Modular algorithms for computing triangular decompositions of polynomial systems

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RTCA 2023, Institut Henri Poincaré, France, October 16
Acknowledgements

- Many thanks to the RTCA organizers for this event and for bringing all of us in this historical site.

- This talk is based on research projects in which many of my former and current graduate students have played an essential role. By alphabetic order: Alexander Brandt (Dalhousie University), Changbo Chen (CIGIT Chinese Academy of Sciences), Juan-Pablo Gonzàlez-Trochez (University of Western Ontario), François Lemaire (Université de Lille), Robert Moir (Earth64), Wei Pan (NVIDIA), Yuzhen Xie (Scotiabank), Haoze Yuan (University of Western Ontario).

- This talk is also based on collaborations with Maplesoft and the following colleagues: François Boulier (Université de Lille), Xavier Dahan (Tohoku University), Éric Schost (University of Waterloo), Wenyuan Wu (CIGIT Chinese Academy of Sciences).
Tentative Plan

- Part 1: Triangular decompositions in polynomial system solving
- Part 2: Modular methods in polynomial system solving
- Part 3: A modular method for triangular decompositions

Part 3 is based on our JSC 2012 paper with Changbo Chen, and our recent CASC 2023 paper with Alexander Brandt, Juan-Pablo González-Trochez and Haoze Yuan.

A proof-of-concept implementation was done with the RegularChains library and an efficient implementation is under development in the BPAS library. See our CASC 2023 paper.

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- Criteria for selecting the **algorithms supporting the solvers**:
  - provide a **comprehensive and coherent** set of tools for manipulating polynomial systems,
  - implement solvers with both **general algorithms** (which may not be the most efficient ones) and **faster algorithms** (which may only work under some assumptions).
The BPAS library

A high-performance polynomial algebra library

- Core of library written in C, wrapped in C++ interface for usability and object-oriented programming

Optimized algorithms and data structures, data locality, and parallelism

- Sparse multivariate polynomials [1], dense univariate and bivariate [7]
- Triangular decomposition of polynomial systems [2, 3]


- A natural encoding of the algebraic hierarchy
- “Dynamic” creation of algebraic types through composition
- Compile-time type safety between algebraic types

Generic support for parallel programming and parallel patterns (this talk)
Outline

1. Triangular decompositions in polynomial system solving

2. Modular methods in polynomial system solving

3. A Modular methods for incremental triangular decompositions

4. Conclusions
Milestones (1/3)

- Let $k$ be a field and $K$ its algebraic closure. Consider $n$ variables $x_1 < \cdots < x_n$. 
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A subset $V \subset K^n$ is a *(affine) variety over $k$* if there exists $F \subset k[x_1, \ldots, x_n]$ such that $V = V(F)$ where

$$V(F) := \{ z \in K^n \mid f(z) = 0 \ (\forall f \in F) \}.$$

The variety $V$ is *irreducible* if for all varieties $V_1, V_2 \subset K^n$

$$V = V_1 \cup V_2 \quad \Rightarrow \quad V = V_1 \text{ or } V = V_2.$$
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- **Theorem** (E. Lasker, 1905) For each variety \( V \subset K^n \) there exist finitely many irreducible varieties \( V_1, \ldots, V_e \subset K^n \) such that

\[
V = V_1 \cup \cdots \cup V_e.
\]

Moreover, if \( V_i \nsubseteq V_j \) for \( 1 \leq i < j \leq e \) then \( \{V_1, \ldots, V_e\} \) is unique. This is the irreducible decomposition of \( V \).
Milestones (2/3)

- **Theorem** (J.F. Ritt, 1932) Let $V \subset K^n$ be an irreducible non-empty variety and let $F \subset k[x_1, \ldots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.

\[
(\forall g \in \langle F \rangle) \quad \text{prem}(g, T) = 0.
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Combined with algebraic factorization one can (in theory) compute irreducible decompositions.
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- **Theorem** (W.T. Wu, 1987) Let $V \subset \mathbb{K}^n$ be a variety and let $F \subset \mathbb{k}[x_1, \ldots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.

  $$(\forall \ g \in F) \ prem(g, T) = 0.$$ 

This leads to a factorization-free algorithm for decomposing varieties (but not into irreducible components).
Example. Applying the charset procedure to

\[ F = \{ x_2^2 - x_1, x_1 x_3^2 - 2x_2 x_3 + 1, (x_2 x_3 - 1)x_4^2 + x_2^2 \} \]

produces \( T = F \). However \( V(F) = \emptyset \). Indeed

\[ x_1 x_3^2 - 2x_2 x_3 + 1 \equiv (x_2 x_3 - 1)^2 \mod x_2^2 - x_1. \]

Thus, the initial \( x_2 x_3 - 1 \) is a zero-divisor modulo \( \langle x_2^2 - x_1, x_1 x_3^2 - 2x_2 x_3 + 1 \rangle \).
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Moreover, for any input \( F \subseteq k[x_1, \ldots, x_n] \) one can compute regular chains \( T_1, \ldots, T_e \) such that a point \( z \in \mathbb{K}^n \) is a zero of \( F \) if and only if
\( z \) is a zero of one of the \( T_1, \ldots, T_e \) (in some technical sense).
(Dong Ming Wang 2000), (Marc Moreno Maza 2000).
A recursive view on polynomials

Let $k$ be a field, $X = x_1 < \cdots < x_n$ be variables and $f, g \in k[X]$ with $g \notin k$.

- $\text{mvar}(g)$: the greatest variable in $g$ is the leader or main variable of $g$,
- $\text{init}(g)$: the leading coefficient of $g$ w.r.t. $\text{mvar}(g)$ is the initial of $g$,
- $\text{mdeg}(g)$: the degree of $g$ w.r.t. $\text{mvar}(g)$,
- $\text{rank}(g) = v^d$ where $v = \text{mvar}(g)$ and $d = \text{mdeg}(g)$,
- $\text{pdivide}(f, g) = (q, r)$ with $q, r \in k[X]$, $\deg(r, v_g) < d_g$ and $h_g^ef = qg + r$ where $h_g = \text{init}(g)$, $e = \max(\deg(f, v) - d_g + 1, 0)$, $v_g = \text{mvar}(g)$ and $d_g = \text{mdeg}(g)$,

Example

Assume $n \geq 3$. If $p = x_1x_3^2 - 2x_2x_3 + 1$, then we have $\text{mvar}(p) = x_3$, $\text{mdeg}(p) = 2$, $\text{init}(p) = x_1$ and $\text{rank}(p) = x_3^2$.

