Around Montgomery’s trick: A taste of a bit hack

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CS 4435 - CS 9624
Introduction

- Let $a, b, p$ be number-like objects (integer numbers, univariate polynomials over a field) and such that $p \not\in \{-1, 0, 1\}$.
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- More formally let $a, b, p$ be elements in an Euclidean domain with $p$ not a unit.
- Computing $(a, b, p) \mapsto (ab) \mod p$ is a fundamental and challenging operation.
- If $a, b, p$ have large sizes, then FFT-based arithmetic and the fast division trick (S. Cook, 1966) (H. T. Kung, 1974) and (M. Sieveking, 1972) provides a practically efficient solution.
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If \( a, b, p \) have small sizes, say are machine integers, then enter Peter Montgomery and his famous reduction (Math. Computation, vol. 44, pp. 519–521, 1985) improved by Xin Li in his PhD thesis (University of Western Ontario 2009).
The Original Montgomery Trick (1/2)

- Let \( x, p \) be integers such that \( p \geq 2 \). In practice \( p \) is a prime. We shall compute \( x \) mod \( p \) in an \textit{indirect way}.
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- Let $x, p$ be integers such that $p \geq 2$. In practice $p$ is a prime. We shall compute $x \mod p$ in an indirect way.
- Consider a positive integer $R \geq p$ such that $\gcd(R, p) = 1$. Hence there exists integers $R^{-1}, p'$ such that

$$RR^{-1} - pp' = 1 \quad \text{and} \quad 0 < p' < R.$$
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Consider the following two Euclidean divisions:

$$
\begin{array}{c|c}
  x & R \\
  d & c \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
  dp' & R \\
  f & e \\
\end{array}
$$

Therefore $x + dp'$ writes $qR$ and thus $x \equiv q \mod p$. 

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- Consider the following two Euclidean divisions:

\[
\begin{array}{c|cc}
  x & R & d \\
  \hline 
  d & c & & R \\
  \end{array}
\quad \text{and} \quad
\begin{array}{c|cc}
  dp' & R \\
  \hline 
  f & e & \\
  \end{array}
\]

- Hence we have:

\[
x + fp = cR + d + (dp' - eR)p = cR + d(1 + pp') - epR.
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  dp' & R \\
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\end{array}.$$

- Hence we have:

$$x + fp = cR + d + (dp' - eR)p = cR + d(1 + pp') - epR.$$ 

- Therefore $x + fp$ writes $qR$ and thus $\frac{x}{R} \equiv q \mod p$. 

Around Montgomery’s trick: A taste of a bit hack
Suppose $p > 2$ is a prime and $R$ is a power of 2. Then we have obtained a procedure computing $\frac{x}{R} \mod p$ for $0 \leq x < p^2$, amounting to 2 multiplications, 2 additions and 3 shifts.
The Original Montgomery Trick (2/2)

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- Recall the three divisions:

\[
\begin{array}{c|c}
x & R \\
\hline
d & c \\
\end{array} \quad \text{and} \quad 
\begin{array}{c|c}
dp' & R \\
\hline
f & e \\
\end{array} \quad \text{and} \quad 
\begin{array}{c|c}
x + fp & R \\
\hline
0 & q \\
\end{array}
\]
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Recall the three divisions:

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\begin{array}{c|c}
\frac{x}{d} & \frac{R}{c} \\
\hline
\frac{dp'}{f} & \frac{R}{e} \\
\hline
\frac{x + fp}{0} & \frac{R}{q}
\end{array}
\]

The result is $q$ or $q - p$ since \(\frac{x}{R} \equiv q \mod p\) and we have:

\[0 \leq x < p^2 \implies 0 \leq q < 2p.\]
The Original Montgomery Trick (2/2)

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- Recall the three divisions:

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  x & R & c \\
  \hline
  d & R & f \\
  \hline
  e & \hline
  \end{array}$

- The result is $q$ or $q - p$ since $\frac{x}{R} \equiv q \mod p$ and we have:

  $0 \leq x < p^2 \Rightarrow 0 \leq q < 2p$.

