Modular algorithms for computing triangular decompositions of polynomial systems

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Acknowledgements

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- Criteria for selecting the **algorithms supporting the solvers**:
  ↳ provide a `comprehensive and coherent` set of tools for manipulating polynomial systems,
  ↳ implement solvers with both `general algorithms` (which may not be the most efficient ones) and `faster algorithms` (which may only work under some assumptions).

The BPAS library

Basic Polynomial Algebra Subprograms

http://www.bpaslib.org/

A high-performance polynomial algebra library

- Core of library written in C, wrapped in C++ interface for usability and object-oriented programming

Optimized algorithms and data structures, data locality, and parallelism

- Sparse multivariate polynomials [1], dense univariate and bivariate [7]
- Triangular decomposition of polynomial systems [2, 3]

User-friendly, object-oriented interface based on template meta-programming [6]

- A natural encoding of the algebraic hierarchy
- “Dynamic” creation of algebraic types through composition
- Compile-time type safety between algebraic types

Generic support for parallel programming and parallel patterns (this talk)

Outline

1. Triangular decompositions in polynomial system solving

2. Modular methods in polynomial system solving

3. A Modular methods for incremental triangular decompositions

4. Conclusions

Milestones (1/3)

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$$V(F) := \{z \in \mathbf{K}^n \ \mid \ f(z) = 0 \ \ (\forall \ f \in F)\}.$$

The variety V is *irreducible* if for all varieties $V_1, V_2 \subset \mathbf{K}^n$

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- **Theorem** (E. Lasker, 1905) *For each variety $V \subset \mathbf{K}^n$ there exist finitely many irreducible varieties $V_1, \ldots, V_e \subset \mathbf{K}^n$* such that

$$V = V_1 \cup \cdots \cup V_e.$$

Moreover, if $V_i \not\subseteq V_j$ for $1 \leq i < j \leq e$ then $\{V_1, \ldots, V_e\}$ is unique. This is the *irreducible decomposition of* V.

Milestones (2/3)

- **Theorem** (J.F. Ritt, 1932) *Let* $V \subset \mathbf{K}^n$ *be an* irreducible *non-empty variety and let* $F \subset \mathbf{k}[x_1, \ldots, x_n]$ *s.t.* $V = V(F)$. *Then, one can compute a (reduced) triangular set* $T \subset \langle F \rangle$ *s.t.*

$$(\forall\ g \in \langle F \rangle) \quad \mathrm{prem}(g, T) = 0.$$

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  This leads to a factorization-free *algorithm for decomposing varieties (but not into irreducible components).*

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- **Example.** Applying the `charset` procedure to $F = \{x_2^2 - x_1, x_1 x_3^2 - 2x_2x_3 + 1, (x_2x_3 - 1)x_4^2 + x_2^2\}$ produces $T = F$. However $V(F) = \varnothing$. Indeed

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- Moreover, for any input $F \subseteq \mathbf{k}[x_1, \ldots, x_n]$ one can compute regular chains $T_1, \ldots, T_e$ such that a point $z \in \mathbf{K}^n$ is a zero of F if and only if z is a zero of one of the $T_1, \ldots, T_e$ (in some technical sense). (Dong Ming Wang 2000), (Marc Moreno Maza 2000).

A recursive view on polynomials

Let $\mathbf{k}$ be a field, $X = x_1 < \cdots < x_n$ be variables and $f, g \in \mathbf{k}[X]$ with $g \notin \mathbf{k}$.

- $\mathrm{mvar}(g)$: the greatest variable in g is the *leader* or *main variable* of g,
- $\mathrm{init}(g)$: the leading coefficient of g w.r.t. $\mathrm{mvar}(g)$ is the *initial* of g,
- $\mathrm{mdeg}(g)$: the degree of g w.r.t. $\mathrm{mvar}(g)$,
- $\mathrm{rank}(g)$ $= v^d$ where $v = \mathrm{mvar}(g)$ and $d = \mathrm{mdeg}(g)$,
- $\mathrm{pdivide}(f, g)$ $= (q, r)$ with $q, r \in \mathbf{k}[X]$, $\deg(r, v_g) < d_g$ and $h_g^e f = qg + r$ where $h_g = \mathrm{init}(g)$, $e = \max(\deg(f, v) - d_g + 1, 0)$, $v_g = \mathrm{mvar}(g)$ and $d_g = \mathrm{mdeg}(g)$,

Example

Assume $n \geq 3$. If $p = x_1 x_3^2 - 2x_2 x_3 + 1$, then we have $\mathrm{mvar}(p) = x_3$, $\mathrm{mdeg}(p) = 2$, $\mathrm{init}(p) = x_1$ and $\mathrm{rank}(p) = x_3^2$.