Go to RegularChains.pdf Section 2.1.
The set $T \subset k[x_n > \cdots > x_1]$ is \textit{triangular set} if it consists of non-constant polynomials with pair-wise different main variables.

Define $h_T := \prod_{t \in T} \text{init}(t)$, where $\text{init}(t) = \text{lc}(t, \text{mvar}(t))$.

The \textit{quasi-component} and \textit{saturated ideal} of $T$ are:

$$W(T) := V(T) \setminus V(h_T) \quad \text{and} \quad \text{sat}(T) = \langle T \rangle : h_T^\infty.$$
Regular chain

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Note that for all triangular set $T$ we have:

- $W(T) = V(\text{sat}(T))$.
- If $\text{sat}(T) \neq \langle 1 \rangle$ then $\text{sat}(T)$ is strongly equi-dimensional.
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Definition (M. Kalkbrner, 1991 - L. Yang, J. Zhang 1991)

$T$ is a regular chain if $T = \emptyset$ or $T := T' \cup \{t\}$ with $\text{mvar}(t)$ maximum s.t.

- $T'$ is a regular chain,
- $\text{init}(t)$ is regular modulo $\text{sat}(T')$. 
Regular chain: alternative definition
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Regular chain: algorithmic properties

Theorem (P. Aubry, D. Lazard, M., 1997)

\( T \) is a regular chain iff \( \{ p \mid \text{prem}(p, T) = 0 \} = \text{sat}(T) \).
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**Definition**

Let \( T \subset k[x_n > \cdots > x_1] \) be a triangular set and \( p \in k[x_n > \cdots > x_1] \). If \( T \) is empty then, the **iterated resultant** of \( p \) w.r.t. \( T \) is \( \text{resultant}(T, p) = p \).

Otherwise, writing \( T = T_{<w} \cup T_w \)

\[
\text{resultant}(T, p) = \begin{cases} 
p & \text{if deg}(p, w) = 0 \\
\text{resultant}(T_{<w}, \text{resultant}(T_w, p, w)) & \text{otherwise}
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**Theorem (L. Yang, J. Zhang 1991)**

\( p \) is regular modulo \( \text{sat}(T) \) iff \( \text{resultant}(T, p) \neq 0 \).
Kalkbrener triangular decomposition

Let $F \subset k[x]$. A family of regular chains $T_1, \ldots, T_e$ of $k[x]$ is called a **Kalkbrener triangular decomposition** of $V(F)$ if

$$V(F) = \bigcup_{i=1}^{e} V(\text{sat}(T_i)).$$
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### Wu-Lazard triangular decomposition

Let $F \subset k[x]$. A family of regular chains $T_1, \ldots, T_e$ of $k[x]$ is called a **Wu-Lazard triangular decomposition** of $V(F)$ if

$$V(F) = \bigcup_{i=1}^{e} W(T_i).$$
Triangularize applied to *sofa* and *cylinder* (1/2)

\[ x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0 \]
Triangularize applied to sofa and cylinder (2/2)

\[ \begin{align*}
R & := \text{PolynomialRing}([z, y, x]): F := [x^2+y^3+z^5, x^4+z^2-1]: \text{dec} := \text{Triangularize}(F, R): \text{map}(\text{Display}, \text{dec}, R); \\
& \begin{bmatrix}
-2x^4 + x^8 + 1 & z + x^2 + y^3 = 0 \\
y^6 + 2x^2y^3 + 10x^{12} - 10x^8 + x^{20} - 5x^{16} + 6x^4 - 1 = 0 \\
-2x^4 + x^8 + 1 \neq 0
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
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\[ \begin{align*}
& \begin{bmatrix}
z = 0 \\
y - 1 = 0 \\
x^2 + 1 = 0
\end{bmatrix} \\
& \begin{bmatrix}
z = 0 \\
y^2 - y + 1 = 0 \\
x + 1 = 0
\end{bmatrix} \\
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In a nutshell, solving bivariate polynomial systems can be done via
Relations with resultants and subresultants (1/3)

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1. resultant computations,
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Example (von zur Gathen & Gerhard, Chapter 6)

Let \( P = (y^2 + 6)(x - 1) - y(x^2 + 1) \) and \( Q = (x^2 + 6)(y - 1) - x(y^2 + 1) \)
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\[
\text{res}(P, Q, y) = 2 (x^2 - x + 4)(x - 2)^2(x - 3)^2.
\]
Relations with resultants and subresultants (1/3)

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- $\gcd(P, Q, x^2 - x + 4 = 0) = (2x - 1)y - 7 - x$. 
Relations with resultants and subresultants (2/3)

In fact, factorizing the resultant is not necessary. Using regularity test and the specialization property of subresultants is sufficient.

Consider the following polynomials $f, g \in \mathbb{Q}[y]$: $f = x^7 - 36x - 22y + 1$, $g = x^6 + 47x^3 - 60x^2y^2 - 6x^2y - 83y^2 - 10y + 50$.

The complete list of subresultants of $\{f, g\}$ w.r.t. $x$ is:

- $S_6 = g$
- $S_5 = 56x^4 + 60x^2y^2 + 6x^2y + 83xy^2 + 10xy + 17x + 81y + 1$
- $S_4 = 46x^4 + 64x^2y^2 + 27x^2y + 13xy^2 + 45xy + 25x + 4y + 56$
- $S_3 = 74x^2y^4 + 7x^3y^2 + 56x^2y^3 + 44xy^4 + 98y^2 + 86y + 53$
- $S_2 = 25x^2y^8 + 10x^2y^7 + 26xy^8 + 62x^2y^6 + 96x + 72y + 43$
- $S_1 = 81xy^{12} + 28xy^{11} + 76y^{12} + 24xy^{10} + 5xy^9 + 4x + 73y + 77$
- $S_0 = 97y^{15} + 82y^{14} + 82y^{13} + 23y^5 + 89y^4 + 31y^3 + y^2 + 54y + 69$.

