Computing limits of real multivariate rational functions: around and beyond the case of an isolated zero of the denominator

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Outline

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Problem

For a multivariate rational function $q \in \mathbb{Q}(X_1, \ldots, X_n)$, we want to decide whether

$$\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1, \ldots, x_n)$$

exists, and if it does, whether it is finite.

Previous works: part I

Univariate functions (including transcendental ones)

D. Gruntz (1993, 1996), B. Salvy and J. Shackell (1999)

- Corresponding algorithms are available in popular computer algebra systems

Multivariate rational functions

S.J. Xiao and G.X. Zeng (2014)

- Given $q \in \mathbb{Q}(X_1, \ldots, X_n)$, they proposed an algorithm deciding whether or not: $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q$ *exists and is zero*.
- No assumptions on the input multivariate rational function
- Techniques used:
  - triangular decomposition of algebraic systems,
  - rational univariate representation,
  - adjoining infinitesimal elements to the base field.

Lagrange multipliers (1/2)

Let q and t be real bivariate functions of class C^1 .

Problem

$$\text{optimize } q(x,y)$$
$$\text{subject to } t(x,y) = 0$$

Solution

1. Assuming $\nabla t(x,y)$ does not vanish on $t(x,y) = 0$, solve the following system of equations:
$$\begin{cases} \nabla q(x,y) & = & \lambda \nabla t(x,y) \\ t(x,y) & = & 0 \end{cases}$$
2. Plug in all (x,y) solutions obtained at Step (1) into $q(x,y)$ and identify the minimum and maximum values, provided that they exist.

Lagrange multipliers (2/2)

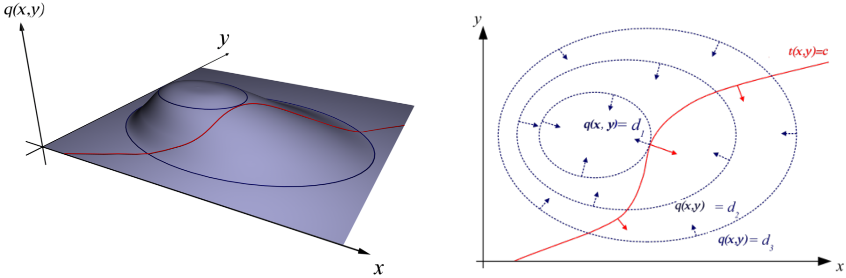


Figure: Optimizing $q(x, y)$ under $t(x, y) = c$

Previous works: bivariate rational functions

C. Cadavid, S. Molina, and J. D. Vélez (2013):

- Assumes that the origin is an isolated zero of the denominator
- Maple built-in command `limit/multi`

Discriminant variety

$$\chi(q) \;=\; \{(x,y) \in \mathbb{R}^2 \;\mid\; y\frac{\partial q}{\partial x} - x\frac{\partial q}{\partial y} = 0\}.$$

Key observation

For determining the existence and possible value of

$$\lim_{(x,y)\to(0,0)} q(x,y),$$

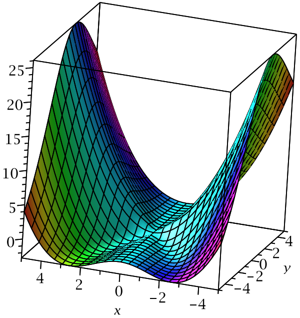
it is sufficient to compute

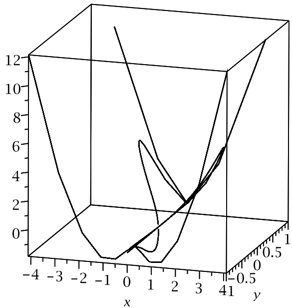
$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in\chi(q)}} q(x,y).$$

Example

Let $q \in \mathbb{Q}(x, y)$ be a rational function defined by $q(x, y) = \frac{x^4+3x^2y-x^2-y^2}{x^2+y^2}$.

$$\chi(q) = \left\{ \begin{array}{rcl} x^4 + 2x^2y^2 + 3y^3 & = & 0 \\ y & < & 0 \end{array} \right. \cup \left\{ \begin{array}{rcl} x & = & 0 \end{array} \right.$$





Previous works: trivariate rational functions

J.D. Vélez, J.P. Hernández, and C.A Cadavid (2015).

- Assumes that the origin is an isolated zero of the denominator
- Ad-hoc method reducing to the case of bivariate rational functions

Similar key observation

For determining the existence and possible value of

$$\lim_{(x,y,z)\to(0,0,0)} q(x,y,z),$$

it is sufficient to compute

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\(x,y,z)\in\chi(q)}} q(x,y,z).$$

Techniques used

- Computation of singular loci
- Variety decomposition into irreducible components

Outline

Our contributions

Sub-routines for computing limits of real rational functions

* Determination of the real branches of a space curve

How?

1. Hensel-Sasaki construction

Limit computation at an isolated zero of the denominator

* Generalize the trivariate algorithm of J.D. Vélez, J.P. Hernández, and C.A Cadavid to arbitrary number of variables
* Avoiding the computation of singular loci and irreducible decompositions

How?

1. Triangular decomposition of semi- algebraic systems

Outline

Regular semi-algebraic system

Notation

- Let $T \subset \mathbb{Q}[X_1 < \ldots < X_n]$ be a regular chain with $\underline{y} := \{\mathrm{mvar}(t) \mid t \in T\}$ and $\mathbf{U} := \underline{x} \setminus \underline{y} = U_1, \ldots, U_d$.
- Let P be a finite set of polynomials, s.t. every $f \in P$ is regular modulo $\mathrm{sat}(T)$.
- Let $\mathcal{Q}$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{U}]$.

