On multivariate Birkhoff rational interpolation

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Outline

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PROBLEM DESCRIPTION

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- The multivariate Birkhoff rational interpolation is one of the most general algebraic interpolation schemes.
- The key character of Birkhoff interpolation is that the orders of the derivative conditions at some nodes are non-continuous.
  For example, $f(x_0) = a$, $\frac{d^2}{dx^2} f(x_0) = b$.
- Without the non-continuity, the problem degenerates into Hermite rational interpolation.
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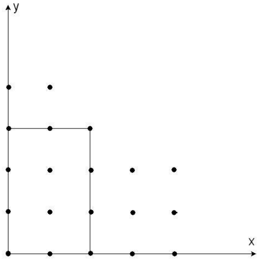
Lower set

Let $L(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in \mathbb{N}_0^n : \beta_i \leq \alpha_i,\ i = 1, \ldots, n\}$.



Let $S \subset \mathbb{N}_0^n$, if $\forall \boldsymbol{\alpha} \in S$, $L(\boldsymbol{\alpha}) \subset S$, then S is a lower set.

PROBLEM DESCRIPTION

A multivariate Birkhoff rational interpolation scheme consists of two components.

a) A set of nodes Z, $Z = \{Y_i\}_{i=1}^m = \{(y_{i,1}, \ldots, y_{i,n})\}_{i=1}^m$, where $Y_i \in K^n$, K is a field.

b) The derivative conditions S_i at each node Y_i, $i = 1, \ldots, m$, where S_i is a subset of $\mathbb{N}_0^n$. Some S_i's ($i = 1, \ldots, m$) may not be lower sets.

PROBLEM DESCRIPTION

The multivariate Birkhoff rational interpolation problem is to find a rational function $r(X) = \frac{p(X)}{q(X)}$ satisfying

$$D^{\boldsymbol{\alpha}} r(Y_i) = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} r(Y_i) = c_{i,\boldsymbol{\alpha}}, \quad \forall \boldsymbol{\alpha} \in S_i, \tag{1}$$

where $p(X) \in \mathscr{P}_{T_1} = \{p \mid p(X) = p(x_1,\ldots,x_n) = \sum_{\boldsymbol{\alpha}_i \in T_1} a_i x_1^{\alpha_1}\cdots x_n^{\alpha_n}\}$, $q(X) \in \mathscr{P}_{T_2} = \left\{q \mid q(X) = q(x_1,\ldots,x_n) = \sum_{\boldsymbol{\beta}_i \in T_2} b_i x_1^{\beta_1}\cdots x_n^{\beta_n}\right\}$, $a_i,\ b_i \in K$, T_1, T_2 are subsets of $\mathbb{N}_0^n$, $c_{i,\boldsymbol{\alpha}} \in K$ are given constants.

PROBLEM DESCRIPTION

Example

Let $Y_1 = (0,0)$, $\quad Y_2 = (0,1)$,

$S_1 = \{(0,0),(0,1),(1,1)\}$, $\quad S_2 = \{(0,0),(1,0),(1,1)\}$,

$V_1 = \{6,5,0\}$, $\quad V_2 = \{7,2,-1\}$.

$$f(X)|_{X=(0,0)} = 6, \quad \frac{\partial}{\partial y} f(X)|_{X=(0,0)} = 5, \quad \frac{\partial^2}{\partial x \partial y} f(X)|_{X=(0,0)} = 0;$$

$$f(X)|_{X=(0,1)} = 7, \quad \frac{\partial}{\partial x} f(X)|_{X=(0,1)} = 2, \quad \frac{\partial^2}{\partial x \partial y} f(X)|_{X=(0,1)} = -1.$$

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- STEP 1: Construct an equivalent parametric Hermite rational interpolation problem;
- STEP 2: Convert the rational system to a parametric polynomial system;
- STEP 3: Solve the parametric polynomial system by triangular decomposition;
- STEP 4: Choose proper parameters to get the Birkhoff rational interpolation functions.

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STEP 1: Construct Hermite problem

- For a given Birkhoff interpolation problem, we add the lacking derivative conditions and set the artificial interpolation values as parameters, then we obtain a parametric Hermite rational interpolation problem.
- Let $\widetilde{S}_i = S_i$. For each $\alpha \in \widetilde{S}_i$, if $\exists \beta \in L(\alpha)$ and $\beta \notin \widetilde{S}_i$, then we add β to $\widetilde{S}_i$, and set $c_{i,\beta}$ as an undetermined parameter. Finally, a parametric Hermite rational system is derived.

$$D^{\alpha}(p/q) = c_{i,\alpha}, \ \forall \alpha \in \widetilde{S}_i, \ i = 1, \ldots, m, \quad (2)$$

where $c_{i,\alpha}$, is a given constant if $\alpha \in S_i$, an undetermined parameter otherwise.