Go to RegularChains.pdf Section 2.1.

Regular chain

Definition

The set $T \subset \mathbf{k}[x_n > \cdots > x_1]$ is triangular set if it consists of non-constant polynomials with pair-wise different main variables.
Define $h_T := \prod_{t \in T} \operatorname{init}(t)$, where $\operatorname{init}(t) = \operatorname{lc}(t, \operatorname{mvar}(t))$.
The *quasi-component* and *saturated ideal* of T are:

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Note that for all triangular set T we have:

- $\overline{W(T)} = V(\mathrm{sat}(T))$.
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Definition (M. Kalkbrner, 1991 - L. Yang, J. Zhang 1991)

T is a *regular chain* if $T = \varnothing$ or $T := T' \cup \{t\}$ with $\mathrm{mvar}(t)$ maximum s.t.

- T' is a regular chain,
- $\mathrm{init}(t)$ is regular modulo $\mathrm{sat}(T')$.

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Let $T \subset \mathbf{k}[x_n > \cdots > x_1]$ be a triangular set and $p \in \mathbf{k}[x_n > \cdots > x_1]$. If T is empty then, the *iterated resultant* of p w.r.t. T is resultant$(T,p) = p$. Otherwise, writing $T = T_{<w} \cup T_w$

$$\text{resultant}(T,p) \;=\; \begin{cases} p & \text{if } \deg(p,w) = 0 \\ \text{resultant}(T_{<w}, \text{resultant}(T_w, p, w)) & \text{otherwise} \end{cases}$$

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T *is a* regular chain *iff* $\{p \mid \mathrm{prem}(p, T) = 0\} \quad = \quad \mathrm{sat}(T)$.

Definition

Let $T \subset \mathbf{k}[x_n > \cdots > x_1]$ be a triangular set and $p \in \mathbf{k}[x_n > \cdots > x_1]$. If T is empty then, the *iterated resultant* of p w.r.t. T is resultant$(T, p) = p$. Otherwise, writing $T = T_{<w} \cup T_w$

$$\text{resultant}(T, p) \quad = \quad \begin{cases} p & \text{if } \deg(p, w) = 0 \\ \text{resultant}(T_{<w}, \text{resultant}(T_w, p, w)) & \text{otherwise} \end{cases}$$

Theorem (L. Yang, J. Zhang 1991)

p *is regular modulo* $\mathrm{sat}(T)$ *iff* resultant$(T, p) \neq 0$.

Triangular decomposition of an algebraic variety

Kalkbrener triangular decomposition

Let $F \subset \mathbf{k}[\mathbf{x}]$. A family of regular chains $T_1, \ldots, T_e$ of $\mathbf{k}[\mathbf{x}]$ is called a Kalkbrener triangular decomposition of $V(F)$ if

$$V(F) \quad = \quad \cup_{i=1}^{e} V(\mathrm{sat}(T_i)).$$

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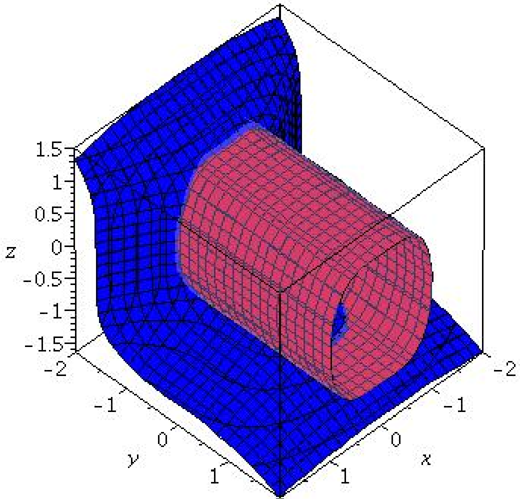
Let $F \subset \mathbf{k}[\mathbf{x}]$. A family of regular chains $T_1, \ldots, T_e$ of $\mathbf{k}[\mathbf{x}]$ is called a Wu-Lazard triangular decomposition of $V(F)$ if

$$V(F) \;=\; \cup_{i=1}^{e} W(T_i).$$

Triangularize applied to *sofa* and *cylinder* (1/2)

$$x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0$$



Triangularize applied to *sofa* and *cylinder* (2/2)

