The Complete Root Classification of a Parametric Polynomial on an Interval

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ABSTRACT

Given a real parametric polynomial $p(x)$ and an interval $(a, b) \subset \mathbb{R}$, the Complete Root Classification (CRC) of $p(x)$ on (a, b) is a collection of all possible cases of its root classification on (a, b), together with the conditions its coefficients must satisfy for each case. In this paper, a new algorithm is proposed for the automatic computation of the complete root classification of a parametric polynomial on an interval. As a direct application, the new algorithm is applied to some real quantifier elimination problems.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*

General Terms

Algorithms

Keywords

Complete root classification, real root, parametric polynomial, interval, real quantifier elimination

1. INTRODUCTION

The counting and classifying of the roots of a polynomial have been the subject of many investigations. This paper concerns the complete root classification of a parametric polynomial on an interval.

RC and CRC. Let $p(x)$ be a real polynomial with constant coefficients. The *root classification (RC) of* $p(x)$ *on* $\mathbb{R}$ is denoted by

$$[L_1, L_2] = [[n_1, n_2, \ldots], [m_1, -m_1, m_2, -m_2, \ldots]],$$

where n_k are the multiplicities of the distinct real roots of $p(x)$ on $\mathbb{R}$, and m_k are the multiplicities of the distinct complex conjugate pairs of $p(x)$, and $L_1 = [n_1, n_2, \ldots]$ is called the *real RC of* $p(x)$ *on* $\mathbb{R}$. Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. The *RC of* $p(x)$ *on* (a, b) is denoted by a list $L = [n_1, n_2, \ldots]$, where $n_1, n_2, \ldots$ are the multiplicities of the distinct real roots of $p(x)$ on (a, b). For a real polynomial $p(x)$ with parametric coefficients, the *complete root classification (CRC) of* $p(x)$ *on* (a, b) is a collection of all possible cases of its RC on (a, b), together with the conditions its coefficients must satisfy for each case.

The history of CRC is short. The CRC of a real parametric quartic polynomial on $\mathbb{R}$ was found by Arnon in 1988 [2]; the first method for establishing the CRC of a real parametric polynomial of any degree on $\mathbb{R}$ was given by Yang, Hou and Zeng in 1996 [10]. They illustrated their method by computing the CRC of a reduced sextic polynomial. The first automatic generation of CRCs was described and implemented by Liang and Zhang [7], with some improvements added in [5]. Further improvements to the algorithm were made in [6] by replacing the 'revised sign lists' (Definition 4 below) with the direct use of 'sign lists'. As well as offering greater efficiency, the new algorithm offers a better filter for eliminating non-realizable conditions.

All works above are on $\mathbb{R}$, and applications often need CRC on an interval. For example, in robust control [1] and problems concerning program termination [12], we have to determine the conditions on the parametric coefficients of $p(x)$ such that $\forall\, x > 0, p(x) > 0$, or the conditions such that $\forall\, x \in (a, b), p(x) \neq 0$. Therefore, it is meaningful to develop an algorithm for computing the CRC of a parametric polynomial on an interval.

However, in order to develop such an algorithm, we have to face two challenging problems. The first problem is the determination of the conditions for a parametric polynomial having a given number of real roots on an interval. One naturally thinks of the well-known Sturm sequence. The Sturm sequence of a polynomial with known, constant coefficients is a good tool for computing the number of real roots on an interval, but it is inconvenient and inefficient when the given polynomial has parametric coefficients. A better solution uses the fact that we know how to determine the conditions for a polynomial having a given number of real roots on $\mathbb{R}$ [6], and converts the problem of the determination of conditions on an interval into a problem on $\mathbb{R}$. This is done in Section 3, where Theorem 4 is given. Let $p \in \mathbb{R}[x]$ with $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $a_n \neq 0$. Let $a, b \in \mathbb{R}$ such that $p(a) \neq 0$ and $p(b) \neq 0$. Let $\Psi_1(x) = (1 - x)^n p\left(\frac{b-ax}{1-x}\right)$ and $\Psi_2(x) = \Psi_1(-x^2) =$

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$(1+x^2)^n p\left(\frac{b+ax^2}{1+x^2}\right)$. Then $L=[r_1,r_2,\ldots,r_k]$ is the RC of $p(x)$ on (a,b), if and only if $L_2=[r_1,r_1,r_2,r_2,\ldots,r_k,r_k]$ is the real RC of $\Psi_2(x)$ on $\mathbb{R}$. Therefore, the conditions for $p(x)$ having L as its RC on an interval can be obtained by computing the conditions for $\Psi_2(x)$ having L_2 as its real RC on $\mathbb{R}$.

The second problem is the computation of the Δ-sequence of Ψ_2 (Definition 5). We try to determine the conditions for Ψ_2 having L_2 as its real RC on $\mathbb{R}$. The set of all possible sign lists of Ψ_2 can be determined by Theorem 2 and 3. Now, in order to make the multiplicities of the $2k$ distinct real roots of Ψ_2 be $r_1,r_1,r_2,r_2,\ldots,r_k,r_k$ respectively, we also have to determine the possible sign lists of the polynomials in the $\Delta-$sequence of Ψ_2. According to Proposition 1, $\Delta^1(\Psi_2)$ can be determined by the maximal index ℓ of non-vanishing members in the sign list of Ψ_2 which actually is the total number of distinct (real and complex) roots of Ψ_2. Since L_2 does not contain information about the number of distinct complex-conjugate roots of Ψ_2, the maximal index ℓ is not uniquely determined. Therefore, unlike the case of RC on $\mathbb{R}$ [6], there may be more than one $\Delta^1(\Psi_2)$ for the real RC L_2, and consequently the conditions for $\Psi_2(x)$ having L_2 as its real RC on $\mathbb{R}$ would be more complicated. So the question is how to determine these $\Delta^1(\Psi_2)$ and corresponding conditions.

In this paper, a new algorithm for the automatic computation of the CRC of a parametric polynomial on an interval is proposed. The new algorithm has been implemented in MAPLE. As an immediate application, the new algorithm has been applied to some real quantifier elimination problems. However, it should be emphasized that the CRC of a parametric polynomial on an interval contains more information than is needed for these problems, and consequently it has more potential applications than the examples given here.

2. PRELIMINARY

In this section, we review some definitions and theorems which mainly come from [10] and [6]. They are necessary for the new algorithm. Let $p(x)\in\mathbb{R}[x]$ with $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ and $a_n\neq 0$.

DEFINITION 1. *The $2n\times 2n$ matrix M*

$$\begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \ldots & a_0 & & & \\ 0 & na_n & (n-1)a_{n-1} & \ldots & a_1 & & & \\ & a_n & a_{n-1} & \ldots & a_1 & a_0 & & \\ & 0 & na_n & \ldots & 2a_2 & a_1 & & \\ & & & \ldots & \ldots & & & \\ & & & \ldots & \ldots & & & \\ & & & & a_n & a_{n-1} & \ldots & a_0 \\ & & & & 0 & na_n & \ldots & a_1 \end{pmatrix}$$

is called the discrimination matrix of p.