The solutions of $f = g = 0$ can be calculated using $S_0, S_1$ only.
Relations with resultants and subresultants (2/3)

In fact, factorizing the resultant is not necessary. Using regularity test and the specialization property of subresultants is sufficient. Consider the following polynomials \( f, g \in \mathbb{Q}[y < x] \):

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Go to RegularChains.pdf Sections 2.2 and 2.3.
Extending the previous ideas to solving $m$ polynomial equations in $n$ variables can be done using

- a cascade of Sylvester resultants (this talk), or
- a combination of Dixon/Macaulay resultants and Sylvester resultants (work in progress).
## Relations with Gröbner bases

### Normalized regular chains

- The regular chain $T \subset \mathbb{k}[x_n > \cdots > x_1]$ is said *normalized* if for every $t, t' \in T$ we have $\deg(\text{init}(t), \text{mvar}(t')) = 0$. 

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Marc Moreno Maza

Modular Algorithms for Triangular Decompositions

RTCA 2023

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Relations with Gröbner bases

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- Let $Y := \{\text{mvar}(t) \mid t \in T\}$ and $U := X \setminus Y$. If $T$ is normalized, then $T$ is a \textit{Gröbner basis} of dimension 0 of the ideal it generates in $\mathbf{k}(U)[Y]$. 
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From lexicographical Gröbner bases to regular chains
Relations with Gröbner bases

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From lexicographical Gröbner bases to regular chains

- Let $G$ be a lexicographical Gröbner basis of a zero-dimensional ideal $\mathcal{I} \subset k[x_n > \cdots > x_1]$. Then, $\text{Lextriangular}(G)$ computes regular chains (optionally normalized) $T_1, \ldots, T_e \subset k[x_n > \cdots > x_1]$ so that $V(G) = \bigcup_{i=1}^e V(T_i)$. (Daniel Lazard, 1992).
Relations with Gröbner bases

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- This is done at a cost which is at most that inverting at most $\#G$ polynomials modulo one of the ideals $\langle T_1 \rangle, \ldots, \langle T_e \rangle$. 
**Relations with Gröbner bases**

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- Let $Y := \{\text{mvar}(t) | t \in T\}$ and $U := X \setminus Y$. If $T$ is normalized, then $T$ is a **Gröbner basis** of dimension 0 of the ideal it generates in $k(U)[Y]$.

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- This is done at a cost which is at most that inverting at most $\#G$ polynomials modulo one of the ideals $\langle T_1 \rangle, \ldots, \langle T_e \rangle$.
- This is practically very effective.
Let \( f \in k[x_1, \ldots, x_n] \) and \( T \subseteq k[x_1, \ldots, x_n] \) be a regular chain.
Let $f \in k[x_1, \ldots, x_n]$ and $T \subseteq k[x_1, \ldots, x_n]$ be a regular chain.

The intersection $V(f) \cap W(T)$ is approximated by the function call $\text{Intersect}(f, T)$, which returns regular chains $T_1, \ldots, T_e \subseteq k[X]$ s.t.:

$$V(f) \cap W(T) \subseteq W(T_1) \cup \cdots \cup W(T_e) \subseteq V(f) \cap \overline{W(T)},$$

where $\overline{W(T)}$ denotes the Zariski closure of $W(T)$. 

Given $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, one can solve $f_1 = \ldots = f_m = 0$ using repeated calls to $\text{Intersect}$. Indeed, if $V(f_1, \ldots, f_{m-1}) = \bigcup_{i=1}^e W(T_i)$, then we have $V(f_1, \ldots, f_m) = \bigcup_{i=1}^e \text{Intersect}(f_m, T_i)$. 

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Triangular decompositions: the incremental approach

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Outline

1. Triangular decompositions in polynomial system solving

2. Modular methods in polynomial system solving

3. A Modular methods for incremental triangular decompositions

4. Conclusions
Computing by homomorphphic images: principles,

Examples

- The computation of the determinant of an integer matrix using the Chinese Remaindering Theorem.
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Computing by homomotphic images: principles,

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- A more advanced example is the computation of the GCD of univariate integer polynomials, again using CRT.

Adantages and issues

- Modular methods (1) may control expression swell, (2) allow sharper implementation (fine control memory), (3) open the door to FFT-based arithmetic, and (4) provide opportunities for concurrency.
- Modular methods are (1) generally harder to implement than direct methods, and (2) usually require change of representations which may come with significant costs in terms of memory consumption.
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Expression swell may sometimes be handled in other ways

Consider the system $F$ (Barry Trager).

\[-x^5 + y^2 - 3y - 1 = 5g^4 - 3 = -20x + y - z = 0\]

We solve it for $z \leq y \leq x$.

$V(F)$ is equiprojective and its Lazard triangular set is

\[
\begin{align*}
\text{Modular Algorithms for Triangular Decompositions} & \\
\text{RTCA 2023} & \\
\text{Expression swell may sometimes be handled in other ways} & \\
\text{Consider the system } F (\text{Barry Trager).} & \\
\text{It} & \\
\text{We solve it for } z \leq y \leq x. & \\
\text{Applying the transformation of Dahan and Schost leads to 1787 characters.} & \\
\text{We compute the regular chain produced by the} & \\
\text{Triangulize algorithm of the RegularChains library, counting 963} & \\
\text{characters.} & \\
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\end{align*}
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Trace algorithms

Consider an algorithm Solver($F$) taking $F \subseteq \mathbb{Z}[x_n > \cdots > x_1]$ computing a finite sequence $\mathcal{G}$ of finite sets $G_1, G_2, \ldots, \subseteq \langle F \rangle$ until $G_i = G_{\text{output}}$ satisfies a property, e.g. Gröbner basis of $\langle F \rangle$ or Wu-characteristic set of $F$. 
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- Endow all such finite sequences $G$ with a rank function so that, for every well-chosen prime number $p$, the sequence computed by $\text{Solver}(F \mod p)$ has maximum rank iff $\text{Solver}(F \mod p) = G_{\text{output}} \mod p$. 
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- For Gröbner bases, one can use the Hilbert function (Carlo Traverso, ISSAC 1988), (Jean-Charles Faugère, PASCO 1994), (Elizabeth Arnold, JSC 2003)
**Trace algorithms**

- Consider an algorithm \( \text{Solver}(F) \) taking \( F \subseteq \mathbb{Z}[x_n > \cdots > x_1] \) computing a finite sequence \( G \) of finite sets \( G_1, G_2, \ldots, \in \langle F \rangle \) until \( G_i = G_{\text{output}} \) satisfies a property, e.g. Gröbner basis of \( \langle F \rangle \) or Wu-characteristic set of \( F \).