- To compute in $\mathbb{Z}/p\mathbb{Z}$, we map each $a \in \mathbb{Z}/p\mathbb{Z}$ to $aR \in \mathbb{Z}/p\mathbb{Z}$. Then the above procedure gives us $\frac{aRbR}{R} \mod p$, that is, the image of $ab$ in this new representation.
The Improved Montgomery Trick (1/5)

- Suppose $p > 2$ is a Fourier prime, that is, $p - 1 = c2^n$ and $\ell \leq 2n$ where $\ell = \lceil \log_2(p) \rceil \leq b$ on $b$-bit machine words.
The Improved Montgomery Trick (1/5)

- Suppose $p > 2$ is a **Fourier prime**, that is, $p - 1 = c2^n$ and $\ell \leq 2n$ where $\ell = \lceil \log_2(p) \rceil \leq b$ on $b$-bit machine words.

- Let $R := 2^\ell$ and $0 \leq x \leq (p - 1)^2$. We get $\frac{x}{R} \mod p$ by:

  $\begin{array}{c|c|c|c|c|c|c}
  x & R & c2^nr_1 & R & c2^nr_2 & R \\
  r_1 & q_1 & & r_2 & q_2 & 0 \\
  \end{array}$
The Improved Montgomery Trick (1/5)

- Suppose \( p > 2 \) is a Fourier prime, that is, \( p - 1 = c2^n \) and \( \ell \leq 2n \) where \( \ell = \lceil \log_2(p) \rceil \leq b \) on \( b \)-bit machine words.

- Let \( R := 2^\ell \) and \( 0 \leq x \leq (p - 1)^2 \). We get \( \frac{x}{R} \mod p \) by:

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\begin{array}{c|c}
    x & R \\
    r_1 & q_1 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
    c2^n r_1 & R \\
    r_2 & q_2 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
    c2^n r_2 & R \\
    0 & q_3 \\
\end{array}
\]

- Using \( c2^n \equiv -1 \mod p \) we have:

\[
\frac{x}{R} \equiv q_1 + \frac{r_1}{R} \equiv q_1 - q_2 - \frac{r_2}{R} \equiv q_1 - q_2 + q_3 \mod p.
\]
The Improved Montgomery Trick (1/5)

- Suppose $p > 2$ is a Fourier prime, that is, $p - 1 = c2^n$ and $\ell \leq 2n$ where $\ell = \lceil \log_2(p) \rceil \leq b$ on $b$-bit machine words.
- Let $R := 2^\ell$ and $0 \leq x \leq (p - 1)^2$. We get $\frac{x}{R}$ mod $p$ by:

  \[
  \begin{array}{c|c c c|c c c|c c c}
  x & R & c2^n r_1 & R & c2^n r_2 & R \\
  r_1 & q_1 & r_2 & q_2 & 0 & q_3 \\
  \end{array}
  \]

- Using $c2^n \equiv -1$ mod $p$ we have:

  \[
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  \]

- The last equality requires a proof. We have:

  \[
  r_2 = c2^n r_1 - q_2 R = c2^n r_1 - q_2 2^\ell.
  \]

  Hence $2^n \mid r_2$ thus $2^{2n} \mid c2^n r_2$ and $R \mid c2^n r_2$. 

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Recall \( R := 2^\ell \) and \( 0 \leq x \leq (p - 1)^2 \). We get \( \frac{x}{R} \mod p \) by:

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\begin{array}{c|c}
 x & R \\
r_1 & q_1 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
 c2^n r_1 & R \\
r_2 & q_2 \\
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\quad \text{and} \quad
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 c2^n r_2 & R \\
0 & q_3 \\
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\]

leading to \( \frac{x}{R} \equiv q_1 - q_2 + q_3 \mod p \).
The Improved Montgomery Trick (2/5)

- Recall $p > 2$ is a Fourier prime, that is, $p - 1 = c2^n$ and $\ell \leq 2n$ where $\ell = \lceil \log_2(p) \rceil \leq b$ on $b$-bit machine words.