DEFINITION 2. *For $1\le k\le 2n$, let M_k be the kth principal minor of M, and let $D_k=M_{2k}$. The n-tuple $D=[D_1,D_2,\ldots,D_n]$ is called the discriminant sequence of p.*

DEFINITION 3. *If $\operatorname{sgn} x$ is the signum function, $\operatorname{sgn} 0=0$, then the list $[s_1,s_2,\ldots,s_n]=[\operatorname{sgn} D_1,\operatorname{sgn} D_2,\ldots,\operatorname{sgn} D_n]$ is called the sign list of p.*

DEFINITION 4. *The revised sign list $[e_1,e_2,\ldots,e_n]$ of $p(x)$ is constructed from the sign list $s=[s_1,s_2,\ldots,s_n]$ of p as follows. If $[s_i,s_{i+1},\ldots,s_{i+j}]$ is a section of s, where $s_i\neq 0$, $s_{i+1}=s_{i+2}=\ldots=s_{i+j-1}=0$ and $s_{i+j}\neq 0$, then we replace the subsection $[s_{i+1},\ldots,s_{i+j-1}]$ by*

$$[-s_i,-s_i,s_i,s_i,-s_i,-s_i,s_i,s_i,\ldots]\,,$$

i.e., let $e_{i+r}=(-1)^{\lfloor (r+1)/2\rfloor}s_i$, for $r=1,2,\ldots,j-1$, and keep other elements unchanged, i.e., let $e_k=s_k$. The revised sign list of p (resp. s) is denoted by $\operatorname{rsl}(p)$ (resp. $\operatorname{rsl}(s)$).

Yang, Hou and Zeng used the following theorem to calculate the number of distinct complex-conjugate roots and real roots.

THEOREM 1. *Suppose a polynomial $p\in\mathbb{R}[x]$ has revised sign list $\operatorname{rsl}(p)$. If the number of non-vanishing members of $\operatorname{rsl}(p)$ is s, and the number of sign changes in $\operatorname{rsl}(p)$ is v, then $p(x)$ has v pairs of distinct complex-conjugate roots and $s-2v$ distinct real roots.*

In order to calculate the multiplicities of roots, Yang, Hou and Zeng used the following definitions and propositions.

DEFINITION 5. *Let $\Delta(p)$ denote $\gcd(p(x),p'(x))$, and let $\Delta^0(p)=p(x)$, $\Delta^j(p)=\Delta(\Delta^{j-1}(p))$, $j=1,2,\ldots$. Then $\Delta^0(p),\Delta^1(p),\Delta^2(p),\ldots$ is called the Δ-sequence of p.*

PROPOSITION 1. *If $\operatorname{rsl}(p)$ contains k zeros, equivalently, $D_n=\ldots=D_{n-k+1}=0$ but $D_{n-k}\neq 0$, then $\gcd(p,p')=P_k(p,p')$, where $P_k(p,p')$ is the kth subresultant of $p(x)$ and $p'(x)$.*

The relationship between the RC of $\Delta^j(p)$ and the RC of its 'repeated part' $\Delta^{j+1}(p)$ is given by the following propositions.

PROPOSITION 2. *If $\Delta^j(p)$ has k distinct roots with respective multiplicities $n_1,n_2,\ldots,n_k$, then $\Delta^{j+1}(p)$ has at most k distinct roots with respective multiplicities $n_1-1,n_2-1,\ldots,n_k-1$.*

PROPOSITION 3. *If $\Delta^j(p)$ has k distinct roots with respective multiplicities $n_1,n_2,\ldots,n_k$, and $\Delta^{j-1}(p)$ has m distinct roots, then $m\ge k$, and the multiplicities of these m distinct roots are $n_1+1,n_2+1\ldots,n_k+1,1,\ldots,1$ respectively.*

However, the old algorithms [5] and the methods above have to work with revised sign list which is a major source of inefficiency, since we have to transfer the output conditions in terms of revised sign lists to conditions in terms of sign lists. The transferring process is usually very difficult and full of opportunities for including non-realizable conditions. This consideration motivated the authors to propose a new algorithm for overcoming these disadvantages [6]. The new algorithm offers improved efficiency and a new test for non-realizable conditions. The improvement lies in the direct use of sign lists, rather than revised sign lists.

The algorithm uses the following definitions and theorems, where "PmV" means "generalized Permanences minus Variations" [3].

DEFINITION 6. *Let $s=[s_n,\ldots,s_0]$ be a finite list of elements in $\mathbb{R}$ such that $s_n\neq 0$. Let $m<n$ such that $s_{n-1}=\cdots=s_{m+1}=0$, and $s_m\neq 0$, and $s'=[s_m,\ldots,s_0]$.*

If there is no such m, then s' is the empty list. We define inductively

$$\mathrm{PmV}(s) = \begin{cases} 0\ , & s' = \emptyset, \\ \mathrm{PmV}(s') + \epsilon_{n-m}\,\mathrm{sgn}(s_n s_m)\ , & n-m \text{ odd}, \\ \mathrm{PmV}(s')\ , & n-m \text{ even}, \end{cases}$$

where $\epsilon_{n-m} = (-1)^{(n-m)(n-m-1)/2}$.

The following theorem gives the number of distinct roots in terms of sign lists.

THEOREM 2. *Let $D = [D_1, \ldots, D_n]$ be the discriminant sequence of a real polynomial $p(x)$ of degree n, and ℓ be the maximal index such that $D_\ell \neq 0$. If $\mathrm{PmV}(D) = r$, then $p(x)$ has $r+1$ distinct real roots and $\frac{1}{2}(\ell - r - 1)$ pairs of distinct complex conjugate roots.*

The next theorem can be used to detect the non-realizable sign lists in output conditions.

THEOREM 3. *Let $S = [s_1, \ldots, s_n]$ and $R = [r_1, \ldots, r_n]$ be the sign list and the revised sign list of $p(x)$ respectively. Then $\mathrm{PmV}(S) = \mathrm{PmV}(R)$.*

At last, we review a result given by Yang and Xia [9][11] for computing the number of real roots on intervals, which gives us some clue for solving the first problem mentioned in Section 1.