- Endow all such finite sequences \( G \) with a *rank function* so that, for every well-chosen prime number \( p \), the sequence computed by \( \text{Solver}(F \mod p) \) has maximum rank iff
  \[
  \text{Solver}(F \mod p) = G_{\text{output}} \mod p.
  \]

- For polynomial GCD computations (with \( n = 1 \)) one can simply use the degree as rank function.

- For Gröbner bases, one can use the Hilbert function (Carlo Traverso, ISSAC 1988), (Jean-Charles Faugère, PASCO 1994), (Elizabeth Arnold, JSC 2003)

- For characteristic sets, one can use the notion of rank as defined by Ritt and Wu (M. ACA 2003).
The case of decomposition algorithms

Consider an algorithm Solver\( (F) \) taking \( F \subseteq \mathbb{Q}[x_n > \cdots > x_1] \) (assumed to be zero-dimensional for simplicity) computing a triangular decomposition into regular chain \( T_1, \ldots, T_e \).
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- The algorithm EquiprojectableDecomposition\( (T_1, \ldots, T_e) \) returns a canonical triangular decomposition of \( V(F) \) based on “geometrical” considerations.
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- Moreover, if the prime $p$ is large enough, then the decompositions $\text{EquiprojectableDecomposition}(\text{Solver}(F \mod p))$ and $\text{EquiprojectableDecomposition}(\text{Solver}(F)) \mod p$ match.
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Using Hensel lifting techniques (Schost 2002) this leads to an effective modular method for Solver\( (F) \) (Dahan, M., Schost, Wu & Xie ISSAC 2005).
Testing regularity of \( p \in k[x_n > \cdots > x_1] \) w.r.t. regular chain \( T \subset k[x_n > \cdots > x_1] \) is equivalent to checking whether \( \text{resultant}(T, p) \) is zero or not.
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Moreover, eliminating variables with pseudo-division (or variants) leads to computing cascade of (pseudo-)remainder sequences and thus (multiples of) iterated resultants.
Issues with iterated subresultant chains (1/2)

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In (C. Chen & M. JSC 2012) we give examples of 3 zero-dimensional systems with $4^3 = 64$ solutions where the extraneous factors have degree in the 1000’s.
Let $C = \{t_1, t_2, \ldots, t_n\}$ be a zero-dimensional regular chain and $f$ be a polynomial all in $k[X]$. 
Issues with iterated subresultant chains (2/2)

- Let \( C = \{t_1, t_2, \ldots, t_n\} \) be a zero-dimensional regular chain and \( f \) be a polynomial all in \( k[X] \).
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Issues with iterated subresultant chains (2/2)

- Let $C = \{t_1, t_2, \ldots, t_n\}$ be a zero-dimensional regular chain and $f$ be a polynomial all in $k[X]$.
- For $i = 1, \ldots, n$, we denote by $h_i$, the initial of $t_i$,
- For $i = 1, \ldots, n - 1$, we define $f_i = \text{res}(\{t_{i+1}, \ldots, t_n\}, f)$ and $e_i = \deg(f_i, x_i)$, with $e_n = \deg(f, X_n)$.
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- Then, the iterated resultant $\text{res}(C, f)$ is given by:

$$
R(C, f) = \left( \prod_{\beta_1 \in V_M(t_1)} h_2(\beta_1) \right)^{e_2} \cdots \left( \prod_{\beta_{n-1} \in V_M(t_1, \ldots, t_{n-1})} h_n(\beta_{n-1}) \right)^{e_n},
$$

where

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R(C, f) = \left( \prod_{\alpha \in V_M(C)} f(\alpha) \right),
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and $V_M(C)$ is the set of the zeros of $C$ counted with multiplicity.
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\begin{align*}
    h_1^{e_1} \left( \prod_{\beta_1 \in V_M(t_1)} h_2(\beta_1) \right)^{e_2} \cdots \left( \prod_{\beta_{n-1} \in V_M(t_1, \ldots, t_{n-1})} h_n(\beta_{n-1}) \right)^{e_n} R(C, f),
\end{align*}
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\]

and \( V_M(C) \) is the set of the zeros of \( C \) counted with multiplicity.
- Thus, if \( h_1 = \cdots = h_n = 1 \), then we simply have:

\[
\text{res}(C, f) = R(C, f).
\]
Outline

1. Triangular decompositions in polynomial system solving

2. Modular methods in polynomial system solving

3. A Modular methods for incremental triangular decompositions

4. Conclusions
Recall the incremental approach and define our goals

- Let \( f \in \mathbb{Q}[X_1, \ldots, X_n] \) and \( T \subseteq \mathbb{Q}[X_1, \ldots, X_n] \) be a regular chain.

The intersection \( \mathcal{V}\{f\} \cap \mathcal{W}\{T\} \) is approximated by the function call \( \text{Intersect}\{f, T\} \), which returns regular chains \( T_1, \ldots, T_e \subseteq \mathcal{K}(X) \) s.t.:

\[ \mathcal{V}\{f\} \cap \mathcal{W}\{T\} \subseteq \mathcal{W}\{T_1\} \cup \mathcal{W}\{T_e\} \subseteq \mathcal{V}\{f\} \cap \mathcal{W}\{T\} \]

Our goals

1. Compute modulo a well-chosen prime as in (Dahan et al., ISSAC 2005)
2. Reduce to the case where \( T \) is zero-dimensional and normalized, by variable specialization
3. Recover the specialized variables, then the rational coefficients.

We want to avoid the recourse to Gröbner bases so as to support:

1. Algorithms in differential algebra, and
2. Positive-dimensional systems for which methods based on regular chains may have smaller output.

This is similar in spirit to (Grégoire Lecerf, J. Complex 2001)

However, we avoid random changes of coordinates and support decompositions in the sense of Lazard.
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Notations

- Let \( f, t_2, \ldots, t_n \in k[X_1 < \cdots < X_n] \) be non-constant.
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- Let \( r_{i-1} \) and \( g_i \) be the subresultants of index 0 and 1 from \( S(t_i, r_i, X_i) \).
- We denote by \( \overline{s} \) the squarefree part of \( s := r_1 \).
Base result

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \notin k$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{\overline{s}, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lc}(t_i, X_i)$ is invertible modulo $\langle \overline{s}, g_2, \ldots, g_{i-1}\rangle$. 