Definition

We say that $R := [\mathcal{Q}, T, P_>]$ is a regular semi-algebraic system if:

(i) $\mathcal{Q}$ defines a non-empty open semi-algebraic set $\mathcal{O}$ in $\mathbb{R}^d$,

(ii) the regular system $[T, P]$ specializes well at every point u of $\mathcal{O}$

(iii) at each point u of $\mathcal{O}$, the specialized system $[T(u), P(u)_>]$ has at least one real solution.

Define

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

Regular semi-algebraic system

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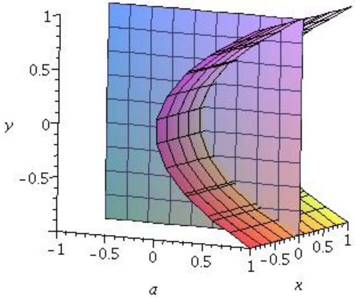
$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

Example

The system $[\mathcal{Q}, T, P_>]$, where

$$\mathcal{Q} := a > 0,\ T := \begin{cases} y^2 - a = 0 \\ x = 0 \end{cases},\ P_> := \{y > 0\}$$

is a regular semi-algebraic system.



Regular semi-algebraic system

Notations

Let $R := [\mathcal{Q}, T, P_>]$ be a regular semi-algebraic system. Recall that $\mathcal{Q}$ defines a non-empty open semi-algebraic set $\mathcal{O}$ in $\mathbb{R}^d$ and

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

Properties

- Each connected component C of $\mathcal{O}$ in $\mathbb{R}^d$ is a real analytic manifold, thus locally homeomorphic to the hyper-cube $(0, 1)^d$
- Above each C, the set $Z_{\mathbb{R}}(R)$ consists of disjoint graphs of semi-algebraic functions forming a real analytic covering of C.
- There is at least one such graph.

Consequences

- R can be understood as a parameterization of $Z_{\mathbb{R}}(R)$
- The Jacobian matrix $\left[\ \nabla t, t \in T\ \right]$ is full rank.

Triangular decomposition of semi-algebraic sets

Proposition

Let $S := [F_=, N_\geq, +, H_\neq]$ be a semi-algebraic system. Then, there exists a finite family of regular semi-algebraic systems $R_1, \ldots, R_e$ such that

$$Z_{\mathbb{R}}(S) = \cup_{i=1}^{e} Z_{\mathbb{R}}(R_i).$$

Triangular decomposition

- In the above decomposition, $R_1, \ldots, R_e$ is called a triangular decomposition of S and we denote by `RealTriangularize` an algorithm computing such a decomposition.
- Moreover, such a decomposition can be computed in an incremental manner with a function `RealIntersect`
  - taking as input a regular semi-algebraic system R and a semi-algebraic constraint $f = 0$ (resp. $f > 0$) for $f \in \mathbb{Q}[X_1, \ldots, X_n]$
  - returning regular semi-algebraic system $R_1, \ldots, R_e$ such that

$$Z_{\mathbb{R}}(f = 0) \cap Z_{\mathbb{R}}(R) \quad = \quad \cup_{i=1}^{e} Z_{\mathbb{R}}(R_i).$$

Outline

Generalization of concepts and basic lemmas (1/3)

Discriminant variety (Cadavid, Molina, and Vélez, 2013)

Let $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a rational function defined on a punctured ball D^*_δ. The discriminant variety $\chi(q)$ of q is the real zero-set of all 2-by-2 minors of

$$\begin{bmatrix} X_1 & \cdots & X_n \\ \frac{\partial q}{\partial X_1} & \cdots & \frac{\partial q}{\partial X_n} \end{bmatrix}$$

Limit along a semi-algebraic set

Let S be a semi-algebraic set of positive dimension (i. e. $\geqslant 1$) such that $\underline{o} \in \overline{S}$ in the Euclidean topology. Let $L \in \mathbb{R}$. We say

$$\lim_{\substack{(x_1,\ldots,x_n) \to (0,\ldots,0) \\ (x_1,\ldots,x_n) \in S}} q(x_1,\ldots,x_n) = L$$

whenever

$(\forall \varepsilon > 0)\,(\exists 0 < \delta)\,(\forall (x_1,\ldots,x_n) \in S \cap D^*_\delta)\;\; |q(x_1,\ldots,x_n) - L| < \varepsilon$

Generalization of concepts and basic lemmas (2/3)

Lemma 1

For all $L \in \mathbb{R}$ the following assertions are equivalent:

- $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1,\ldots,x_n)$ exists and equals L,
- $\lim_{\substack{(x_1,\ldots,x_n)\to(0,\ldots,0)\\(x_1,\ldots,x_n)\in\chi(q)}} q(x_1,\ldots,x_n)$ exists and equals L.

Lemma 2

Let $R_1,\ldots,R_e$ be regular semi-algebraic systems forming a triangular decomposition of $\chi(q)$. Then, for all $L \in \mathbb{R}$ the following are equivalent:

- $\lim_{\substack{(x_1,\ldots,x_n)\to(0,\ldots,0)\\(x_1,\ldots,x_n)\in\chi(q)}} q$ exists and equals L.
- for all $i \in \{1,\ldots,e\}$ such that $Z_{\mathbb{R}}(R_i)$ has dimension at least 1 and the origin belongs to $\overline{Z_{\mathbb{R}}(R_i)}$, we have $\lim_{\substack{(x_1,\ldots,x_n)\to(0,\ldots,0)\\(x_1,\ldots,x_n)\in Z_{\mathbb{R}}(R_i)}} q$ exists and equals L.