Let $p \in \mathbb{R}[x]$ with $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $a_n \neq 0$. Let $a, b \in \mathbb{R}$ such that $p(a) \neq 0$ and $p(b) \neq 0$. Let $\Psi_1(x) = (1-x)^n p\left(\frac{b-ax}{1-x}\right)$ and $\Psi_2(x) = \Psi_1(-x^2) = (1+x^2)^n p\left(\frac{b+ax^2}{1+x^2}\right)$. Then, it is easy to see that $\mathrm{coeff}(\Psi_1, x, n) = (-1)^n p(a) \neq 0$, $\mathrm{coeff}(\Psi_2, x, 2n) = p(a) \neq 0$ and $\Psi_1(0) = \mathrm{coeff}(\Psi_1, x, 0) = \mathrm{coeff}(\Psi_2, x, 0) = p(b) \neq 0$. Furthermore

PROPOSITION 4. $\#\{x \in (a,b) | p(x) = 0\} = \#\{x < 0 | \Psi_1(x) = 0\} = \frac{1}{2}\#\{x \in \mathbb{R} | \Psi_2(x) = 0\}$.

3. BASIS OF THE ALGORITHM

In this section, we establish the basis for the new algorithm. The main idea is that we transfer the computation of CRC for a parametric polynomial on an interval to the computation of CRC for a parametric polynomial on $\mathbb{R}$.

THEOREM 4. *Let $p(x), \Psi_1(x), \Psi_2(x)$ be defined as in Section 2. Then, $[r_1, r_2, \ldots, r_k]$ is the RC of $p(x)$ on (a,b), if and only if $[r_1, r_1, r_2, r_2, \ldots, r_k, r_k]$ is the real RC of $\Psi_2(x)$ on $\mathbb{R}$.*

PROOF. Since $[r_1, r_2, \ldots, r_k]$ is the RC of $p(x)$ on (a,b), we can decompose $p(x)$ in $\mathbb{C}$ as

$$p(x) = a_n \prod_{i=1}^{k}(x-\alpha_i)^{r_i} \prod_{j=1}^{s}(x-\beta_j)^{f_j} \prod_{u=1}^{t}(x-\gamma_u)^{h_u}(x-\overline{\gamma_u})^{h_u},$$

where, $\alpha_i (i = 1, \ldots, k), \beta_j (j = 1, \ldots, s) \in \mathbb{R}$ are all distinct real roots of $p(x)$, $\gamma_u, \overline{\gamma_u} (u = 1, \ldots, t) \in \mathbb{C}$ are all pairs of distinct complex-conjugate roots of $p(x)$, and $a < \alpha_i < b$, $\beta_j \notin [a,b]$. Then, since a, b are not roots of $p(x)$, we have $\Psi_1(x) =$

$$a_n \prod_{i=1}^{k}(\alpha_i - a)^{r_i}\left(x - \frac{\alpha_i - b}{\alpha_i - a}\right)^{r_i} \prod_{j=1}^{s}(\beta_j - a)^{f_j}\left(x - \frac{\beta_j - b}{\beta_j - a}\right)^{f_j} \cdot$$

$$\prod_{u=1}^{t}(\gamma_u - a)^{h_u}\left(x - \frac{\gamma_u - b}{\gamma_u - a}\right)^{h_u}(\overline{\gamma_u} - a)^{h_u}\left(x - \frac{\overline{\gamma_u} - b}{\overline{\gamma_u} - a}\right)^{h_u}.$$

Notice that $\frac{\alpha_i - b}{\alpha_i - a} \neq \frac{\alpha_j - b}{\alpha_j - a} (i \neq j)$, $\frac{\alpha_i - b}{\alpha_i - a} < 0$, $\frac{\beta_j - b}{\beta_j - a} > 0$ and $\frac{\gamma_u - b}{\gamma_u - a}, \frac{\overline{\gamma_u} - b}{\overline{\gamma_u} - a} \notin \mathbb{R}$ because of $\alpha_i \neq \alpha_j$, $a < \alpha_i < b$, $\beta_j \notin [a,b]$ and $a \neq b$ respectively.

Based on the facts above, we can conclude that $\Psi_1(x)$ in $(-\infty, 0)$ has exactly k distinct real roots of respective multiplicities $r_1, \ldots, r_k$. Therefore, the RC of $\Psi_1(x)$ on $(-\infty, 0)$ is $[r_1, r_2, \ldots, r_k]$, if and only if the RC of $p(x)$ on (a,b) is $[r_1, r_2, \ldots, r_k]$. Noticing $\Psi_1(0) = p(b) \neq 0$, by similar discussion, we can conclude that the real RC of $\Psi_2(x)$ on $\mathbb{R}$ is $[r_1, r_1, r_2, r_2, \ldots, r_k, r_k]$, if and only if the RC of $\Psi_1(x)$ on $(-\infty, 0)$ is $[r_1, r_2, \ldots, r_k]$. Finally, the desired result follows. $\square$

From Theorem 4, for intervals $(-\infty, a)$ and $(a, +\infty)$, we have the following corollary.

COROLLARY 1. *Let notations be as in Theorem 4. Then, $[r_1, r_2, \ldots, r_k]$ is the RC of $p(x)$ on $(-\infty, a)$ iff $[r_1, r_1, r_2, r_2, \ldots, r_k, r_k]$ is the real RC of $\Psi_2(x) := p(-x^2 + a)$ on $\mathbb{R}$; $[r_1, r_2, \ldots, r_k]$ is the RC of $p(x)$ on $(a, +\infty)$ iff $[r_1, r_1, r_2, r_2, \ldots, r_k, r_k]$ is the real RC of $\Psi_2(x) := p(x^2 + a)$ on $\mathbb{R}$.*

Based on Theorem 4 and Corollary 1, the conditions for $p(x)$ having $[r_1, r_2, \ldots, r_k]$ as its RC on an interval can be obtained by computing the conditions for $\Psi_2(x)$ having $[r_1, r_1, r_2, r_2, \ldots, r_k, r_k]$ as its real RC on $\mathbb{R}$.

4. THE ALGORITHM

Let $p(x), \Psi_2(x)$ be defined as in Section 2. In this section, we propose an algorithm for computing the CRC of $p(x)$ on (a,b), where $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. We need some functions to present the algorithm.

- ClassifySL: Input a set of sign lists and a positive integer i; output the subset of sign lists of which the maximal index of non-vanishing members is i. For example, if the input is $S = \{[1,1,1,-1,0,0], [1,0,0,-1,0,0], [1,1,1,-1,1,1], [1,-1,0,0,1,1]\}$, then $\mathrm{ClassifySL}(S,4) = \{[1,1,1,-1,0,0], [1,0,0,-1,0,0]\}$, $\mathrm{ClassifySL}(S,6) = \{[1,1,1,-1,1,1], [1,-1,0,0,1,1]\}$, but $\mathrm{ClassifySL}(S,1) = \mathrm{ClassifySL}(S,2) = \mathrm{ClassifySL}(S,3) = \mathrm{ClassifySL}(S,5) = \emptyset$.
- MinusOne: Input an RC $[r_1, \ldots, r_k]$; output $[r_1 - 1, \ldots, r_k - 1]$ (all elements with value 0 are removed).
- Op: Input a set $\{a_1, \ldots, a_n\}$ or a list $[a_1, \ldots, a_n]$; output the sequence $a_1, \ldots, a_n$.
- DRC: input an RC $[r_1, \ldots, r_k]$; output $[r_1, r_1, \ldots, r_k, r_k]$.
- AllRC: Input $n \in \mathbb{N}$; output the union of the sets of partitions of $0, 1, \ldots, n$.