Theorem With our four Hypotheses, we have:

$V\{f, t_2, \ldots, t_n\} = V\{s, g_2, \ldots, g_n\}$. 

Under Hypotheses 1, 2, 3 and 4, the regular chain $C$ can be computed by a cascade of subresultant chain computations.

This modular method can be enhanced so that the 4 Hypotheses are no longer necessary (as we will see later).
Base result

**H1** for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i$,

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- Under Hypotheses 1, 2, 3 and 4, the regular chain \(C\) can be computed by a cascade of subresultant chain computations.
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- This modular method can be enhanced so that the 4 Hypotheses are no longer necessary (as we will see later).
$R := \text{PolynomialRing}([x3, x2, x1])$:

$f := (x2 + x1) \cdot x3^2 + x3 + 1$; \hspace{1cm} f := (x2 + x1) x3^2 + x3 + 1 \hspace{1cm} (1)$

$t3 := x1 \cdot x3^2 + x2 \cdot x3 + 1$; \hspace{1cm} t3 := x1 x3^2 + x2 x3 + 1 \hspace{1cm} (2)$

$t2 := (x1 + 1) \cdot x2^2 + x2 + 2$; \hspace{1cm} t2 := (x1 + 1) x2^2 + x2 + 2 \hspace{1cm} (3)$
\[ R := \text{PolynomialRing}([x_3, x_2, x_1]) : \]
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\[ f := (x_2 + x_1)x_3^2 + x_3 + 1 \] \hspace{1cm} (1)
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\[ \text{src1} := \text{SubresultantChain}(f, t_3, x_3, R) : \]
\[ g_3 := \text{SubresultantOfIndex}(1, \text{src1}, R); r := \text{SubresultantOfIndex}(0, \text{src1}, R); \]
\[ g_3 := x_1x_2x_3 + x_2^2x_3 - x_1x_3 + x_2 \]
\[ r := x_1x_2^2 + x_2^3 - 2x_1x_2 + x_1 \] \hspace{1cm} (4)
\[ R := \text{PolynomialRing}([x3, x2, x1]) : \]
\[ f := (x2 + x1) \cdot x3^2 + x3 + 1; \]
\[ t3 := x1 \cdot x3^2 + x2 \cdot x3 + 1; \]
\[ t2 := (x1 + 1) \cdot x2^2 + x2 + 2; \]
\[ f := (x2 + x1) \cdot x3^2 + x3 + 1 \quad (1) \]
\[ t3 := x1 \cdot x3^2 + x2 \cdot x3 + 1 \quad (2) \]
\[ t2 := (x1 + 1) \cdot x2^2 + x2 + 2 \quad (3) \]
\[ \text{src1} := \text{SubresultantChain}(f, t3, x3, R) : \]
\[ g3 := \text{SubresultantOfIndex}(1, \text{src1}, R); r := \text{SubresultantOfIndex}(0, \text{src1}, R); \]
\[ g3 := x1 \cdot x2 \cdot x3 + x2^2 \cdot x3 - x1 \cdot x3 + x2 \]
\[ r := x1 \cdot x2^2 + x2^3 - 2 \cdot x1 \cdot x2 + x1 \quad (4) \]
\[ \text{src2} := \text{SubresultantChain}(r, t2, x2, R) : \]
\[ g2 := \text{SubresultantOfIndex}(1, \text{src2}, R); s := \text{SubresultantOfIndex}(0, \text{src2}, R); \]
\[ g2 := -2 \cdot x1^3 \cdot x2 + x1^3 - 5 \cdot x1^2 \cdot x2 - 5 \cdot x1 \cdot x2 - x1 - x2 + 2 \]
\[ s := x1^5 + 9 \cdot x1^4 + 24 \cdot x1^3 + 38 \cdot x1^2 + 13 \cdot x1 + 8 \quad (5) \]
\[ \text{sol} := \text{Chain}([s], \text{Empty}(R), R) : \text{IsRegular}(\text{Initial}(g2, R), \text{sol}, R); \text{true} \] (6)

\[ \text{sol2} := \text{Chain}([g2], \text{sol}, R) : \text{IsRegular}(\text{Initial}(g3, R), \text{sol2}, R); \text{true} \] (7)

\[ \text{IsRegular}(\text{Initial}(t3, R), \text{sol2}, R); \text{true} \] (8)
\text{sol} := \text{Chain}([s, \text{Empty}(R), R]) : \text{IsRegular}(\text{Initial}(g2, R), \text{sol}, R);
true
\text{sol2} := \text{Chain}([g2, \text{sol}, R]) : \text{IsRegular}(\text{Initial}(g3, R), \text{sol2}, R);
true
\text{IsRegular}(\text{Initial}(t3, R), \text{sol2}, R);
true
\text{sol3} := \text{Chain}([g3, \text{sol2}, R]) : \text{Display}(\text{sol3}, R);
\begin{align*}
(x^2 + x^2 x - x) x^3 + x^2 &= 0 \\
(-2 x^3 - 5 x^2 - 5 x - 1) x^2 + x^3 - x + 2 &= 0 \\
x^5 + 9 x^4 + 24 x^3 + 38 x^2 + 13 x + 8 &= 0 \\
x^2 + x - x &\neq 0 \\
-2 x^3 - 5 x^2 - 5 x - 1 &\neq 0
\end{align*}
\[ \text{sol} := \text{Chain}([s, \text{Empty}(R), R] : \text{IsRegular}(\text{Initial}(g2, R), \text{sol}, R); \]
\[ \text{true} \quad (6) \]
\[ \text{sol2} := \text{Chain}([g2, \text{sol}, R] : \text{IsRegular}(\text{Initial}(g3, R), \text{sol2}, R); \]
\[ \text{true} \quad (7) \]
\[ \text{IsRegular}(\text{Initial}(t3, R), \text{sol2}, R); \]
\[ \text{true} \quad (8) \]
\[ \text{sol3} := \text{Chain}([g3, \text{sol2}, R] : \text{Display}(\text{sol3}, R); \]
\[ \begin{align*}
(x^2 + x_2 \cdot x_1 - x_1) \cdot x_3 + x_2 &= 0 \\
(-2 \cdot x_1^3 - 5 \cdot x_1^2 - 5 \cdot x_1 - 1) \cdot x_2 + x_1^3 - x_1 + 2 &= 0 \\
x_1^5 + 9 \cdot x_1^4 + 24 \cdot x_1^3 + 38 \cdot x_1^2 + 13 \cdot x_1 + 8 &= 0 \\
x_2^2 + x_2 \cdot x_1 - x_1 &\neq 0 \\
-2 \cdot x_1^3 - 5 \cdot x_1^2 - 5 \cdot x_1 - 1 &\neq 0 
\end{align*} \quad (9) \]
\[ \text{dec3} := \text{Triangularize}([f, t3, t2], R) : \text{Display}(\text{dec3}[1], R); \]
\[ \begin{align*}
(x^2 + x_2 \cdot x_1 - x_1) \cdot x_3 + x_2 &= 0 \\
(2 \cdot x_1^3 + 5 \cdot x_1^2 + 5 \cdot x_1 + 1) \cdot x_2 - x_1^3 + x_1 - 2 &= 0 \\
x_1^5 + 9 \cdot x_1^4 + 24 \cdot x_1^3 + 38 \cdot x_1^2 + 13 \cdot x_1 + 8 &= 0 \\
x_2^2 + x_2 \cdot x_1 - x_1 &\neq 0 \\
2 \cdot x_1^3 + 5 \cdot x_1^2 + 5 \cdot x_1 + 1 &\neq 0 
\end{align*} \quad (10) \]
The modular algorithm in $\mathbb{F}_p$ under our hypotheses