- Recall $R := 2^\ell$ and $0 \leq x \leq (p - 1)^2$. We get $\frac{x}{R}$ mod $p$ by:

$$
\begin{align*}
x & \equiv \frac{R}{q_1} \text{ and } \frac{c2^n r_1}{r_2} \equiv \frac{R}{q_2} \text{ and } \frac{c2^n r_2}{0} \equiv \frac{R}{q_3}
\end{align*}
$$

leading to $\frac{x}{R} \equiv q_1 - q_2 + q_3 \mod p$.

- Moreover we have:

$$-(p - 1) < q_1 - q_2 + q_3 < 2(p - 1).$$

Hence the desired output is either $(q_1 - q_2 + q_3) + p$, or $q_1 - q_2 + q_3$ or $(q_1 - q_2 + q_3) - p$. 
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- Recall $R := 2^\ell$ and $0 \leq x \leq (p - 1)^2$. We get $\frac{x}{R} \mod p$ by:

$$
\begin{array}{c|c|c|c|c|c|}
 & R & & R & & R \\
q_1 & r_1 & c2^n & r_2 & c2^n & r_3 \\
\end{array}
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leading to $\frac{x}{R} \equiv q_1 - q_2 + q_3 \mod p$.
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- Indeed $0 \leq x \leq (p - 1)^2$ and $p \leq R$ imply $q_1 = \frac{x}{R} \mod R \leq (p - 1)^2/R < p - 1$.

Next, we have: $q_2 = c2^n r_1 \mod R < c2^n = p - 1$, since $r_1 < R$. Similarly, we have $q_3 < p - 1$. 
The Improved Montgomery Trick (3/5)

We describe now the C implementation for 32-bit machine integer assuming that we have at hand the following function:

```c
/**
 * Input : The addresses of two unsigned machine integers a, b
 * Output : Store (a * b) quo 2^32 into a, and
 *          store (a * b) mod 2^32 into b
 */

inline void MulHiLoUnsigned (uint32_t *a, uint32_t *b) {
    uint64_t prod;
    prod = (uint64_t)(*a) * (uint64_t)(*b);

    *a = (uint32_t) (prod >> 32);
    *b = (uint32_t) prod;
}
```
The Improved Montgomery Trick (4/5)

- Recall $p > 2$ is a Fourier prime, that is, $p - 1 = c2^n$ and $\ell \leq 2n$ where $\ell = \lceil \log_2(p) \rceil$. Recall $R := 2^\ell$.

- Let $a, b$ be non-negative 32-bit machine integers less than $p$. We show how to compute $\frac{ab}{R} \mod p$. 

- Let $A := q_1 - q_2 + q_3$. Then we execute the following code:

  \[ A += (A >> 31) \& p; \]
  \[ A -= p; \]
  \[ A += (A >> 31) \& p; \]

  Finally we have performed 6 shifts, 5 additions, 2 64-bit multiplications and 1 32-bit multiplication.
The Improved Montgomery Trick (4/5)

- Recall $p > 2$ is a Fourier prime, that is, $p - 1 = c2^n$ and $\ell \leq 2n$ where $\ell = \lceil \log_2(p) \rceil$. Recall $R := 2^\ell$.
- Let $a, b$ be non-negative 32-bit machine integers less than $p$. We show how to compute $\frac{ab}{R} \mod p$.
- $q_1, 2^{32-\ell}r_1 := \text{MulHiLoUnsigned}(a, 2^{32-\ell}b)$
Recall $p > 2$ is a Fourier prime, that is, $p - 1 = c2^n$ and $\ell \leq 2n$ where $\ell = \lceil \log_2(p) \rceil$. Recall $R := 2^\ell$.