By Theorem 4 and Corollary 1, the conditions for p having $L = [r_1, r_2, \ldots, r_k]$ as its RC on (a,b) can be obtained by computing the conditions for Ψ_2 having $L_2 = [r_1, r_1, r_2, r_2, \ldots, r_k, r_k]$ as its real RC on $\mathbb{R}$. So, what we have to do is to find the conditions for Ψ_2 having L_2 as its real RC on $\mathbb{R}$.

We first compute all possible sign lists of Ψ_2 for Ψ_2 having L_2 as its real RC on $\mathbb{R}$.

PolySL.
Input: Ψ_2 and L_2.
Output: The set of all possible sign lists of Ψ_2.
Procedure:

- Compute the discriminant sequence $D = [D_1, \ldots, D_{2n}]$ of Ψ_2.
- Compute the set S_0 of all possible sign lists from D: for $1 \le k \le 2n$, if $D_k \in \mathbb{R}$, then $D_k \to \mathrm{sgn}(D_k)$; otherwise, $D_k \to \{-1, 0, 1\}$. For example, if $D = [1, -2, a]$, then $S_0 = \{[1, -1, -1], [1, -1, 0], [1, -1, 1]\}$.
- Compute $S = \{s \in S_0 \,|\, \mathrm{PmV}(s) = \mathrm{PmV}(\mathrm{rsl}(s)) = 2k - 1\}$, where $\#L_2 = 2k$.
- Return S.

Then $S = \mathrm{PolySL}(\Psi_2, L_2)$ is the set of all possible sign lists of Ψ_2 for Ψ_2 having L_2 as its real RC on $\mathbb{R}$. In order to make the multiplicities of the $2k$ distinct real roots of Ψ_2 be $r_1, r_1, r_2, r_2, \ldots, r_k, r_k$ respectively, we also have to determine the possible sign lists of the polynomials in the $\Delta-$sequence of Ψ_2. According to Proposition 1, $\Delta^1(\Psi_2)$ can be determined by the maximal index ℓ of non-vanishing members in the sign list of Ψ_2, which actually is the total number of distinct (real and complex) roots of Ψ_2. Since L_2 does not contain information about the number of distinct complex roots of Ψ_2, the maximal index ℓ is not uniquely determined. Therefore, unlike the case of RC on $\mathbb{R}$ [6], there may be more than one $\Delta^1(\Psi_2)$ for the real RC L_2, and consequently the conditions for $\Psi_2(x)$ having L_2 as its real RC on $\mathbb{R}$ would be more complicated. The following observations can be used to solve the problem.

Let $G_i = \mathrm{ClassifySL}(S, i)$. Then $S = \bigcup_{i=k}^{n} G_{2i}$ because Ψ_2 has at least $2k$ distinct roots and, by Theorem 4 and its proof, the number of distinct roots of Ψ_2 is always a multiple of 2. For $k \le i \le n$, if the sign list of Ψ_2 belongs to G_{2i}, then the number of distinct roots of Ψ_2 is $2i$. So by Proposition 1, $\Delta^1(\Psi_2) = P_{2n-2i}(\Psi_2, \Psi_2')$. It is a polynomial of degree $2n - 2i$. By Proposition 2, the real RC of $\Delta^1(\Psi_2)$ on $\mathbb{R}$ is $\mathrm{MinusOne}(L_2)$. Then the computation above can be repeated for $\Delta^1(\Psi_2)$ and $\mathrm{MinusOne}(L_2)$ until the termination conditions (see below) are reached. Therefore, unlike the case of RC on $\mathbb{R}$, we have to divide all possible sign lists of Ψ_2 into different groups. For each group, there is a distinct $\Delta^1(\Psi_2)$.

For the following three *termination conditions*, we can determine just by S that the real RC of Ψ_2 on $\mathbb{R}$ is L_2 without further computation for the sign lists of other polynomials in the Δ-sequence of Ψ_2.

1. $k = n$,
2. $k = 0$,
3. the last entry of the sign list of Ψ_2 is nonzero.

Therefore, the algorithm for generating the necessary and sufficient conditions for $p(x)$ having $L = [r_1, r_2, \ldots, r_k]$ as its RC on (a, b) is as follows. Let Ψ_2 and L_2 be defined as above. The output conditions are a sequence of *mixed lists*. Each mixed list consists of a polynomial in the Δ-sequence of Ψ_2, followed by all of its possible sign lists. We denote the empty sequence by NULL. Notice that if the empty sequence is returned, then L_2, and consequently L are not realizable.

IntCond
Input: Ψ_2 and a real RC L_2.
Output: A sequence of mixed lists (the conditions for Ψ_2 having L_2 as its real RC on $\mathbb{R}$).
Procedure:

```
t ← #L2
m ← deg(Ψ2)
S ← PolySL(Ψ2, L2)
if S = ∅ then
    return NULL
elif t = 0 or t = m or ClassifySL(S, m) = S then
    return [Ψ2, Op(S)]    (*)
else
    for i from t to m by 2 do
        Gi ← ClassifySL(S, i)
        if Gi ≠ ∅ then
            if i = m then
                return [Ψ2, Op(Gi)]    (**)
            else
                Ci ← IntCond(Δ^1(Ψ2), MinusOne(L2))
                if Ci ≠ NULL then
                    return [Ψ2, Op(Gi)], Ci
                else
                    return NULL
```

Remark 1. Notice that $\Delta^1(\Psi_2)$ depends on i. By Proposition 1, it is a polynomial of degree $2n - i$. In the algorithm, t is a nonnegative integer representing the number of distinct real roots of Ψ_2. m is a positive integer representing the degree of Ψ_2. S is a set of sign lists representing all possible sign lists of Ψ_2 for Ψ_2 having L_2 as its real RC on $\mathbb{R}$. G_i is the subset of S such that the maximal index of non-vanishing members in the sign lists of G_i is i. Finally, C_i is a sequence of mixed lists representing the conditions for $\Delta^1(\Psi_2)$ having $\mathrm{MinusOne}(L_2)$ as its real RC on $\mathbb{R}$.

Proof. First, observe that the number of recursions in the algorithm is finite. So the algorithm will terminate in finite steps.

Second, the output conditions can be viewed as a couple of trees whose nodes consist of mixed lists. Their roots are those mixed lists of which the first member is Ψ_2, and their leaves are the mixed lists at points (*) or (**) in the algorithm.