Applications Places System Sun Feb 13, 10:36 PM changbo

/home/changbo/src/maple/triade/mapleP4/lib/cylinder.mw - [Server 1] - Maple 14

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

```
> R := PolynomialRing([z, y, x]): F := [x^2+y^3+z^5, x^4+z^2-1]: dec :=
  Triangularize(F, R): map(Display, dec, R);
```

$$\left[\left\{\begin{array}{c} \left(-2\,x^4+x^8+1\right)z+x^2+y^3=0 \\ y^6+2\,x^2y^3+10\,x^{12}-10\,x^8+x^{20}-5\,x^{16}+6\,x^4-1=0 \\ -2\,x^4+x^8+1\neq 0 \end{array}\right.\right]$$

```
> dec := Triangularize(F, R, output=lazard): map(Display, dec, R);
```

$$\left[\left\{\begin{array}{c} \left(-2\,x^4+x^8+1\right)z+x^2+y^3=0 \\ y^6+2\,x^2y^3+10\,x^{12}-10\,x^8+x^{20}-5\,x^{16}+6\,x^4-1=0 \\ -2\,x^4+x^8+1\neq 0 \end{array}\right., \left\{\begin{array}{c} z=0 \\ y^2+y+1=0 \\ x^2+1=0 \end{array}\right.,\right.$$

$$\left\{\begin{array}{c} z=0 \\ y-1=0 \\ x^2+1=0 \end{array}\right., \left\{\begin{array}{c} z=0 \\ y^2-y+1=0 \\ x+1=0 \end{array}\right., \left\{\begin{array}{c} z=0 \\ y^2-y+1=0 \\ x-1=0 \end{array}\right., \left\{\begin{array}{c} z=0 \\ y+1=0 \\ x+1=0 \end{array}\right.,$$

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Let $P = \left(y^2 + 6\right)(x - 1) - y\left(x^2 + 1\right)$ and $Q = \left(x^2 + 6\right)(y - 1) - x\left(y^2 + 1\right)$

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- $\mathrm{gcd}(\mathrm{P}, \mathrm{Q}, \mathrm{x}-3=0) = (\mathrm{y}-2)(\mathrm{y}-3)$.
- $\mathrm{gcd}(\mathrm{P}, \mathrm{Q}, \mathrm{x}^2-\mathrm{x}+4=0) = (2\,\mathrm{x}-1)\,\mathrm{y} - 7 - \mathrm{x}$.

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Consider the following polynomials $f, g \in \mathbb{Q}[y < x]$:

$$\begin{array}{rcl} f & = & x^7 - 36x - 22y + 1, \\ g & = & x^6 + 47x^3 - 60xy^2 - 6xy - 83y^2 - 10y + 50. \end{array}$$

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The complete list of subresultants of (f, g) w.r.t. x is:

$$\begin{array}{rcl} S_6 &=& g, \\ S_5 &=& 56x^4 + 60x^2y^2 + 6x^2y + 83xy^2 + 10xy + 17x + 81y + 1, \\ S_4 &=& 46x^4 + 64x^2y^2 + 27x^2y + 13xy^2 + 45xy + 25x + 4y + 56, \\ S_3 &=& 74x^2y^4 + 7x^3y^2 + 56x^2y^3 + 44xy^4 + \cdots + 98y^2 + 86y + 53, \\ S_2 &=& 25x^2y^8 + 10x^2y^7 + 26xy^8 + 62x^2y^6 + \cdots + 96x + 72y + 43, \\ S_1 &=& 81xy^{12} + 28xy^{11} + 76y^{12} + 24xy^{10} + 5xy^9 + \cdots + 4x + 73y + 77, \\ S_0 &=& 97y^{15} + 82y^{14} + 82y^{13} + \cdots + 23y^5 + 89y^4 + 31y^3 + y^2 + 54y + 69. \end{array}$$

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The solutions of $f = g = 0$ can be calculated using S_0, S_1 only.
Go to RegularChains.pdf Sections 2.2 and 2.3.

Relations with resultants and subresultants (3/3)

Extending the previous ideas to solving m polynomial equations in n variables can be done using

- a cascade of Sylvester resultants (this talk), or
- a combination of Dixon/Macaulay resultants and Sylvester resultants (work in progress).