Lemma 3

- Assume $n \geq 3$. Let $R = [\mathcal{Q}, \{t_n\}, P_>]$ be a regular semi-algebraic system of $\mathbb{Q}[X_1, \ldots, X_n]$ such that $Z_{\mathbb{R}}(R)$ has dimension $d := n - 1$, and $\underline{o} \in \overline{Z_{\mathbb{R}}(R)}$. W.l.o.g. we assume that $\mathrm{mvar}(t_n) = X_n$ holds.
- Let $\mathcal{M} := \begin{bmatrix} X_1 & \cdots & X_n \\ \frac{\partial t_n}{\partial X_1} & \cdots & \frac{\partial t_n}{\partial X_n} \end{bmatrix}$

Then, there exists a non-empty set $\mathcal{U} \subset D^*_\rho \cap Z_{\mathbb{R}}(R)$, which is open relatively to $Z_{\mathbb{R}}(R)$, such that $\mathcal{M}$ is full rank at any point of $\mathcal{U}$, and $\underline{o} \in \overline{\mathcal{U}}$.

Overview of `RationalFunctionLimit`

Input: a rational function $q \in \mathbb{Q}(X_1, \ldots, X_n)$ such that origin is an isolated zero of the denominator.

Output: $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1, \ldots, x_n)$

1. Apply `RealTriangularize` on $\chi(q)$, obtaining rsas $R_1, \ldots, R_e$
2. Discard R_i if either $\dim(R_i) = 0$ or $\underline{o} \notin \overline{Z_{\mathbb{R}}(R_i)}$
   - QuantifierElimination checks whether $\underline{o} \in \overline{Z_{\mathbb{R}}(R_i)}$ or not.
3. Apply LimitInner(R) on each regular semi algebraic system of dimension higher than one.
   - main task: solving constrained optimization problems
4. Apply `LimitAlongCurve` on each one-dimensional regular semi algebraic system resulting from Step 3
   - main task: Puiseux series expansions

Principles of LimitInner

Input: a rational function q and a regular semi algebraic system $R := [Q, T, P_>]$ with $\dim(Z_{\mathbb{R}}(R)) \geq 1$ and $\underline{o} \in \overline{Z_{\mathbb{R}}(R)}$

Output: limit of q at the origin along $Z_{\mathbb{R}}(R)$

1. if $\dim(Z_{\mathbb{R}}(R)) = 1$ then return LimitAlongCurve(q, R)
2. otherwise build $\mathcal{M} := \begin{bmatrix} X_1 & \cdots & X_n \\ \multicolumn{3}{c}{\nabla t, t \in T} \end{bmatrix}$
3. For all $m \in$ Minors$(\mathcal{M})$ such that $Z_{\mathbb{R}}(R) \not\subseteq Z_{\mathbb{R}}(m)$ build

   $\mathcal{M}' := \begin{bmatrix} \frac{\partial E_r}{\partial X_1} & \cdots & \frac{\partial E_r}{\partial X_n} \\ X_1 & \cdots & X_n \\ \multicolumn{3}{c}{\nabla t, t \in T} \end{bmatrix}$ with $E_r := \sum_{i=1}^{n} A_i X_i^2 - r^2$

   For all $m' \in$ Minors$(\mathcal{M}')$ $\mathcal{C} :=$ `RealIntersect`$(R, m' = 0, m \neq 0)$

   For all $C \in \mathcal{C}$ such that $\dim(Z_{\mathbb{R}}(C)) > 0$ and $\underline{o} \in \overline{Z_{\mathbb{R}}(C)}$

   1. compute $L =$ LimitInner(q, C);
   2. if L is `no_finite_limit` or L is finite but different from a previously found finite L then return `no_finite_limit`
4. If the search completes then a unique finite was found and is returned.

Principles of LimitAlongCurve

Input: a rational function q and a curve C given by $[Q, T, P_>]$
Output: limit of q at the origin along C

1. Let f, g be the numerator and denominator of q
2. Let $T' := \{gX_{n+1} - f\} \cup T$ with X_{n+1} a new variable
3. Compute the real branches of $W_{\mathbb{R}}(T') := Z_{\mathbb{R}}(T') \setminus Z_{\mathbb{R}}(h_{T'})$ in $\mathbb{R}^n$ about the origin via Puiseux series expansions
4. If no branches escape to infinity and if $W_{\mathbb{R}}(T')$ has only one limit point $(x_1, \ldots, x_n, x_{n+1})$ with $x_1 = \cdots = x_n = 0$, then x_{n+1} is the desired limit of q
5. Otherwise return `no_finite_limit`

Example

Let $q(x, y, z, w) = \frac{z\,w + x^2 + y^2}{x^2 + y^2 + z^2 + w^2}$.
`RealTriangularize` $(\chi(q))$:

$$Z_{\mathbb{R}}(\chi(q)) = Z_{\mathbb{R}}(R_1) \cup Z_{\mathbb{R}}(R_2) \cup Z_{\mathbb{R}}(R_3) \cup Z_{\mathbb{R}}(R_4),$$

where

$$R_1 := \begin{cases} x = 0 \\ y = 0 \\ z = 0 \\ w = 0 \end{cases}, R_2 := \begin{cases} x = 0 \\ y = 0 \\ z + w = 0 \end{cases},$$

$$R_3 := \begin{cases} x = 0 \\ y = 0 \\ z - w = 0 \end{cases}, R_4 := \begin{cases} z = 0 \\ w = 0 \end{cases}.$$

Example

- $\dim(Z_{\mathbb{R}}(R_1)) = 0$
- $\dim(Z_{\mathbb{R}}(R_2)) = 1 \Longrightarrow$ LimitAlongCurve$(q, R_2) = \frac{-1}{2}$
- $\dim(Z_{\mathbb{R}}(R_3)) = 1 \Longrightarrow$ LimitAlongCurve$(q, R_3) = \frac{1}{2}$
- $\dim(Z_{\mathbb{R}}(R_4)) = 2 \Longrightarrow$ LimitInner(q, R_4)
  -
$$R_5 := \begin{cases} z = 0 \\ w = 0 \\ x = 0 \\ y \neq 0 \end{cases}, R_6 := \begin{cases} z = 0 \\ w = 0 \\ y = 0 \\ x \neq 0 \end{cases}$$

  - $\dim(Z_{\mathbb{R}}(R_5)) = 1 \Longrightarrow$ LimitAlongCurve$(q, R_5) = 1$
  - $\dim(Z_{\mathbb{R}}(R_6)) = 1 \Longrightarrow$ LimitAlongCurve$(q, R_6) = 1$

$\Longrightarrow$ the limit does not exists.