For the case of RC on $\mathbb{R}$ [6], the necessary and sufficient conditions for a polynomial having a given RC is a single chain of mixed lists connected using conjunction ($\wedge$). Now suppose the leaves are labeled by i ($i = 1, \ldots, N$). Let Θ_i be the chain of mixed lists obtained by connecting the nodes from the ith leaf to its root using conjunction. Then, the output conditions can be expressed as the disjunction of these chains, $\bigvee_{i=1}^{N} \Theta_i$.

Using Theorem 2 and Propositon 2 and 3, as we did in [6], we can prove that $\bigvee_{i=1}^{N} \Theta_i$ is the necessary and sufficient conditions for Ψ_2 having L_2 as its real RC on $\mathbb{R}$.

Finally, by Theorem 4, the correctness of IntCond follows. $\square$

Based on the algorithm IntCond above, we can now propose an algorithm for generating the CRC of a parametric polynomial $p(x)$ on an interval (a, b).

IntCRC
Input: A real parametric polynomial $p(x)$, and $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$.
Output: The CRC of $p(x)$ on (a, b).
Procedure:

$\mathcal{L} \leftarrow$ AllRC(deg(p))
compute Ψ_2
for L in $\mathcal{L}$ do
    $C \leftarrow$ IntCond(Ψ_2, DRC(L))
    if $C \neq$ NULL then
        return L and C

Optimization of algorithm. Finally we discuss the optimization of the algorithm. In comparison with the case on $\mathbb{R}$, the output conditions of the CRC of a parametric polynomial on an interval is usually large, especially when the parametric polynomial has a general form. So there remains the work of condensing the output conditions. Suppose $[D_1, \ldots, D_{2n}]$ is the discriminant sequence of Ψ_2, and S is the set of all possible sign lists of Ψ_2 for Ψ_2 having $L_2 = [r_1, r_1, \ldots, r_k, r_k]$ as its real RC on $\mathbb{R}$. Notice that the length of any sign list in S is $2n$ and the first element of it can always be assumed to be 1.

For $S_1, S_2, S_3 \in S$, if there is an index i such that for any index $j \neq i$, we have $S_1[j] = S_2[j] = S_3[j]$ and $\{S_1[i], S_2[i], S_3[i]\} = \{-1, 0, 1\}$, then S_1, S_2, S_3 can be represented by a single list $[\,S_1[1], \ldots, S_1[i-1], *, S_1[i+1], \ldots, S_1[2n]\,]$, where '$*$' means that D_i is unconstrained.

For $S_1, S_2 \in S$, if there is an index i such that for any index $j \neq i$, we have $S_1[j] = S_2[j]$ and $\{S_1[i], S_2[i]\} = \{0, 1\}$, then S_1, S_2 can be represented by a single list $[\,S_1[1], \ldots, S_1[i-1], 0+, S_1[i+1], \ldots, S_1[2n]\,]$, where '0+' means that $D_i \geq 0$.

Similarly, for $S_1, S_2 \in S$, if there is an index i such that for any index $j \neq i$, we have $S_1[j] = S_2[j]$ and $\{S_1[i], S_2[i]\} = \{0, -1\}$, then S_1, S_2 can be represented by a single list $[\,S_1[1], \ldots, S_1[i-1], 0-, S_1[i+1], \ldots, S_1[2n]\,]$, where '0−' means that $D_i \leq 0$.

At last, for $S_1, S_2 \in S$, if there is an index i such that for any index $j \neq i$, we have $S_1[j] = S_2[j]$ and $\{S_1[i], S_2[i]\} = \{1, -1\}$, then S_1, S_2 can be represented by a single list $[\,S_1[1], \ldots, S_1[i-1], <>, S_1[i+1], \ldots, S_1[2n]\,]$, where '<>' means that $D_i \neq 0$.

In this way, S can be condensed into a smaller set S'. For the sake of simplicity, we still call the elements in S' sign lists. For example, if $S = \{[1,0,1,1], [1,0,0,1], [1,0,-1,1]\}$, then $S' = \{[1,0,*,1]\}$.

Complexity analysis. We give an upper bound for the running time of the algorithm **IntCond**. We count the number of operations $T(n)$ on the coefficients, regarding all polynomials as univariate in x. We recall that the input polynomial Ψ_2 has degree $2n$. We first estimate the call $S = \text{PolySL}(\Psi_2, L_2)$ and make three observations: (1) computing the discriminant sequence D of Ψ_2 can be done in $O(n^2)$ coefficient operations, (2) computing the set S_0 of all possible sign lists from D requires $O(3^{2n})$ operations on coefficients (and integers), (3) for each $s \in S_0$, checking whether s belongs to the output set S can be done in linear time (w.r.t.n), and there are 3^{2n} sign lists in S_0. So computing S requires $O(n3^{2n})$ operations

Consequently the call $S = \text{PolySL}(\Psi_2, L_2)$ runs in $O(n3^{2n})$ operations on coefficients. Returning to the body of the algorithm **IntCond**, we observe that all the calls ClassifySL(S, i) together have a linear cost w.r.t. the size of S, hence these costs can be absorbed in the cost for creating S. The remaining costs in the body of **IntCond** are (at most) $n-1$ recursive calls with respective costs $T(n-1), T(n-2), \ldots, T(1)$. Therefore, we have

$$T(n) \leq O(n3^{2n}) + T(n-1) + T(n-2) + \ldots + T(2) + T(1).$$

Unrolling the recurrence, performing elementary computations we obtain: $T(n) \in O(n9^n)$.

5. EXAMPLES

In this section, we present some examples of the CRCs for some parametric polynomials on intervals. As a direct application, we show how to apply the CRC to questions concerning real quantifier elimination. As with many symbolic computations, solving general cases of high degree polynomials usually would result in very large solutions. Therefore, we only focus on sparse parametric polynomials. All computations were performed with Maple 10 running on a 1.6 GHz Pentium CPU. Notice that, in the output conditions below, different chains of mixed lists (see the proof of **IntCond**) are separated by semicolons (;).

Example 1. The following is the CRC of $p_3 = x^3 + ax + b$ on $(0, 2)$, where the initial conditions correspond to $p_3(0) \neq 0$ and $p_3(2) \neq 0$. It takes 1.18 seconds to generate the CRC. After the output, there is an explanation of its interpretation.

```
(*) p3:=x^3+a*x+b
The CRC of p3 on (0,2) is:
([polynomial,its all possible sign lists])

(1) [ ], if and only if [P6,[1,-1,0,0,<>,-1],
[1,-1,-1,1,0,0],[1,-1,-1,0+,0,-1],[1,-1,-1,1,-1,-1],
[1,-1,-1,*,1,-1],[1,0,0,-1,1,-1],[1,1,1,-1,1,-1]]

(2) [1], if and only if [P6,[1,1,1,-1,0,0],
[1,0,0,-1,0,0],[1,-1,-1,-1,0,0]],[P62,[1,-1]];
[P6,[1,1,1,-1,1,1],[1,-1,0,0,1,1],[1,0-,-1,-1,1,1],
[1,1,1,<>,-1,1],[1,1,1,0,0,1],[1,-1,-1,1,1,1],
[1,-1,-1,0,0,1],[1,-1,-1,-1,-1,1],[1,0,0,-1,<>,1]]

(3) [2], if and only if [P6,[1,1,1,-1,0,0],
[1,0,0,-1,0,0],[1,-1,-1,-1,0,0]], [P62, [1,1]]

(4) [1,1], if and only if [P6,[1,-1,-1,-1,-1,-1],
[1,1,1,-1,-1,-1],[1,0,0,-1,-1,-1]],
where,
P6:=8+2*a+4*a*x^2+2*a*x^4+b+3*b*x^2+3*b*x^4+b*x^6,
P62:=b*a^2*x^2+9*b^2*x^2+2*a^3*x^2+6*b*a*x^2
        +b*a^2+8*a^2+2*a^3.
And the initial conditions are
 b <> 0, 8+2*a+b <> 0
```