Relations with Gröbner bases

Normalized regular chains

- The regular chain $T \subset \mathbf{k}[x_n > \cdots > x_1]$ is said *normalized* if for every $t, t' \in T$ we have $\deg(\mathrm{init}(t), \mathrm{mvar}(t')) = 0$.

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- This is done at a cost which is at most that inverting at most $\#G$ polynomials modulo one of the ideals $\langle T_1 \rangle, \ldots, \langle T_e \rangle$.
- This is practically very effective.

Triangular decompositions: the incremental approach

- Let $f \in \mathbf{k}[x_1, \ldots, x_n]$ and $T \subseteq \mathbf{k}[x_1, \ldots, x_n]$ be a regular chain.

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$$V(f) \cap W(T) \subseteq W(T_1) \cup \cdots \cup W(T_e) \subseteq V(f) \cap \overline{W(T)},$$

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- Given $f_1, \ldots, f_m \in \mathbf{k}[x_1, \ldots, x_n]$, one can solve $f_1 = \cdots = f_m = 0$ using repeated calls to Intersect.
- Indeed, if $V(f_1, \ldots, f_{m-1}) = \cup_{i=1}^{e} W(T_i)$, then we have

$$V(f_1, \ldots, f_m) = \cup_{i=1}^{e} \, \mathsf{Intersect}(f_m, T_i).$$

(Daniel Lazard 1991), (M. 2000), (Changbo Chen & M. 2011-2012).

Outline

1. Triangular decompositions in polynomial system solving

2. Modular methods in polynomial system solving

3. A Modular methods for incremental triangular decompositions

4. Conclusions

Computing by homomotphic images: principles,

Examples

- The computation of the determinant of an integer matrix using the Chinese Remaindering Theorem.

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- Modular methods (1) may control expression swell, (2) allow sharper implementation (fine control memory), (3) open the door to FFT-based arithmetic, and (4) provide opportunities for concurrency.

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Advantages and issues

- Modular methods (1) may control expression swell, (2) allow sharper implementation (fine control memory), (3) open the door to FFT-based arithmetic, and (4) provide opportunities for concurrency.
- Modular methods are (1) generally harder to implement than direct methods, and (2) usually require change of representations which may come with significant costs in terms of memory consumption.

Expression swell may sometimes be handled in other ways

- Consider the system F (Barry Trager).

$$-x^5 + y^5 - 3y - 1 = 5y^4 - 3 = -20x + y - z = 0$$

We solve it for $z < y < x$.

- $V(F)$ is equiprojectable and its Lazard triangular set is
  - [illegible]
  - [illegible]
  - $3125z^{20} - 9375z^{16} - 40000000000z^{15} - 2015999988750z^{12} - 1560000000000z^{11} + 192000000000000000z^{10} - 121651253568000006750z^{8} - 1474560223200000000z^{7} - 6528000000000000000z^{6} - 4096000000000000000000000z^{5} - 16986908639233347839997975z^{4} - 141557671526403024000000000z^{3} - 589823873280000000000000z^{2} - 12288000000000000000000000z - 619530361923198287873244160024 3$
- Applying the transformation of Dahan and Schost leads to 1787 characters.
  - $(20z^{19} + (-48z^{15}) + (-192000000z^{14}) + (-(38707199784/5)z^{11}) + (-5491200000z^{10}) + 614400000000000z^{9} + (-(7785680228352004 32/25)z^{7}) + (-3303014899968000 0z^{6}) + (-125337600000000000z^{5}) + (-65536000000000000000z^{4}) + (-(27179053822773356543996 76/125)z^{3}) + (-1358953646653469030 4000z^{2}) + (-377487278899200000000z) - 3932160000000000000 00)x + 3200000z^{15} + 161280000z^{12} + 124800000z^{11} + (-30720000000000z^{10}) + 194641962854400 0z^{8} + 2359296178560000z^{7} + 10444800000000000z^{6} + 98304000000000000000z^{5} + 407685987827722782720 0z^{4} + 33973848244224241920 00z^{3} + 1415577397248000000000z^{2} + 294912000000000000000z + 198249699507965678059619 5328$
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  - $z^20 + (-3z^16) + (-12800000z^{15}) + (-(3225599982/5)z^{12}) + (-499200000z^{11}) + 61440000000000z^{10} + (-(97321002854400054/25)z^{8}) + (-4718592714240000z^{7}) + (-2088960000000000z^{6}) + (-131072000000000000000z^{5}) + (-(6794763455693339135999 19/125)z^{4}) + (-4529845488844896768000z^{3}) + (-188743639449600000000z^{2}) + (-393216000000000000000z) + (-619530361923198287873244160024 3/3125)$
- One can do better! Here's the regular chain produced by the `Triangularize` algorithm of the `RegularChains` library, counting 963 characters.
  - $20x - 1y + z$
  - $\left((4375z^{12} + 52800011625z^{8} + 32000000000z^{7} + 110591902080002925z^{4} + 614399808000000000z^{3} + 128000000000000000z^{2} + 566231172710410368000 27\right)y - 1875z^{13} - 9600010125z^{9} + 2000000000z^{8} - 73727147520045 45z^{5} + 307200024000000000z^{4} + 128000000000000000z^{3} - 22118403456000135z + 235929636864001440000 00$
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Trace algorithms