Outline

Hensel-Sasaki construction: ultimate goal

Ideally:

For $F \in \overline{\mathbb{Q}}[x, y]$ (and in fact, even for $F(x, y) \in \mathbb{C}\langle y \rangle[x]$) we aim at factorizing $F(x, y)$ as

$$F(x, y) \quad = \quad G_1(x, y) \cdots G_r(x, y)$$

where

1. this factorization holds in $\mathbb{C}((y^*))[x]$, and
2. $\deg_x(G_i) = 1$ holds for all $i = 1, \ldots, r$.

In practice:

1. We truncate the coefficients of $G_1(x, y), \ldots, G_r(x, y)$ (as polynomials in y), that is, we factor $F(x, y)$ modulo an ideal.
2. The main algorithm (next slide) may not guarantee $\deg_x(G_i) = 1$ for all $i = 1, \ldots, r$. To overcome this difficulty, the main algorithm is applied repeatedly on those factors G_i for which $\deg_x(G_i) > 1$ holds, after a change of coordinates.

Hensel-Sasaki construction: main algorithm

Initial phase

1. Let $F(x,y) \in \mathbb{C}[x,y]$ be square-free, monic in x and let $d := \deg_x(F)$.
2. The "south-west-most" terms $c\,x^{e_x}y^{e_y}$ of $F(x,y)$ satisfy an equation of the form $e_x/d + e_y/\delta = 1$, with $\delta \in \mathbb{Q}$ and form the Newton polynomial $F^{(0)}(x,y)$.
3. Let $\hat{\delta}, \hat{d} \in \mathbb{Z}^{>0}$ such that: $\hat{\delta}/\hat{d} = \delta/d, \quad \gcd(\hat{\delta}, \hat{d}) = 1$ Choosing such integers $\hat{\delta}, \hat{d}$ is possible since $\delta \in \mathbb{Q}$ and $d \in \mathbb{N}^{>0}$.
4. Define $S_k = \langle x^d y^{(k+0)/\hat{d}}, x^{d-1}y^{(k+\hat{\delta})/\hat{d}}, x^{d-2}y^{(k+2\hat{\delta})/\hat{d}}, \ldots, x^0 y^{(k+d\hat{\delta})/\hat{d}} \rangle$ for $k = 1, 2, \ldots$.
5. $F^{(0)}(x,y)$ is homogeneous $(x, y^{\delta/d})$ of degree d and can be factorized in $\mathbb{C}((y^*))[x]$.
6. This yields a factorization of $F(x,y)$ in $\mathbb{C}((y^*))[x]$ modulo S_1, say:
$$F^{(0)}(x,y) \;\equiv\; G_1^{(0)}(x,y) \cdots G_r^{(0)}(x,y) \mod S_1$$

Inductive phase

For any positive integer k, we can construct $G_i^{(k)}(x,y) \in \mathbb{C}\langle y^{1/\hat{d}} \rangle [x]$, for $i = 1, \ldots, r$, satisfying

1. $F(x,y) = G_1^{(k)}(x,y) \cdots G_r^{(k)}(x,y) \mod S_{k+1}$,
2. $G_i^{(k)}(x,y) = G_i^{(0)}(x,y) \mod S_1$.

Hensel-Sasaki construction: The Yun-Moses polynomials

For simplicity, we write $\hat{y} = y^{\hat{\delta}/\hat{d}}$.

Lemma

Let $\hat{G}_i(x, \hat{y}) \in \mathbb{C}[x, \hat{y}]$, for $i = 1, \ldots, r$, be homogeneous polynomials in $(x, \hat{y})$, that we regard in $\mathbb{C}\langle\hat{y}\rangle[x]$, such that

- $r \geq 2$ and $d = \deg_x \left(\hat{G}_1 \cdots \hat{G}_r\right)$,
- $\gcd_x(\hat{G}_i, \hat{G}_j) = 1$ for any $i \neq j$.

Then, for each $\ell \in \{0, \ldots, d-1\}$, there exists only one set of polynomials $\{W_i^{(\ell)}(x, \hat{y}) \in \mathbb{C}\langle\hat{y}\rangle[x] \mid i = 1, \ldots, r\}$ satisfying

1. $W_1^{(\ell)} \left(\left(\hat{G}_1 \cdots \hat{G}_r\right) / \hat{G}_1\right) + \cdots + W_r^{(\ell)} \left(\left(\hat{G}_1 \cdots \hat{G}_r\right) / \hat{G}_r\right) = x^\ell \hat{y}^{d-\ell}$,
2. $\deg_x \left(W_i^{(\ell)}(x, \hat{y})\right) < \deg_x \left(\hat{G}_i(x, \hat{y})\right)$, for $i = 1, \ldots, r$.

Moreover, the polynomials $W_i^{(0)}, \ldots, W_i^{(d-1)}$, for $i = 1, \ldots, r$ are homogeneous in $(x, \hat{y})$ of degree $\deg_x \hat{G}_i$.

