_x^2 - 4u] : \{u_x\}^\infty$. This indicates that even in the case of a single polynomial, the problem is much harder in the differential setting since the case of a single polynomial in the algebraic setting is obvious (line 3 of Algorithm 1). It is obvious to show that $u_{xx} - 2 \in [u_x^2 - 4u] : \{u_x\}^\infty$ since $dp/dx = 2u_x(u_{xx} - 2)$. However $u_{xx} - 2 \notin [u_x^2 - 4u]$ holds for the following reason: the solution $u = 0$ for $[u_x^2 - 4u]$ does not cancel $u_{xx} - 2$ which implies: $u_{xx} - 2 \notin [u_x^2 - 4u]$. Thus, we have; $[u_x^2 - 4u] \subsetneq [u_x^2 - 4u] : \{u_x\}^\infty$

In general a primitive regular chain is *not* a lex Gröbner basis with the same variable ordering. Over the ring $\mathbb{Q}[x \succ y \succ z \succ v \succ u \succ r \succ t]$, the following regular chain T is primitive:

$$T = \begin{cases} v - rt, \\ ztu - 1, \\ yr - t^2u - 1, \\ xu - xr - 2x - u - 2t^4 + 1. \end{cases}$$

The ideal $\text{sat}(T) = \langle T \rangle$ has the reduced lex Gröbner basis G in lex order $x \succ y \succ z \succ$

$v \succ u \succ r \succ t$, which consists of 6 polynomials.

$$G = \begin{cases} v - rt, \\ ztu - 1, \\ yr - t^2u - 1, \\ xu - xr - 2x - u - 2t^4 + 1, \\ rtzx + 1 - zt - x + 2ztx + 2t^5z, \\ ztx + t^2x - yzt - xy + y + 2yzt + 2t^5yz \end{cases}$$

Moreover, in the same variable ordering, its reduced degree reverse lex Gröbner basis (omitted here) consists of 12 polynomials. Clearly, when T is a primitive regular chain, no other generating set of the saturated ideal $\text{sat}(T)$ can have fewer elements than T ; in addition T provides nice algorithmic properties (membership test, regularity test) as Gröbner bases do.

As discussed in Section 7, an application of Algorithm 1 is in the removal of redundant components for triangular decompositions in the sense of Kalkbrener. However, this Algorithm 2 provides only a criterion for removing redundant components. Obtaining a decision algorithm, free of Gröbner basis computations, for testing the inclusion of saturated ideals, remains an open problem.

Another possible research direction is to investigate the relations between primitive regular chains and the minimum number of generators of saturated ideals. For instance, it is natural to ask whether a prime ideal \mathcal{P} of height h can be generated by h elements if and only if there exists a variable ordering and a primitive regular chain C (w.r.t. this variable ordering) such that C generates \mathcal{P} .

References

- J. Arnold and R. Gilmer. On the contents of polynomials. *Proc. Amer. Math. Soc.*, 24:556–562, 1970.
- P. Aubry, D. Lazard, and M. Moreno Maza. On the theories of triangular sets. *J. Symb. Comput.*, 28(1-2):105–124, 1999.
- P. Aubry and M. Moreno Maza. Triangular sets for solving polynomial systems: A comparative implementation of four methods. *J. Symb. Comp.*, 28(1-2):125–154, 1999.
- F. Boulier, F. Lemaire, and M. Moreno Maza. Well known theorems on triangular systems and the D5 Principle. In *Transgressive Computing 2006*, Universidad de Granada, 2006.
- C. Chen, F. Lemaire, O. Golubitsky, M. Moreno Maza, and W. Pan. Comprehensive triangular decomposition. In Proc. of CASC 2007. *Lecture Notes in Computer Science*, volume 4770, pages 73–101. Springer-Verlag, 2007.
- C. Chen, F. Lemaire, M. Moreno Maza, W. Pan, and Y. Xie. Efficient computations of irredundant triangular decompositions with the REGULARCHAINS library. In Proc. of CASA2007. *Lecture Notes in Computer Science*, volume 4488, pages 268–271. Springer-Verlag, 2007.
- S.C. Chou and X.S. Gao. On the dimension of an arbitrary ascending chain. *Chinese Bull. of Sci.*, 38:799–804, 1991.

- T. Coquand, L. Ducos, H. Lombardi, and C. Quitté. L'idéal des coefficients du produit de deux polynômes. *Revue des Mathématiques de l'enseignement Supérieur*, 113(3):25–39, 2003.
- A. Corso, W. V. Vasconcelos, and R. H. Villarreal. On the contents of polynomials. *J. Pure. Appl. Algebra*, 125(1-3):117–127, 1998.
- D. Eisenbud. *Commutative Algebra*. Springer-Verlag, 1994.
- E. Hubert. Notes on triangular sets and triangulation-decomposition algorithms I: Polynomial systems. In *SNSC*, pages 1–39, 2001.
- F. Ischebeck and R. A. Rao. Ideals and Reality, projective modules and number of generators of ideals. Springer-Verlag, 2005.
- M. Kalkbrener. A generalized euclidean algorithm for computing triangular representations of algebraic varieties. *J. Symb. Comp.*, 15:143–167, 1993.
- M. Kalkbrener. Algorithmic properties of polynomial rings. *J. Symb. Comput.*, 26(5):525–581, 1998.
- F. Lemaire, M. Moreno Maza, and Y. Xie. The REGULARCHAINS library in MAPLE 10. In Ilias S. Kotsireas, editor, *Maple Conference 2005*, pages 355–368, 2005.
- F. Lemaire, M. Moreno Maza, W. Pan and Y. Xie. When does (T) equal Sat(T)? *Proc. ISSAC '08*, ACM Press, pages 207–214, 2008.
- M. Monagan and R. Pearce. Rational simplification modulo a polynomial ideal. In *ISSAC '06*, pages 239–245. ACM, 2006.
- M. Moreno Maza. On triangular decompositions of algebraic varieties. Technical Report TR 4/99, NAG Ltd, Oxford, UK, 1999. Presented at the MEGA-2000 Conference, Bath, England. <http://www.csd.uwo.ca/~moreno>.
- M. Moreno Maza and R. Rioboo. Polynomial gcd computations over towers of algebraic extensions. In Proc. AAEECC-11, *Lecture Notes in Computer Science*, volume 948, pages 365–382. Springer, 1995.
- Joseph Fels Ritt. *Differential Algebra*. Dover Publications Inc., New York, 1950. Available at http://www.ams.org/online_bks/coll133.
- M. Şahin. On the minimal number of elements generating an algebraic set. Master thesis, Bilkent University, 2002.
- B. Sturmfels. Solving Systems of Polynomial Equations. *Amer. Math. Soc.*, 2002.
- D. Wang. Computing triangular systems and regular systems. *J. Symb. Comput.*, 30(2):221–236, 2000.
- W.T. Wu. On zeros of algebraic equations – an application of Ritt principle. *Kexue Tongbao*, 31(1):1–5, 1986.
- L. Yang and J. Zhang. Searching dependency between algebraic equations: an algorithm applied to automated reasoning. Technical Report IC/91/6, International Atomic Energy Agency, Miramare, Trieste, Italy, 1991.