Let $a, b$ be non-negative 32-bit machine integers less than $p$. We show how to compute $\frac{ab}{R} \mod p$.

$q_1, 2^{32-\ell}r_1 := \text{MulHiLoUnsigned}(a, 2^{32-\ell}b)$

$q_2, 2^{32-\ell}r_2 := \text{MulHiLoUnsigned}(2^{32-\ell}r_1, 2^n c)$

Let $A := q_1 - q_2 + q_3$. Then we execute the following code:

$A += (A >> 31) & p$
$A -= p$
$A += (A >> 31) & p$

Finally we have performed 6 shifts, 5 additions, 2 64-bit multiplications and 1 32-bit multiplication.
Recall $p > 2$ is a Fourier prime, that is, $p - 1 = c 2^n$ and $\ell \leq 2n$ where $\ell = \lceil \log_2(p) \rceil$. Recall $R := 2^\ell$.

Let $a, b$ be non-negative 32-bit machine integers less than $p$. We show how to compute $\frac{ab}{R} \mod p$.

- $q_1, 2^{32-\ell} r_1 := \text{MulHiLoUnsigned}(a, 2^{32-\ell} b)$
- $q_2, 2^{32-\ell} r_2 := \text{MulHiLoUnsigned}(2^{32-\ell} r_1, 2^n c)$
- $q_3 := c \frac{r_2}{2^{\ell-n}}$. The division $\frac{r_2}{2^{\ell-n}}$ is exact and the multiplication $c \frac{r_2}{2^{\ell-n}}$ is correct on 32 bits.
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- Let $a, b$ be non-negative 32-bit machine integers less than $p$. We show how to compute $\frac{ab}{R} \mod p$.
- $q_1, 2^{32-\ell}r_1 := \text{MulHiLoUnsigned}(a, 2^{32-\ell}b)$
- $q_2, 2^{32-\ell}r_2 := \text{MulHiLoUnsigned}(2^{32-\ell}r_1, 2^nc)$
- $q_3 := c \frac{r_2}{2^{\ell-n}}$. The division $\frac{r_2}{2^{\ell-n}}$ is exact and the multiplication $c \frac{r_2}{2^{\ell-n}}$ is correct on 32 bits.
- Let $A := q_1 - q_2 + q_3$. Then we execute the following code:
  - $A += (A >> 31) \& p$;
  - $A -= p$;
  - $A += (A >> 31) \& p$;
- Finally we have performed 6 shifts, 5 additions, 2 64-bit multiplications and 1 32-bit multiplication.
Consider $p = 257 = 1 + 2^8$. Hence $c = 1$, $n = 8$, $\ell = 9$ and $R = 2^9$.

Take $a = 131$ and $b = 187$.

Compute $2^{32-\ell} b = 1568669696$.

Compute $q_1 = 47$ and $2^{32-\ell} r_1 = 3632267264$.

Compute $q_2 = 216$ and $2^{32-\ell} r_2 = 2147483648$.

Compute $q_3 = c \frac{r_2}{2^{\ell-n}} = 128$.

Compute $A = q_1 - q_2 + q_3 = -41$.

Ajust to get $\frac{ab}{R} \equiv 216 \mod p$. 

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**The Improved Montgomery Trick (5/5)**

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- Take $a = 131$ and $b = 187$.
- Compute $2^{32-\ell} b = 1568669696$.
- Compute $q_1 = 47$ and $2^{32-\ell} r_1 = 3632267264$.
- Compute $q_2 = 216$ and $2^{32-\ell} r_2 = 2147483648$.
- Compute $q_3 = c \frac{r_2}{2^{\ell-n}} = 128$.
- Compute $A = q_1 - q_2 + q_3 = -41$.
- Ajust to get $\frac{ab}{R} \equiv 216 \mod p$. 