Hensel-Sasaki construction: the inductive phase

Main steps

1. Compute $\Delta F^{(k)}(x,y) := F(x,y) - G_1^{(k-1)} \cdots G_r^{(k-1)} \mod S_{k+1}$.
2. From (T. Sasaki & F. Kako, 1999), we have

$$\Delta F^{(k)}(x,y) = f_{d-1}^{(k)} x^{d-1} y^{\hat{\delta}/\hat{d}} + \cdots + f_0^{(k)} x^0 y^{d\hat{\delta}/\hat{d}}$$
$$f_\ell^{(k)} = c_\ell^{(k)} y^{k/\hat{d}},\ c_\ell^{(k)} \in \mathbb{C} \quad \text{for } \ell = 0, \ldots, d-1$$

3. Fix $i \in \{1, \ldots, r\}$. Construct $G_i^{(k)}(x,y)$ by writing

$$G_i^{(k)}(x,y) = G_i^{(k-1)}(x,y) + \Delta G_i^{(k)}(x,y),\ \ \Delta G_i^{(k)}(x,y) \equiv 0 \mod S_k.$$

4. From (T. Sasaki & F. Kako, 1999), we have

$$\Delta G_i^{(k)}(x,y) = \sum_{\ell=0}^{d-1} W_i^{(\ell)}(x,y) f_\ell^{(k)}(y) \quad i = 1, \ldots, r$$

Hensel-Sasaki construction: an example

1. $F(x, y) = x^5 + x^4\, y - 2\, x^3\, y - 2\, x^2\, y^2 + x\,(y^2 - y^3) + y^3$
2. $F^{(0)} = x^5 - 2\, x^3\, y + x\, y^2 = x\,(x + y^{1/2})^2\,(x - y^{1/2})^2$
3. $G_1^{(0)} = x, G_2^{(0)} = (x + y^{1/2})^2, G_3^{(0)} = (x - y^{1/2})^2.$
4. Yun-Moses polynomials:

$$
\begin{array}{lll}
W_1^{(0)} = y^{1/2} & W_2^{(0)} = -\frac{1}{2}x\, y^{1/2} - \frac{3}{4}y & W_3^{(0)} = -\frac{1}{2}xy^{1/2} + \frac{3}{4}y \\
W_1^{(1)} = 0 & W_2^{(1)} = \frac{1}{4}x\, y^{1/2} + \frac{1}{2}y & W_3^{(1)} = -\frac{1}{4}x\, y^{1/2} + \frac{1}{2}y \\
W_1^{(2)} = 0 & W_2^{(2)} = -\frac{1}{4}y & W_3^{(2)} = \frac{1}{4}y \\
W_1^{(3)} = 0 & W_2^{(3)} = -\frac{1}{4}x\, y^{1/2} & W_3^{(3)} = \frac{1}{4}x\, y^{1/2} \\
W_1^{(4)} = 0 & W_2^{(4)} = \frac{1}{2}x\, y^{1/2} + \frac{1}{4}y & W_3^{(4)} = \frac{1}{2}x\, y^{1/2} - \frac{1}{4}y
\end{array}
$$

5. From the computation of $\Delta F^{(1)} = F - G_1^{(0)} G_2^{(0)} G_3^{(0)} \mod S_2$:

$$f_4^{(1)} = y^{1/2}, f_2^{(1)} = -2y^{1/2}, f_0^{(1)} = y^{1/2}, f_3^{(1)} = f_1^{(1)} = 0$$

6. $G_1^{(1)} \equiv G_1^{(0)} + W_1^{(0)} f_0^{(1)} \equiv x + y \mod S_2$
7. $G_2^{(1)} \equiv G_2^{(0)} + W_2^{(4)} f_4^{(1)} + W_2^{(0)} f_0^{(1)} + W_2^{(2)} f_2^{(1)} \equiv$ $(x + y^{1/2})^2 - (\frac{1}{4}x\, y^{3/2} + \frac{1}{2}y^2)$
8. $G_3^{(1)} \equiv G_3^{(0)} + W_3^{(4)} f_4^{(1)} + W_3^{(0)} f_0^{(1)} + W_3^{(2)} f_2^{(1)} \equiv$ $(x - y^{1/2})^2 + (\frac{1}{4}x\, y^{3/2} - \frac{1}{2}y^2)$

Hensel-Sasaki construction: our observations

Inductive phase: recall

1. Compute $\Delta F^{(k)}(x,y) := F(x,y) - G_1^{(k-1)} \cdots G_r^{(k-1)} \mod S_{k+1}$.
2. From (T. Sasaki & F. Kako, 1999), we have
$$\Delta F^{(k)}(x,y) = f_{d-1}^{(k)} x^{d-1} y^{\hat{\delta}/\hat{d}} + \cdots + f_0^{(k)} x^0 y^{d\hat{\delta}/\hat{d}}$$
$$f_\ell^{(k)} = c_\ell^{(k)} y^{k/\hat{d}},\ c_\ell^{(k)} \in \mathbb{C} \quad \text{for } \ell = 0, \ldots, d-1$$
3. $\Delta G_i^{(k)}(x,y) := G_i^{(k)}(x,y) - G_i^{(k-1)}(x,y)$, $\Delta G_i^{(k)}(x,y) \equiv 0 \mod S_k$
4. We have: $\Delta G_i^{(k)}(x,y) = \sum_{\ell=0}^{d-1} W_i^{(\ell)}(x,y) f_\ell^{(k)}(y)$

Proposition 1

1. If $F(x,y) \in \mathbb{Q}[x,y]$ holds then $c_\ell^{(k)} \in \mathbb{Q}$ holds for $\ell = 0, \ldots, d-1$ and all $k > 0$.
2. If $F(x,y) \in \mathbb{Q}[x,y]$ and $F^{(0)}(x,y)$ factors in $\overline{\mathbb{Q}}((y^*))[x]$ as $(x - \zeta_1 y^{\delta/d})^{m_1} \cdots (x - \zeta_r y^{\delta/d})^{m_r}$, then we have $W_i^{(\ell)} \in \mathbb{Q}(\zeta_i)[\hat{y}, x]$, where $\hat{y} := y^{\hat{\delta}/\hat{d}}$.