1. Evaluate $f$ and $T$ at sufficiently many (use the Bézout bound or the mixed volume) values $a$ of $X_1$ so that $T$ specializes well at $X_1 = a$ to a zero-dimensional regular chain $T_a$. 
The modular algorithm in $\mathbb{F}_p$ under our hypotheses

1. Evaluate $f$ and $T$ at sufficiently many (use the Bézout bound or the mixed volume) values $a$ of $X_1$ so that $T$ specializes well at $X_1 = a$ to a zero-dimensional regular chain $T_a$

2. For each good specialization $X_1 = a$
The modular algorithm in $\mathbb{F}_p$ under our hypotheses

1. Evaluate $f$ and $T$ at sufficiently many (use the Bézout bound or the mixed volume) values $a$ of $X_1$ so that $T$ specializes well at $X_1 = a$ to a zero-dimensional regular chain $T_a$.

2. For each good specialization $X_1 = a$:
   1. Replace $T_a$ by a normalized (= monic) regular chain $N_a$.
The modular algorithm in $\mathbb{F}_p$ under our hypotheses

1. Evaluate $f$ and $T$ at sufficiently many (use the Bézout bound or the mixed volume) values $a$ of $X_1$ so that $T$ specializes well at $X_1 = a$ to a zero-dimensional regular chain $T_a$

2. For each good specialization $X_1 = a$
   1. Replace $T_a$ by a normalized (= monic) regular chain $N_a$
   2. Compute the images of the polynomials $r_{i-1}$ and $g_i$ at $X_1 = a$

Technical details:
1. Specializations $X_1 = a$, $X_1 = b$, ... must produce faithful images of resultants $r_i$, that is, resultants of maximum degree.
2. The implementation uses a priori bounds for the number of non-faithful specializations, and the degree of $s$; see the details in our CASC 2023 paper.
3. We stop combining those images of the $r_i$'s when the recombination of the images stabilizes (Monagan’s probabilistic idea, ISSAC 2005).
The modular algorithm in $\mathbb{F}_p$ under our hypotheses

1. Evaluate $f$ and $T$ at sufficiently many (use the Bézout bound or the mixed volume) values $a$ of $X_1$ so that $T$ specializes well at $X_1 = a$ to a zero-dimensional regular chain $T_a$

2. For each good specialization $X_1 = a$
   1. Replace $T_a$ by a normalized (=$\text{monic}$) regular chain $N_a$
   2. Compute the images of the polynomials $r_{i-1}$ and $g_i$ at $X_1 = a$

3. Recover $X_1$ (by interpolation and rational function reconstruction) and deduce $s, g_2, \ldots, g_n$
The modular algorithm in \( \mathbb{F}_p \) under our hypotheses

1. Evaluate \( f \) and \( T \) at sufficiently many (use the Bézout bound or the mixed volume) values \( a \) of \( X_1 \) so that \( T \) specializes well at \( X_1 = a \) to a zero-dimensional regular chain \( T_a \)

2. For each good specialization \( X_1 = a \)
   1. Replace \( T_a \) by a normalized (= monic) regular chain \( N_a \)
   2. Compute the images of the polynomials \( r_{i-1} \) and \( g_i \) at \( X_1 = a \)

3. Recover \( X_1 \) (by interpolation and rational function reconstruction) and deduce \( s, g_2, \ldots, g_n \)

Technical details:

- specializations \( X_1 = a, X_1 = b, \ldots \) must produce faithful images of resultant \( r_i \), that is, resultants of maximum degree.
  - good \( \neq \) faithful.
The modular algorithm in $\mathbb{F}_p$ under our hypotheses

1. Evaluate $f$ and $T$ at sufficiently many (use the Bézout bound or the mixed volume) values $a$ of $X_1$ so that $T$ specializes well at $X_1 = a$ to a zero-dimensional regular chain $T_a$

2. For each good specialization $X_1 = a$
   
   1. Replace $T_a$ by a normalized (= monic) regular chain $N_a$
   2. Compute the images of the polynomials $r_{i-1}$ and $g_i$ at $X_1 = a$

3. Recover $X_1$ (by interpolation and rational function reconstruction) and deduce $s, g_2, \ldots, g_n$

Technical details:

- Specializations $X_1 = a, X_1 = b, \ldots$ must produce faithful images of resultants $r_i$, that is, resultants of maximum degree. Good $\neq$ faithful.