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- For characteristic sets, one can use the notion of rank as defined by Ritt and Wu (M. ACA 2003).

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- Using Hensel lifting techniques (Schost 2002) this leads to an effective modular method for Solver(F) (Dahan, M., Schost, Wu & Xie ISSAC 2005).

Issues with iterated subresultant chains (1/2)

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- In (C. Chen & M. JSC 2012) we give examples of 3 zero-dimensional systems with $4^3 = 64$ solutions where the extraneous factors have degree in the 1000's.

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- Thus, if $h_1 = \cdots = h_n = 1$, then we simply have:

$$\mathrm{res}(C, f) = R(C, f).$$

Outline

1. Triangular decompositions in polynomial system solving
2. Modular methods in polynomial system solving
3. A Modular methods for incremental triangular decompositions
4. Conclusions

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- However, we avoid random changes of coordinates and support decompositions in the sense of Lazard.

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- Let r_{i-1} and g_i be the subresultants of index 0 and 1 from $S(t_i, r_i, X_i)$.
- We denote by $\overline{s}$ the squarefree part of $s := r_1$.

Base result

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- C can be computed in a modular fashion (as we see shortly) improving performance w.r.t. the direct and non-modular approach.
- This modular method can be enhanced so that the 4 Hypotheses are no longer necessary (as we will see later).

$R := PolynomialRing([x3,\ x2,\ x1]):$

$f := (x2 + x1) \cdot x3^2 + x3 + 1;$

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$src1 := SubresultantChain(f, t3, x3, R):$
$g3 := SubresultantOfIndex(1, src1, R); r := SubresultantOfIndex(0, src1, R);$

$$g3 := x1\, x2\, x3 + x2^2\, x3 - x1\, x3 + x2$$
$$r := x1\, x2^2 + x2^3 - 2\, x1\, x2 + x1 \tag{4}$$

$R := PolynomialRing([x3,\ x2,\ x1]):$
$f := (x2 + x1) \cdot x3^2 + x3 + 1;$

$$f := (x2 + x1)\, x3^2 + x3 + 1 \tag{1}$$

$t3 := x1 \cdot x3^2 + x2 \cdot x3 + 1;$

$$t3 := x1\, x3^2 + x2\, x3 + 1 \tag{2}$$

$t2 := (x1 + 1) \cdot x2^2 + x2 + 2;$

$$t2 := (x1 + 1)\, x2^2 + x2 + 2 \tag{3}$$

$src1 := SubresultantChain(f, t3, x3, R):$
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$$r := x1\, x2^2 + x2^3 - 2\, x1\, x2 + x1 \tag{4}$$

$src2 := SubresultantChain(r, t2, x2, R):$
$g2 := SubresultantOfIndex(1, src2, R);\ s := SubresultantOfIndex(0, src2, R);$

$$g2 := -2\, x1^3\, x2 + x1^3 - 5\, x1^2\, x2 - 5\, x1\, x2 - x1 - x2 + 2$$
$$s := x1^5 + 9\, x1^4 + 24\, x1^3 + 38\, x1^2 + 13\, x1 + 8 \tag{5}$$

$$sol := Chain([s], Empty(R), R) : IsRegular(Initial(g2, R), sol, R);$$
$$true \tag{6}$$
$$sol2 := Chain([g2], sol, R) : IsRegular(Initial(g3, R), sol2, R);$$
$$true \tag{7}$$
$$IsRegular(Initial(t3, R), sol2, R);$$
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$sol3 := Chain([g3], sol2, R) : Display(sol3, R);$

$$\begin{cases} (x2^2 + x2\,x1 - x1)\,x3 + x2 = 0 \\ (-2\,x1^3 - 5\,x1^2 - 5\,x1 - 1)\,x2 + x1^3 - x1 + 2 = 0 \\ x1^5 + 9\,x1^4 + 24\,x1^3 + 38\,x1^2 + 13\,x1 + 8 = 0 \\ x2^2 + x2\,x1 - x1 \neq 0 \\ -2\,x1^3 - 5\,x1^2 - 5\,x1 - 1 \neq 0 \end{cases} \tag{9}$$