Computing the $W_i^{(\ell)}$'s and proving $W_i^{(\ell)} \in \mathbb{Q}(\zeta_i)[\hat{y}, x]$

Recall: $\sum_{i=1}^{r} W_i^{(\ell)} \frac{F^{(0)}}{F_i^{(0)}} = x^\ell \hat{y}^{d-\ell}$ where $W_i^{(\ell)} = \sum_{j=1}^{m_i-1} w_{i,j}^{(\ell)}(\hat{y}) x^j$.

- Take μ-th derivative of first equation for $\mu = 0, 1, \cdots, m_i - 1$, and evaluate $x = \hat{y}\zeta_i$ where ζ_i is a root of $F_i^{(0)}(x, 1)$
- We have $\frac{\partial^\mu}{\partial x^\mu}(W_i^{(\ell)} \frac{F^{(0)}}{F_i^{(0)}})|_{x=\zeta_i \hat{y}} = \frac{\partial^\mu}{\partial x^\mu}(x^\ell \hat{y}^{d-\ell})|_{x=\zeta_i \hat{y}}$
- This is a system of linear equations in $\mathbb{Q}\ (\zeta_i)[\hat{y}]$ with unknowns $w_{i,j}^{(\ell)}$ and coefficient matrix is a Wronskian matrix

$$W = \left[\frac{\partial^\mu}{\partial x^\mu}(x^j \frac{F^{(0)}}{F_i^{(0)}})|_{x=\zeta_i \hat{y}} \right]_{j,\mu=0,1,\cdots,m_i-1}$$

- A Wronskian matrix is invertible if the functions in first row of the matrix are analytic and linearly independent (J. Bôcher 1900).

$\det(W) = c(\frac{F^{(0)}}{F_i^{(0)}})^{m_i}$ where $c = \prod_{k=1}^{m_i}(k-1)!$ and the inverse of W is

$$W^{-1} = \left[\frac{(-1)^{m_i+j+k-1}}{(j-1)!(m_i-j)!(\frac{F^{(0)}}{F_i^{(0)}})^{m_i}} \frac{\partial^{m_i-1-k}}{\partial x^{m_i-1-k}}(x^{m_i-1-j}(\frac{F^{(0)}}{F_i^{(0)}})^{m_i-1}))|_{x=\zeta_i \hat{y}} \right]$$

where $j, k = 0, 1, \cdots, m_i - 1$. From there, one derives the desired properties: $W_i^{(\ell)} \in \mathbb{Q}(\zeta_i)(\hat{y})[x]$ and $W_i^{(\ell)} \in \mathbb{Q}(\zeta_i)[x, \hat{y}]$.

Outline

Branches of Space Curves

Let

$$T := \begin{cases} x_1^4\, x_3 + x_2^3 - x_2^2 \\ -x_2^3 + x_2^2 + x_1^5 \end{cases}$$

Puiseux Parametrizations corresponding to T

$$\phi_1 = \begin{cases} x_3 = -\frac{1}{8}t^2\,(-t^{20} + 6\,\sqrt{-1}\,t^{15} + 10\,t^{10} + 8) \\ x_2 = \frac{1}{2}t^5(-t^5 + 2\,\sqrt{-1}) \\ x_1 = t^2 \end{cases}$$

$$\phi_2 = \begin{cases} x_3 = \frac{1}{8}t^2\,(t^{20} + 6\,\sqrt{-1}\,t^{15} - 10\,t^{10} - 8) \\ x_2 = -\frac{1}{2}t^5(t^5 + 2\,\sqrt{-1}) \\ x_1 = t^2 \end{cases}$$

$$\phi_3 = \begin{cases} x_3 = -t\,(t^{10} + 2\,t^5 + 1) \\ x_2 = t^5 + 1 \\ x_1 = t \end{cases}$$

Branches of Space Curves

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$$\phi_3 = \begin{cases} x_3 = -t\left(t^{10} + 2\, t^5 + 1\right) \\ x_2 = t^5 + 1 \\ x_1 = t \end{cases}$$

Real Puiseux Expansions

Proposition (Characterization of Puiseux Expansions)

Let $f(U,Y) \in \mathbb{Q}\langle U\rangle[Y]$ be square-free, monic w.r.t Y and of degree $s > 0$ in Y. Then, for each $\ell = 1, \ldots, s$, we can compute a positive integer σ_ℓ and algebraic numbers $\Theta_\ell^1, \ldots, \Theta_\ell^{\sigma_\ell}$ over $\mathbb{Q}$ such that

1. for $i = 1, \ldots, \sigma_\ell$, the algebraic number Θ_ℓ^i has a minimal polynomial of the form $h_\ell^i(Y) \in \mathbb{Q}[\Theta_\ell^1, \ldots, \Theta_\ell^{i-1}][Y]$,
2. $f(U,Y)$ factorizes as $(Y - \chi_1(U)) \cdots (Y - \chi_s(U))$ where $\chi_\ell(U) \in \mathbb{Q}((U^*))[\Theta_\ell^1, \ldots, \Theta_\ell^{\sigma_\ell}]$ holds.

Note that $\mathbb{Q}((U^*))$ stands for the field of Puiseux series over $\mathbb{Q}$.