- The implementation uses a priori bounds for
  
  1. The number of non-faithful specializations, and
  2. The degree of $s$; see the details in our CASC 2023 paper.
The modular algorithm in $\mathbb{F}_p$ under our hypotheses

1. Evaluate $f$ and $T$ at sufficiently many (use the Bézout bound or the mixed volume) values $a$ of $X_1$ so that $T$ specializes well at $X_1 = a$ to a zero-dimensional regular chain $T_a$

2. For each good specialization $X_1 = a$
   1. Replace $T_a$ by a normalized (≡ monic) regular chain $N_a$
   2. Compute the images of the polynomials $r_{i-1}$ and $g_i$ at $X_1 = a$

3. Recover $X_1$ (by interpolation and rational function reconstruction) and deduce $\overline{s}, g_2, \ldots, g_n$

Technical details:

- specializations $X_1 = a, X_1 = b, \ldots$ must produce faithful images of resultants $r_i$, that is, resultants of maximum degree. good ≠ faithful.

- the implementation uses a priori bounds for
  1. the number of non-faithful specializations, and
  2. the degree of $\overline{s}$; see the details in our CASC 2023 paper.

- we stop combining those images of the $r_i$’s when the recombination of the images stabilizes (Monagan’s probabilistic idea, ISSAC 2005).
The full modular algorithm: relaxing the hypotheses (1/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i,$

H2 For $2 \leq i \leq n,$ we have $g_i \notin k$ and $\text{mvar}(g_i) = X_i,$

H3 The polynomial set $C := \{s, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n,$ $\text{lcm}(t_i, X_i)$ is invertible modulo $\langle s, g_2, \ldots, g_{i-1} \rangle.$
The full modular algorithm: relaxing the hypotheses (1/3)

**H1** for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i$,

**H2** For $2 \leq i \leq n$, we have $g_i \notin k$ and $\text{mvar}(g_i) = X_i$,

**H3** The polynomial set $C := \{\bar{s}, g_2, \ldots, g_n\}$ is a regular chain,

**H4** For every $2 \leq i \leq n$, $\text{lcm}(t_i, X_i)$ is invertible modulo $\langle \bar{s}, g_2, \ldots, g_{i-1} \rangle$.

Relaxing the $r_i \notin k$ part of H1 implies that the last computed resultant, say $s$, could be constant:
The full modular algorithm: relaxing the hypotheses (1/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \not\in k$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \not\in k$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{\bar{s}, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lct}(t_i, X_i)$ is invertible modulo $\langle \bar{s}, g_2, \ldots, g_{i-1} \rangle$.

Relaxing the $r_i \not\in k$ part of H1 implies that the last computed resultant, say $s$, could be constant:

1 if $s = 0$, then its “parents” (say $r_j$ and $t_j$) have a non-trivial GCD over $k$ which must be added to the chain $C'$,
The full modular algorithm: relaxing the hypotheses (1/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \notin k$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{s, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lc}(t_i, X_i)$ is invertible modulo $\langle s, g_2, \ldots, g_{i-1} \rangle$.

Relaxing the $r_i \notin k$ part of H1 implies that the last computed resultant, say $s$, could be constant:

1. if $s = 0$, then its “parents” (say $r_j$ and $t_j$) have a non-trivial GCD over $k$ which must be added to the chain $C$,
2. if $s \neq 0$, then $\text{Intersect}(f, T) = \emptyset$. 
The full modular algorithm: relaxing the hypotheses (1/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \not\in k$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \not\in k$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{s, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lcm}(t_i, X_i)$ is invertible modulo $\langle s, g_2, \ldots, g_{i-1} \rangle$.

- Relaxing the $r_i \not\in k$ part of H1 implies that the last computed resultant, say $s$, could be constant:
  1. if $s = 0$, then its “parents” (say $r_j$ and $t_j$) have a non-trivial GCD over $k$ which must be added to the chain $C$,
  2. if $s \neq 0$, then $\text{Intersect}(f, T) = \emptyset$.

- Handling this modification only requires to possibly computing this GCD, whose cost is negligible.
The full modular algorithm: relaxing the hypotheses (2/3)

H1 for \(1 \leq i \leq n - 1\), we have \(r_i \notin \mathbf{k}\) and \(\text{mvar}(r_i) = X_i\),

H2 For \(2 \leq i \leq n\), we have \(g_i \notin \mathbf{k}\) and \(\text{mvar}(g_i) = X_i\),

H3 The polynomial set \(C := \{\bar{s}, g_2, \ldots g_n\}\) is a regular chain,

H4 For every \(2 \leq i \leq n\), \(\text{lcm}(t_i, X_i)\) is invertible modulo \(\langle \bar{s}, g_2, \ldots, g_{i-1} \rangle\).
The full modular algorithm: relaxing the hypotheses (2/3)

H1  for $1 \leq i \leq n - 1$, we have $r_i \notin \mathbf{k}$ and $\text{mvar}(r_i) = X_i$,

H2  For $2 \leq i \leq n$, we have $g_i \notin \mathbf{k}$ and $\text{mvar}(g_i) = X_i$,

H3  The polynomial set $C := \{s, g_2, \ldots g_n\}$ is a regular chain,

H4  For every $2 \leq i \leq n$, $\text{lc}(t_i, X_i)$ is invertible modulo $\langle s, g_2, \ldots, g_{i-1} \rangle$.

Relaxing the $\text{mvar}(r_i) = X_i$ part of H1 implies that the successive resultants $r_n, \ldots$ may have a gap in their sequence of main variables.
The full modular algorithm: relaxing the hypotheses (2/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin \mathbb{k}$ and \(\text{mvar}(r_i) = X_i\),

H2 For $2 \leq i \leq n$, we have $g_i \notin \mathbb{k}$ and \(\text{mvar}(g_i) = X_i\),

H3 The polynomial set $C := \{\bar{s}, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lc}(t_i, X_i)$ is invertible modulo $\langle \bar{s}, g_2, \ldots, g_{i-1} \rangle$.

- Relaxing the \(\text{mvar}(r_i) = X_i\) part of H1 implies that the successive resultants $r_n, \ldots$ may have a gap in their sequence of main variables,
- we simply use the appropriate polynomials from $T = \{t_n, \ldots, t_2\}$ to fill those gaps.
The full modular algorithm: relaxing the hypotheses (2/3)

H1 for \(1 \leq i \leq n - 1\), we have \(r_i \notin k\) and \(\text{mvar}(r_i) = X_i\),

H2 For \(2 \leq i \leq n\), we have \(g_i \notin k\) and \(\text{mvar}(g_i) = X_i\),

H3 The polynomial set \(C := \{s, g_2, \ldots, g_n\}\) is a regular chain,

H4 For every \(2 \leq i \leq n\), \(\text{lc}(t_i, X_i)\) is invertible modulo \(\langle s, g_2, \ldots, g_{i-1} \rangle\).