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Let $Y_1 = (0,0)$, $\tilde{S}_1 = \{(0,0),(0,1),(1,0),(1,1)\}$, $\tilde{V}_1 = \{6,5,c_1,0\}$,

$Y_2 = (0,1)$, $\tilde{S}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$, $\tilde{V}_2 = \{7,c_2,2,-1\}$.

we add two interpolation conditions

$$\frac{\partial}{\partial x} f(X)|_{X=(0,0)} = c_1, \qquad \frac{\partial}{\partial y} f(X)|_{X=(0,1)} = c_2.$$

KEY IDEA

STEP 2: Convert to polynomial system

Theorem

If $q(Y_i) \neq 0$ $(i = 1, \ldots, m)$, the Hermite rational interpolation system

$$D^{\boldsymbol{\alpha}}(p/q)(Y_i) = c_{i,\boldsymbol{\alpha}},\ i = 1, \ldots, m,\ \boldsymbol{\alpha} \in \tilde{S}_i \tag{3}$$

is equivalent to the polynomial system

$$D^{\boldsymbol{\alpha}} p(Y_i) = \sum_{\boldsymbol{\sigma} \in L(\boldsymbol{\alpha})} c_{i,\boldsymbol{\sigma}} D^{\boldsymbol{\alpha}-\boldsymbol{\sigma}} q(Y_i),\ i = 1, \ldots, m,\ \boldsymbol{\alpha} \in \tilde{S}_i, \tag{4}$$

where $\tilde{S}_i$, $i = 1, \ldots, m$, are lower sets, $c_{i,\boldsymbol{\sigma}},\ \boldsymbol{\sigma} \in L(\boldsymbol{\alpha}),\ i = 1, \ldots, m$, are the given derivative values.

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STEP 3: Solve the polynomial system

- The original problem is reduced to solving a parametric polynomial system;
- Set the constant term of the denominator as 1 unless 0 is a pole point of the desired rational function.
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STEP 4: Choose parameters to get the interpolation function

Theorem

If p/q is a solution of (1), then there exist some parameters $c_{i,\boldsymbol{\beta}}$ such that p, q satisfy

$$D^{\boldsymbol{\alpha}} p(Y_i) = \sum_{\boldsymbol{\sigma} \in L(\boldsymbol{\alpha})} c_{i,\boldsymbol{\sigma}} D^{\boldsymbol{\alpha}-\boldsymbol{\sigma}} q(Y_i), \; i = 1, \ldots, m, \; \boldsymbol{\alpha} \in \widetilde{S}_i. \tag{5}$$

Conversely, if $p, q \in K[X]$ is a solution of (5), and q satisfies $q(Y_i) \neq 0$, $i = 1, \ldots, m$, then p/q satisfies (1).

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STEP 4: Choose parameters to get the interpolation function

- The above theorem guarantees the solution provides a Birkhoff rational interpolation function as long as there exist proper parameters such that the denominator does not vanish at each node.
- We check each of the parameters to pick out all the proper ones such that the denominator does not vanish at any node.

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- **Calling sequence**
  BirkhoffRationalInterpolation(Y,F,Option)
- **Parameters**
  Y–list of nodes. Each node is represented as a row vector.
  F–list of matrices. The i-th matrix is determined by the interpolation conditions corresponding to the i-th node Y_i. The number of the rows of the i-th matrix equals to the number of the interpolation conditions according to the i-th node. Each row of the i-th matrix $[\alpha_1, \ldots, \alpha_n, c_{i,\boldsymbol{\alpha}}]$ denotes a interpolation condition $D^{\boldsymbol{\alpha}} r(Y_i) = c_{i,\boldsymbol{\alpha}}$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.
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- The BirkhoffRationalInterpolation command constructs the multivariate Birkhoff rational interpolation functions in a field K. The output of this command is a list of the rational functions with real or complex coefficients.
- The package “RegularChains" is required.
- So far the input can only be rational numbers.

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Given a interpolation problem as follows:

Table: Interpolation problem

Y_i	(0,0)	(0,1)	(1,0)	(1,1)
S_i	$\{(0,0),(0,1),(1,1)\}$;	$\{(0,0),(1,0),(1,1)\}$;	$\{(0,0),(1,1)\}$;	$\{(0,0),(1,0),(0,1)\}$
$c_{i,\alpha}$	{ 6 , 5 , 0 };	{ 7 , 2 , -2 };	{ 6 , -5/2 };	{20/3, -7/9 , 16/9}

EXAMPLE

Let

$$Y := [[0,0],[0,1],[1,0],[1,1]];$$

$$F_1 := \text{Matrix}([[0,0,6],[0,1,5],[1,1,0]]),$$

$$F_2 := \text{Matrix}([[0,0,7],[1,0,2],[1,1,-2]]),$$

$$F_3 := \text{Matrix}([[0,0,6],[1,1,-\tfrac{5}{2}]]),$$

$$F_4 := \text{Matrix}([[0,0,\tfrac{20}{3}],[1,0,\tfrac{16}{9}],[0,1,-\tfrac{7}{9}]]).$$

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The output of the command BirkhoffRationalInterpolation(Y, [F_1, F_2, F_3, F_4]),"real" is a list $[r_1(x,y), r_2(x,y)]$, where

$$r_1(x,y) = \frac{6-44.217y+233.040x+77.917y^2-221.333xy-108.216x^2}{1-8.203y+35.048x+12.874y^2-34.997xy-14.244x^2},$$

$$r_2(x,y) = \frac{6-37.464y+2887.787x-196.995y^2-261.344xy-2552.415x^2}{1-7.077y+430.953x+-26.560y^2-46.423xy-375.057x^2}.$$

Thank you!