Remark

The Puiseux expansion $\chi_\ell(U)$ of $f(U,Y)$ is real if Θ_ℓ^i is a real algebraic number over $\mathbb{Q}[\Theta_\ell^1, \ldots, \Theta_\ell^{i-1}]$, for $i = 1, \ldots, \sigma_\ell$.

Real Branches of Space Curves (Example)

Let

$$T := \begin{cases} x_1^4\, x_3 + x_2^3 - x_2^2 \\ -x_2^3 + x_2^2 + x_1^5 \end{cases}$$

Puiseux Parametrizations corresponding to T

$$\phi_1 = \begin{cases} x_3 = -\frac{1}{8}t^2\,(-t^{20} + 6\,\sqrt{-1}\,t^{15} + 10\,t^{10} + 8) \\ x_2 = \frac{1}{2}t^5(-t^5 + 2\,\sqrt{-1}) \\ x_1 = t^2 \end{cases} \in \mathbb{Q}(\sqrt{-1})[t]$$

$$\phi_2 = \begin{cases} x_3 = \frac{1}{8}t^2\,(t^{20} + 6\,\sqrt{-1}\,t^{15} - 10\,t^{10} - 8) \\ x_2 = -\frac{1}{2}t^5(t^5 + 2\,\sqrt{-1}) \\ x_1 = t^2 \end{cases} \in \mathbb{Q}(\sqrt{-1})[t]$$

$$\phi_3 = \begin{cases} x_3 = -t\,(t^{10} + 2\,t^5 + 1) \\ x_2 = t^5 + 1 \\ x_1 = t \end{cases} \in \mathbb{Q}[t]$$

Splitting field representation in Maple

- Let $h(Y) \in \mathbb{Q}[Y]$ be an irreducible and monic with degree s.
- Let $g_1 := h(X_1)$.

Then, there exists a positive integer $s' \leq s$ and monic polynomials $g_i \in \mathbb{Q}[X_1, \ldots, X_{i-1}]/\langle g_1, \ldots, g_{i-1}\rangle[X_i]$, for $i = 2, \ldots, s'$ such that

$$\mathbb{Q}[Y] \subset \frac{\mathbb{Q}[X_1]}{\langle g_1 \rangle}[Y] \subset \cdots \subset \frac{\mathbb{Q}[X_1, \ldots, X_{s'}]}{\langle g_1, \ldots, g_{s'} \rangle}[Y],$$

where $h(Y)$ admits at least one linear factor over $\frac{\mathbb{Q}[X_1,\ldots,X_i]}{\langle g_1,\ldots,g_i\rangle}[Y]$, for each i; furthermore, $\frac{\mathbb{Q}[X_1,\ldots,X_{s'}]}{\langle g_1,\ldots,g_{s'}\rangle}[Y]$ is the splitting field of $h(Y)$.

Consequence

- Let Θ be a root of $h(Y)$.
- Let j be the smallest integer for which $\Theta \in \frac{\mathbb{Q}[X_1,\ldots,X_j]}{\langle g_1,\ldots,g_j\rangle}$.

Then $\mathcal{H} := \{g_1, \ldots, g_j\}$ is a zero-dimensional regular chain that we call encoding of Θ.

Overview of `RealPuiseuxExpansions`

Input: monic irreducible polynomial $f(U, Y) \in \mathbb{Q}[U, Y]$ w.r.t Y;

Output: the real Puiseux expansions of $f(U, Y)$ at origin

1. Compute Puiseux expansions of $f(U, Y)$ at origin and obtaining $\mathcal{B} := \{\chi_1(U), \ldots, \chi_s(U)\}$
2. $\mathcal{R} := \emptyset$
3. for each $\chi(U) \in \mathcal{B}$ do
   1. let $\chi(U) \in \mathbb{Q}\langle U^* \rangle[\Theta^1, \ldots, \Theta^\sigma]$
   2. let $\mathcal{H}^i \subset \mathbb{Q}[X_{i,1}, \ldots, X_{i,j_i}]$ be the zero-dimensional regular chain encoding the algebraic number Θ^i
   3. $\mathcal{F} := \mathcal{H}^1 \cup \cdots \cup \mathcal{H}^\sigma$;
   4. if RealTriangularize($\mathcal{F}$) $\neq \emptyset$ then
      - $\mathcal{R} := \mathcal{R} \cup \{\chi(U)\}$
4. return $\mathcal{R}$

Example 1

```
> R := PolynomialRing([x, y, z]):
  rc := Chain([y^(3)-2*y^(3) + y^(2) + z^(5), z^(4)*x + y^(3)-y^(2)], Empty(R), R): Display(rc, R);
  br := RegularChainBranches(rc, R, [z], coefficient = complex);
```

$$\begin{cases} z^4 x + y^3 - y^2 = 0 \\ -y^3 + y^2 + z^5 = 0 \\ z^4 \neq 0 \end{cases}$$

$$br := \left[\left[z = T^2, y = \frac{1}{2} T^5 \left(-T^5 + 2\, RootOf(\_Z^2 + 1)\right), x = -\frac{1}{8} T^2 \left(-T^{20} + 6\, T^{15}\, RootOf(\_Z^2 + 1) + 10\, T^{10} + 8\right)\right], \left[z = T^2, y = -\frac{1}{2} T^5 \left(T^5 + 2\, RootOf(\_Z^2 + 1)\right), x = \frac{1}{8} T^2 \left(T^{20} + 6\, T^{15}\, RootOf(\_Z^2 + 1) - 10\, T^{10} - 8\right)\right], \left[z = T, y = T^5 + 1, x = -T \left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

```
> br := RegularChainBranches(rc, R, [z], coefficient = real);
```

$$br := \left[\left[z = T, y = T^5 + 1, x = -T \left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

- The `PowerSeries` library provides the Hensel-Sasaki construction.
- From there, the `RegularChains` library deduces the real branches of the curve at the given point.