Let us explain the CRC of p_3 on $(0, 2)$. We assume that the initial conditions $b \neq 0$ and $2a + b + 8 \neq 0$ hold. All possible cases of RC for p_3 on $(0, 2)$ are $[\,], [1], [2], [1,1], [3], [1,2], [1,1,1]$.

The conditions for p_3 having $[1]$ (one single root) as its RC in $(0, 2)$ can be obtained by computing the conditions for P_6 (which is Ψ_2 in the algorithm) having $[1, 1]$ as its real RC on $\mathbb{R}$.

All possible sign lists of P_6 would be $[1,1,1,-1,0,0]$, $[1,0,0,-1,0,0]$,$[1,-1,-1,-1,0,0]$, $[1,1,1,-1,1,1]$, $[1,-1,0,0,1,1]$, $[1,0-,-1,-1,1,1]$,$[1,1,1,<>,-1,1]$, $[1,1,1,0,0,1]$, $[1,-1,-1,1,1,1]$, $[1,-1,-1,0,0,1]$, $[1,-1,-1,-1,-1,1]$, $[1,0,0,-1,<>,1]$.

Now these sign lists of P_6 can be divided into two groups: $G_4 = \{[1,1,1,-1,0,0],[1,0,0,-1,0,0],[1,-1,-1,-1,0,0]\}$ and $G_6 = \{[1,1,1,-1,1,1],[1,-1,0,0,1,1],[1,0-,-1,-1,1,1],[1,1,1,<>,-1,1],[1,1,1,0,0,1],[1,-1,-1,1,1,1],[1,-1,-1,0,0,1],[1,-1,-1,-1,-1,1],[1,0,0,-1,<>,1]\}$.

If the sign list of P_6 belongs to G_4, then the number of distinct roots of P_6 is 4. So the 'repeated part' $\Delta^1(P_6) = P_{62}$ and the RC of P_{62} is MinusOne($[1,1]$) = $[\,]$. For P_{62} and $[\,]$, IntCond is called again, obtaining that the condition for P_{62} having $[\,]$ as its real RC on $\mathbb{R}$ is its sign list being $[1,-1]$.

At this point, the termination condition 2 is satisfied, so IntCond terminates. If the sign list of P_6 belongs to G_6, then the termination condition 3 is satisfied, and IntCond terminates.

In summary, p_3 has $[1]$ as its RC on $(0,2)$, if and only if the sign list of P_6 belongs to G_4 and the sign list of P_{62} is $[1,-1]$, or the sign list of P_6 belongs to G_6. The cases $[\,], [2]$ and $[1,1]$ can be explained similarly.

For the cases $[3], [1,2], [1,1,1]$, since the output of IntCond is the empty sequence NULL, they are not realizable. Based on the CRC of p_3, we can answer some questions concerning real quantifier elimination. The discriminant sequence of P_6 is $[1, D_2, D_3, D_4, D_5, D_6]$, where

$$\begin{aligned}
&D_2 = -3b^2 - 2ab,\\
&D_3 = -a^2b(2a+3b),\\
&D_4 = a^2b(a^2b + 9b^2 + 2a^3 + 6ab),\\
&D_5 = -b(a^2b + 2a^3 + 6ab + 9b^2)(4a^3 + 27b^2),\\
&D_6 = -(8 + 2a + b)b(4a^3 + 27b^2)^2.
\end{aligned}$$

The necessary and sufficient condition for $\forall x \in (0,2)[p_3 \neq 0]$ is that case (1) holds, and case (1) holds iff the sign list of P_6 be one of the following: $[1,-1,0,0,<>,-1]$, $[1,-1,-1,1,0,0],[1,-1,-1,0+,0,-1],[1,-1,-1,1,-1,-1]$, $[1,-1,-1,*,1,-1],[1,0,0,-1,1,-1],[1,1,1,-1,1,-1]$. Therefore, the necessary and sufficient condition for $\forall x \in (0,2)[p_3 \neq 0]$ is $[D_2 < 0 \wedge D_3 = 0 \wedge D_4 = 0 \wedge D_5 \neq 0 \wedge D_6 < 0] \vee [D_2 < 0 \wedge D_3 < 0 \wedge D_4 > 0 \wedge D_5 = 0 \wedge D_6 = 0] \vee [D_2 < 0 \wedge D_3 < 0 \wedge D_4 \geq 0 \wedge D_5 = 0 \wedge D_6 < 0] \vee [D_2 < 0 \wedge D_3 < 0 \wedge D_4 > 0 \wedge D_5 < 0 \wedge D_6 < 0] \vee [D_2 < 0 \wedge D_3 < 0 \wedge D_5 > 0 \wedge D_6 < 0] \vee [D_2 = 0 \wedge D_3 = 0 \wedge D_4 < 0 \wedge D_5 > 0 \wedge D_6 < 0] \vee [D_2 > 0 \wedge D_3 > 0 \wedge D_4 < 0 \wedge D_5 > 0 \wedge D_6 < 0]$.

Similarly, the necessary and sufficient condition for p_3 having two distinct real roots of multiplicities 1 on $(0,2)$ is that case (4) holds. That is $[D_2 < 0 \wedge D_3 < 0 \wedge D_4 < 0 \wedge D_5 < 0 \wedge D_6 < 0] \vee [D_2 > 0 \wedge D_3 > 0 \wedge D_4 < 0 \wedge D_5 < 0 \wedge D_6 < 0] \vee [D_2 = 0 \wedge D_3 = 0 \wedge D_4 < 0 \wedge D_5 < 0 \wedge D_6 < 0]$.