- Relaxing the \(\text{mvar}(r_i) = X_i\) part of H1 implies that the successive resultants \(r_n, \ldots\) may have a gap in their sequence of main variables.
- We simply use the appropriate polynomials from \(T = \{t_n, \ldots, t_2\}\) to fill those gaps.
- Handling this modification comes at no cost.
The full modular algorithm: relaxing the hypotheses (3/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \notin k$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{\bar{s}, g_2, \ldots, g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lce}(t_i, X_i)$ is invertible modulo $\langle \bar{s}, g_2, \ldots, g_{i-1} \rangle$. 
The full modular algorithm: relaxing the hypotheses (3/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin \mathbf{k}$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \notin \mathbf{k}$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{\bar{s}, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lcm}(t_i, X_i)$ is invertible modulo $\langle \bar{s}, g_2, \ldots, g_{i-1} \rangle$.

- When either H2, H3, or H4 fails
The full modular algorithm: relaxing the hypotheses (3/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \notin k$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{\overline{s}, g_2, \ldots, g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lc}(t_i, X_i)$ is invertible modulo $\langle \overline{s}, g_2, \ldots, g_{i-1} \rangle$.

- When either H2, H3, or H4 fails
  1 the “candidate” regular chain $C$ must split, and
The full modular algorithm: relaxing the hypotheses (3/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \notin k$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{s, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lc}(t_i, X_i)$ is invertible modulo $\langle s, g_2, \ldots, g_{i-1} \rangle$.

- When either H2, H3, or H4 fails
  1. the “candidate” regular chain $C$ must split, and
  2. some subresultants of index higher than 1 must be used.
The full modular algorithm: relaxing the hypotheses (3/3)

H1 for \(1 \leq i \leq n - 1\), we have \(r_i \notin \mathbf{k}\) and \(\text{mvar}(r_i) = X_i\),

H2 For \(2 \leq i \leq n\), we have \(g_i \notin \mathbf{k}\) and \(\text{mvar}(g_i) = X_i\),

H3 The polynomial set \(C := \{s, g_2, \ldots, g_n\}\) is a regular chain,

H4 For every \(2 \leq i \leq n\), \(\text{lcm}(t_i, X_i)\) is invertible modulo \(\langle s, g_2, \ldots, g_{i-1} \rangle\).

- When either H2, H3, or H4 fails
  1. the “candidate” regular chain \(C\) must split, and
  2. some subresultants of index higher than 1 must be used.

- Costs for handling this:
The full modular algorithm: relaxing the hypotheses (3/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin \mathbf{k}$ and $\text{mvar}(r_i) = X_i$,
H2 For $2 \leq i \leq n$, we have $g_i \notin \mathbf{k}$ and $\text{mvar}(g_i) = X_i$,
H3 The polynomial set $C := \{s, g_2, \ldots g_n\}$ is a regular chain,
H4 For every $2 \leq i \leq n$, $\text{lc}(t_i, X_i)$ is invertible modulo $\langle s, g_2, \ldots, g_{i-1} \rangle$.

- When either H2, H3, or H4 fails
  1. the “candidate” regular chain $C$ must split, and
  2. some subresultants of index higher than 1 must be used.
- Costs for handling this:
  1. computing resultants and GCDs modulo regular chains by evaluation and interpolation, which is what this whole algorithm is about,
The full modular algorithm: relaxing the hypotheses (3/3)

H1 for \(1 \leq i \leq n - 1\), we have \(r_i \notin k\) and \(\text{mvar}(r_i) = X_i\),

H2 For \(2 \leq i \leq n\), we have \(g_i \notin k\) and \(\text{mvar}(g_i) = X_i\),

H3 The polynomial set \(C := \{\bar{s}, g_2, \ldots, g_n\}\) is a regular chain,

H4 For every \(2 \leq i \leq n\), \(\text{lc}(t_i, X_i)\) is invertible modulo \(\langle \bar{s}, g_2, \ldots, g_{i-1} \rangle\).

- When either H2, H3, or H4 fails
  1. the “candidate” regular chain \(C\) must split, and
  2. some subresultants of index higher than 1 must be used.

- Costs for handling this:
  1. computing resultants and GCDs modulo regular chains by evaluation and interpolation, which is what this whole algorithm is about,
The full modular algorithm: relaxing the hypotheses (3/3)

H1 for $1 \leq i \leq n - 1$, we have $r_i \notin k$ and $\text{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \notin k$ and $\text{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{s, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\text{lc}(t_i, X_i)$ is invertible modulo $\langle s, g_2, \ldots, g_{i-1} \rangle$.

When either H2, H3, or H4 fails

1. the “candidate” regular chain $C$ must split, and
2. some subresultants of index higher than 1 must be used.

Costs for handling this:

1. computing resultants and GCDs modulo regular chains by evaluation and interpolation, which is what this whole algorithm is about,
2. interpolating those subresultants of higher index
Outline

1. Triangular decompositions in polynomial system solving

2. Modular methods in polynomial system solving

3. A Modular methods for incremental triangular decompositions

4. Conclusions
Conclusions

- We have discussed \( \text{Intersect}(f, T) \) which computes \( V(f) \cap W(T) \) and which is at the core of the incremental method for triangular decompositions.

- We have presented a modular method for \( \text{Intersect}(f, T) \) focusing on the case where \( T \) is dimension one.

- This method allows us to get rid off of the large extraneous factors occurring in iterated resultant computations.

- For technical details (in particular degree bounds) see our CASC 2023.

- The experimentation reported there is based on an implementation which does not support yet the relaxation of our hypotheses (thus providing no benefits when those hypotheses do not hold).

- This modular method is designed to take advantage of FFT-based algorithms (speculative methods for computing subresultant chains, see our CASC 2022 paper).

- Parallel execution: multiple specialization can be done concurrently.
Thank You!

http://www.bpaslib.org/
References