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$dec3 := Triangularize([f, t3, t2], R) : Display(dec3[1], R);$

$$\begin{cases} (x2^2 + x2\,x1 - x1)\,x3 + x2 = 0 \\ (2\,x1^3 + 5\,x1^2 + 5\,x1 + 1)\,x2 - x1^3 + x1 - 2 = 0 \\ x1^5 + 9\,x1^4 + 24\,x1^3 + 38\,x1^2 + 13\,x1 + 8 = 0 \\ x2^2 + x2\,x1 - x1 \neq 0 \\ 2\,x1^3 + 5\,x1^2 + 5\,x1 + 1 \neq 0 \end{cases} \tag{10}$$

The modular algorithm in $\mathbb{F}_p$ under our hypotheses

1. Evaluate f and T at sufficiently many (use the Bézout bound or the mixed volume) values a of X_1 so that T specializes well at $X_1 = a$ to a zero-dimensional regular chain T_a

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Technical details:

- specializations $X_1 = a$, $X_1 = b$, ... must produce faithful images of resultants r_i, that is, resultants of maximum degree. good $\neq$ faithful.

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  1. the number of non-faithful specializations, and
  2. the degree of $\overline{s}$; see the details in our CASC 2023 paper.
- we stop combining those images of the r_i's when the recombination of the images stabilizes (Monagan's probabilistic idea, ISSAC 2005).

The full modular algorithm: relaxing the hypotheses (1/3)

H1 for $1 \leq i \leq n-1$, we have $r_i \notin \mathbf{k}$ and $\mathrm{mvar}(r_i) = X_i$,

H2 For $2 \leq i \leq n$, we have $g_i \notin \mathbf{k}$ and $\mathrm{mvar}(g_i) = X_i$,

H3 The polynomial set $C := \{\overline{s}, g_2, \ldots g_n\}$ is a regular chain,

H4 For every $2 \leq i \leq n$, $\mathrm{lc}(t_i, X_i)$ is invertible modulo $\langle \overline{s}, g_2, \ldots, g_{i-1} \rangle$.

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  2. if $s \neq 0$, then Intersect$(f, T) = \varnothing$.
- Handling this modification only requires to possibly computing this GCD, whose cost is negligible.

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- Relaxing the $\mathrm{mvar}(r_i) = X_i$ part of H1 implies that the successive resultants $r_n, \ldots$ may have a gap in their sequence of main variables

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- we simply use the appropriate polynomials from $T = \{t_n, \ldots, t_2\}$ to fill those gaps.

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- we simply use the appropriate polynomials from $T = \{t_n, \ldots, t_2\}$ to fill those gaps.
- Handling this modification comes at no cost.

The full modular algorithm: relaxing the hypotheses (3/3)

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  1. the "candidate" regular chain C must split, and
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  2. interpolating those subresultants of higher index

Outline

1. Triangular decompositions in polynomial system solving
2. Modular methods in polynomial system solving
3. A Modular methods for incremental triangular decompositions
4. Conclusions

Conclusions

- We have discussed Intersect(f, T) which computes $V(f) \cap W(T)$ and which is at the core of the incremental method for triangular decompositions
- We have presented a modular method for Intersect(f, T) focusing on the case where T is dimension one.
- This method allows us to get rid off of the large extraneous factors occurring in iterated resultant computations
- For technical details (in particular degree bounds) see our CASC 2023.
- The experimentation reported there is based on an implementation which does not support yet the relaxation of our hypotheses (thus providing no benefits when those hypotheses do not hold).

- This modular method is designed to take advantage of FFT-based algorithms (speculative methods for computing subresultant chains, see our CASC 2022 paper).
- Parallel execution: multiple specialization can be done concurrently.

Thank You!

Basic Polynomial Algebra Subprograms

http://www.bpaslib.org/

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