Example 2. The following is the CRC of $p_4 = x^4 + ax + b$ on $(-\infty, -1)$, where the initial condition corresponds to $p_4(-1) \neq 0$. It takes 0.75 second to generate the CRC.

```
(*) p4:=x^4+a*x+b
The CRC of p4 on (-Infinity,-1) is:
([polynomial,its all possible sign lists])

(1) [ ], if and only if
[P8,[1,-1,0,0,0,0,0,0],[1,-1,0,0,-1,-1,-1,1],
     [1,-1,0,0,0,0,<>,1],[1,-1,0,0,<>,-1,0,0],
     [1,-1,0,0,1,*,-1,1],[1,-1,0,0,<>,-1,1,1]]

(2) [1], if and only if
[P8,[1,-1,0,0,0,0,-1,-1],[1,-1,0,0,1,-1,-1,-1],
    [1,-1,0,0,1,0,0,-1],[1,-1,0,0,1,1,1,-1],
    [1,-1,0,0,-1,-1,-1,-1]]

(3) [1,1], if and only if
[P8, [1, -1, 0, 0, 1, 1, 1, 1]]
Where,
(#1) P8:=1+4*x^2+6*x^4+4*x^6+x^8-a-a*x^2+b,
and the initial condition is
 1-a+b <> 0
```

Example 3. Find the conditions on a, b, c such that $\forall\, x > 0, p_5 = x^5 + ax^2 + bx + c > 0$.

This problem was studied by Yang in [9]. Here we try to use the CRC method to solve this problem. First, we compute the CRC of p_5 on $(0, +\infty)$. It takes 6.33 seconds.

```
(*) p5:=x^5+a*x^2+b*x+c
The CRC of p5 on (0,+Infinity) is:
([polynomial,its all possible sign lists])

(1)[ ], if and only if
[P10,[1,0,0,0,0,-1,1,*,-1,-1],[1,0,0,0,0,*,-1,1,
 0,0],[1,0,0,0,0,-1,1,1,1,-1],[1,0,0,0,0,-1,1,1,0,0],
  [1,0,0,0,0,-1,0,0,-1,-1],[1,0,0,0,0,*,-1,0+,0,-1],
  [1,0,0,0,0,0+,0,-1,1,-1],[1,0,0,0,0,0-,0,1,*,-1],
  [1,0,0,0,0,*,-1,*,1,-1],[1,0,0,0,0,-1,0,0,0,0],
  [1,0,0,0,0,1,1,-1,1,-1],[1,0,0,0,0,*,-1,1,-1,-1],
  [1,0,0,0,0,*,0,0,1,-1],[1,0,0,0,0,0-,0,0,0,-1],
  [1,0,0,0,0,-1,1,0+,0,-1],[1,0,0,0,0,0-,0,1,0,0]]

(2) [1,2], if and only if
[P10,[1,0,0,0,0,1,1,1,0,0]],[P102,[1,1]]

(3) [1], if and only if
[P10,[1,0,0,0,0,1,0,0,0,0]],[P104,[1,-1,0,0],[1,-1,
1,1],[1,0,0,1],[1,-1,0,1],[1,1,-1,1],[1,0,-1,1],
[1,-1,-1,1]];[P10,[1,0,0,0,0,1,1,-1,0,0],
[1,0,0,0,0,*,-1,-1,0,0],[1,0,0,0,0,0+,0,-1,0,0]],
[P102,[1,-1]];[P10,
  [1,0,0,0,0,-1,1,1,1,1],[1,0,0,0,0,1,1,0-,0,1],
  [1,0,0,0,0,1,1,-1,1,1],[1,0,0,0,0,*,-1,0-,0,1],
  [1,0,0,0,0,1,0,0,-1,1],[1,0,0,0,0,1,1,*,-1,1],
  [1,0,0,0,0,*,0,0,1,1],[1,0,0,0,0,0-,0,1,1,1],
  [1,0,0,0,0,*,-1,*,1,1],[1,0,0,0,0,*,-1,-1,-1,1],
  [1,0,0,0,0,0+,0,0,0,1],[1,0,0,0,0,0+,0,-1,*,1]]

(4) [2], if and only if
[P10,[1,0,0,0,0,1,0,0,0,0]],[P104,[1,0,-1,-1],[1,-1,
-1,-1],[1,1,-1,-1],[1,1,1,-1],[1,1,0,-1],[1,0,0,-1]];
[P10,[1,0,0,0,0,0,0,-1,0,0],[1,0,0,0,0,1,-1,-1,0,0],
  [1,0,0,0,0,0,-1,-1,0,0],[1,0,0,0,0,-1,-1,-1,0,0],
  [1,0,0,0,0,1,1,-1,0,0],[1,0,0,0,0,1,0,-1,0,0]],
  [P102, [1, -1]]

(5) [3], if and only if [P10,[1,0,0,0,0,1,0,0,0,0]],
[P104,[1,1,0,0]],[P1042, [1,1]]

(6) [1,1], if and only if
```

```
[P10,[1,0,0,0,0,1,1,1,0,0]],[P102,[1,-1]];[P10,
 [1,0,0,0,0,1,1,1,1,-1],[1,0,0,0,0,1,0,0,-1,-1],
 [1,0,0,0,0,1,1,*,-1,-1],[1,0,0,0,0,1,1,0+,0,-1],
[1,0,0,0,0,0+,0,-1,-1,-1],[1,0,0,0,0,*,-1,-1,-1,-1]]

(7) [1,1,1], if and only if
[P10,[1,0,0,0,0,1,1,1,1,1]]
Where,
(#1) P1042:=-2*b*x^2-5*c
(#2) P102:=54*a^4*c+27*b*a^4*x^2-225*x^2*c^2*a^2
+600*a*c^2*b+720*a*x^2*c*b^2-320*c*b^3-256*x^2*b^4,
(#3) P10:=x^10+a*x^4+b*x^2+c,
(#4) P104:=-3*a*x^4-4*b*x^2-5*c,
 and the initial condition is
 c <> 0
```

The discriminant sequence of P_{10} is $[1, 0, 0, 0, 0, D_6, D_7, D_8, D_9, D_{10}]$, where

$$\begin{aligned}
D_6 &= -a^5, \qquad D_7 = -a^3(27a^4 + 300abc - 160b^3),\\
D_8 &= (300bac - 160b^3 + 27a^4)(720acb^2 - 256b^4 + 27a^4b\\
&\quad -225a^2c^2),\\
D_9 &= -(720acb^2 - 256b^4 + 27a^4b - 225a^2c^2)(-1600b^3ca\\
&\quad +256b^5 - 27a^4b^2 + 2250ba^2c^2 + 3125c^4 + 108a^5c),\\
D_{10} &= -c(-1600b^3ca + 256b^5 - 27a^4b^2 + 2250ba^2c^2\\
&\quad +3125c^4 + 108a^5c)^2
\end{aligned}$$

Again, we assume that the initial condition $c \neq 0$ holds. Then $(\forall x > 0)[p_5 = x^5 + ax^2 + bx + c > 0]$ iff case (1) holds. That is $[D_6 < 0 \wedge D_7 > 0 \wedge D_9 < 0 \wedge D_{10} < 0] \vee [D_7 < 0 \wedge D_8 > 0 \wedge D_9 = 0 \wedge D_{10} = 0] \vee [D_6 < 0 \wedge D_7 > 0 \wedge D_8 > 0 \wedge D_9 > 0 \wedge D_{10} < 0] \vee [D_6 < 0 \wedge D_7 > 0 \wedge D_8 > 0 \wedge D_9 = 0 \wedge D_{10} = 0] \vee [D_6 < 0 \wedge D_7 = 0 \wedge D_8 = 0 \wedge D_9 < 0 \wedge D_{10} < 0] \vee [D_7 < 0 \wedge D_8 \geq 0 \wedge D_9 = 0 \wedge D_{10} < 0] \vee [D_6 \geq 0 \wedge D_7 = 0 \wedge D_8 < 0 \wedge D_9 > 0 \wedge D_{10} < 0] \vee [D_6 \leq 0 \wedge D_7 = 0 \wedge D_8 > 0 \wedge D_{10} < 0] \vee [D_7 < 0 \wedge D_9 > 0 \wedge D_{10} < 0] \vee [D_6 < 0 \wedge D_7 = 0 \wedge D_8 = 0 \wedge D_9 = 0 \wedge D_{10} = 0] \vee [D_6 > 0 \wedge D_7 > 0 \wedge D_8 < 0 \wedge D_9 > 0 \wedge D_{10} < 0] \vee [D_7 < 0 \wedge D_8 > 0 \wedge D_9 < 0 \wedge D_{10} < 0] \vee [D_7 = 0 \wedge D_8 = 0 \wedge D_9 > 0 \wedge D_{10} < 0] \vee [D_6 \leq 0 \wedge D_7 = 0 \wedge D_8 = 0 \wedge D_9 = 0 \wedge D_{10} < 0] \vee [D_6 < 0 \wedge D_7 > 0 \wedge D_8 \geq 0 \wedge D_9 = 0 \wedge D_{10} < 0] \vee [D_6 \leq 0 \wedge D_7 = 0 \wedge D_8 > 0 \wedge D_9 = 0 \wedge D_{10} = 0]$.

This solution appears to be different from that given by Yang [9]. To obtain the same form as he gave, one might apply additional transformations to the result. For example, one writes $D_7 = -a^3(27a^4 + 300abc - 160b^3) = a^3d_8$ with d_8 taking the obvious value. Then the condition $D_7 > 0$ can be written as $(a^3 > 0 \wedge d_8 > 0) \vee (a^3 < 0 \wedge d_8 < 0)$. It is now possible to use Boolean algebra to transform the above result into Yang's form. Furthermore, the CRC of p_5 on $(0, +\infty)$ can be used to solve other problems easily. For example, the necessary and sufficient condition for p_5 having two distinct positive roots of multiplicities 1 is that case (6) holds. Thus the condition is
$[D_6 > 0 \wedge D_7 > 0 \wedge D_8 > 0 \wedge D_9 = 0 \wedge D_{10} = 0 \wedge E_2 < 0] \vee [D_6 > 0 \wedge D_7 > 0 \wedge D_8 > 0 \wedge D_9 > 0 \wedge D_{10} < 0] \vee [D_6 > 0 \wedge D_7 = 0 \wedge D_8 = 0 \wedge D_9 < 0 \wedge D_{10} < 0] \vee [D_6 > 0 \wedge D_7 > 0 \wedge D_9 < 0 \wedge D_{10} < 0] \vee [D_6 > 0 \wedge D_7 > 0 \wedge D_8 \geq 0 \wedge D_9 = 0 \wedge D_{10} < 0] \vee [D_6 \geq 0 \wedge D_7 = 0 \wedge D_8 < 0 \wedge D_9 < 0 \wedge D_{10} < 0] \vee [D_7 < 0 \wedge D_8 < 0 \wedge D_9 < 0 \wedge D_{10} < 0]$,
where $[1, E_2]$ is the discriminant sequence of P_{102}, and $E_2 = -c(300bac - 160b^3 + 27a^4)(720acb^2 - 256b^4 + 27a^4b - 225a^2c^2)$.

Also, we can conclude that it is impossible for p_5 to have four distinct positive roots.

6. CONCLUSION

In this paper, we have proposed a new algorithm for the automatic computation of the complete root classification of a parametric polynomial on an interval. However, some issues deserve further consideration. Although Theorem 3 is used in the algorithm to filter non-realizable sign lists, it is not guaranteed that all non-realizable sign lists are detected and deleted. Furthermore, the output conditions are basically equalities and inequalities in terms of the parametric coefficients. A further step would be to determine what are the possible values of the parametric coefficients such that the conditions described are satisfied. This is essentially the problem of solving semi-algebraic systems, a problem well-known to be difficult. This problem may be addressed using interval analysis [4] or method based on Gröbner basis [8]. We will leave these issues in further work.

7. REFERENCES

[1] H. Anai and H. Yanami. SyNRAC: A Maple-package for solving real algebraic constraints. *Lecture Notes in Computer Science*, 2657:828-837,2003.

[2] D. S. Arnon. Geometric reasoning with logic and algebra. *Artificial Intelligence*, 37:37-60, 1988.

[3] S. Basu, R. Pollack, M.-F. Roy. *Algorithms in Real Algebraic Geometry,* 2nd edition. Springer-Verlag, 2003.

[4] A. Colagrossi and A. M. Miola Computing real zeros of polynomials with parametric coefficients. *ACM SIGSAM Bulletin*, 17(1):12-15, 1983.

[5] S. Liang and D. J. Jeffrey. An Algorithm for computing the complete root classification of a parametric polynomial. *Lecture Notes in Computer Science*, 4120:116-130,2006.

[6] S. Liang and D. J. Jeffrey. The automatic computation of the complete root classification for a parametric polynomial. Electronic proceedings of MEGA 2007, `www.ricam.oeaw.ac.at/mega2007/electronic/30.pdf`.

[7] S. Liang and J. Zhang. A complete discrimination system for polynomials with complex coefficients and its automatic generation. *Science in China (Series E)*, 42(2):113-128, 1999.

[8] F. Rouillier. On solving parametric systems. In Workshop on *Challenges in Linear and Polynomial Algebra in Symbolic Computation Software*. Banff International Research Center, 2005.

[9] L. Yang. Recent advances on determining the number of real roots of parametric polynomials. *Journal of Symbolic Computation*, 28:225-242, 1999.

[10] L. Yang, X. Hou, and Z. Zeng. Complete discrimination system for polynomials. *Science in China (Series E)*, 39(6):628-646, 1996.

[11] L. Yang and B. Xia. Explicit criterion to determine the number of positive roots of a polynomial. *MM Research Preprints*, 15:134-145, 1997.

[12] L. Yang and B. Xia. Quantifier elimination for quartics. *Lecture Notes in Computer Science*, 4120:131-145,2006.