Example 2

```
> R := PolynomialRing([x, y, z]);
  rc := Chain([y^3 + z*y^2 + 3*z^3, x + 2*y^2 + 6*z^2], R);
  Display(rc, R);
  br := RegularChainBranches(rc, R, [z]);
```

$$R := polynomial\_ring$$

$$rc := regular\_chain$$

$$\begin{cases} x + 2\,y^2 + 6\,z^2 = 0 \\ y^3 + z\,y^2 + 3\,z^3 = 0 \end{cases}$$

$$br := \Big[\Big[z = T,\ y = \frac{1}{3}\,T\,\big(\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + \mathrm{RootOf}\big(\_Z^2 + \_Z\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + \mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 - 3\big) - 1\big),\ x = -\frac{2}{9}\,T^2\,\big(\mathrm{RootOf}\big(\_Z^2 + \_Z\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + \mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 - 3\big)\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) - 2\,\mathrm{RootOf}\big(\_Z^2 + \_Z\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + \mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 - 3\big) - 2\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + 31\big)\Big],\ \Big[z = T,\ y = -\frac{1}{3}\,T\,\big(\mathrm{RootOf}\big(\_Z^2 + \_Z\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + \mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 - 3\big) + 1\big),\ x = \frac{2}{9}\,T^2\,\big(\mathrm{RootOf}\big(\_Z^2 + \_Z\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + \mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 - 3\big)\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + \mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 - 2\,\mathrm{RootOf}\big(\_Z^2 + \_Z\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + \mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 - 3\big) - 31\big)\Big],\ \Big[z = T,\ y = -\frac{1}{3}\,T\,\big(\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + 1\big),\ x = -\frac{2}{9}\,T^2\,\big(\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 + 2\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + 28\big)\Big]\Big]$$

```
> real_br := RegularChainBranches(rc, R, [z], coefficient = real);
```

$$real\_br := \Big[\Big[z = T,\ y = -\frac{1}{3}\,T\,\big(\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + 1\big),\ x = -\frac{2}{9}\,T^2\,\big(\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83)^2 + 2\,\mathrm{RootOf}(\_Z^3 - 3\,\_Z - 83) + 28\big)\Big]\Big]$$

Outline

Notations and preliminaries

1. Let again $f, g \in \mathbb{Q}[X_1, \ldots, X_n]$ such that the fraction $q := f/g$ is irreducible and not constant.
2. Let $Z_{\mathbb{R}}(f) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f(x_1, \ldots, x_n) = 0\}$. Similarly, we define $Z_{\mathbb{R}}(g)$.
3. We assume that $\underline{o} := (0, \ldots, 0) \in Z_{\mathbb{R}}(f) \cap Z_{\mathbb{R}}(g)$ holds.
4. Let $C_{f,\underline{o}}$ and $C_{g,\underline{o}}$ be the connected components of $Z_{\mathbb{R}}(f)$ and $Z_{\mathbb{R}}(g)$ to which $\underline{o}$ belongs.
5. Assume $\dim(C_{g,\underline{o}}) > 0$.

What holds over $\mathbb{C}$ may break over $\mathbb{R}$

- $C_{g,\underline{o}} = C_{f,\underline{o}}$ might occur with f, g different and irreducible; consider $f := (X - Y)^2 + (X^2 + Z^2T^2)^2$ and $g := (X - Y)^2 + (Y^2 + Z^2T^2)^2$
- $C_{g,\underline{o}} \cap C_{f,\underline{o}}$ may consist of a single point; consider $f := (X - Y)^2 + (Z - Y)^2$ and $g := (X - Y)^2 + (Z - X)^2$.

Computing rational function limits often reduces to path tracking

Curve selection lemma[J. Milnor]

Let $f_1, \ldots, f_m, g_1, \ldots, g_p \in \mathbb{Q}[X_1, \ldots, X_n]$ such that the origin $\underline{o}$ is in the closure of the semi-algebraic set S defined by:

$$f_1 = \cdots = f_m = 0, \; g_1 > 0, \ldots, g_p > 0.$$

Then, there exists a real analytic curve $\gamma : [0, \varepsilon) \to \mathbb{R}^n$, with $\gamma(0) = \underline{o}$, and $\gamma(t) \in S$ for $t > 0$.

Remark

- Testing $\underline{o} \in \overline{S}$ can be phrased as a quantifier elimination problem and thus solved by CAD:

$$\underline{o} \in \overline{S} \quad \Longleftrightarrow \quad (\forall \varepsilon > 0)\, (\exists \underline{x} \in \mathbb{R}^n)\; \|\underline{x}\| < \varepsilon \Longrightarrow \underline{x} \in S.$$

- For $n = 2$, one can use "lighter" methods for this test. For instance, computing the real branches (thus Puiseux series, which form an ordered field) of $f(x_1, x_2) = 0$ about $(x_1, x_2) = (0, 0)$ and check which ones satisfy $g(x_1, x_2) > 0$.

Condition for $\lim_{(x_1,\dots,x_n)\to(0,\dots,0)} q(x_1,\dots,x_n)$ not to be finite

Proposition 1

Assume that $\underline{o}$ belongs to the closure of $\{g = 0, f > 0\}$. Then, $\lim_{(x_1,\dots,x_n)\to(0,\dots,0)} q(x_1,\dots,x_n)$ cannot be finite.

- Fix $\varepsilon > 0$. Assume by contradiction that the limit exists and equals $\ell \in \mathbb{R}$. Then, there exists $r > 0$ such that for all $\underline{x} \in B(\underline{o}, r)$ we have $\ell - \varepsilon \leq q(\underline{x}) \leq \ell + \varepsilon$. Thus, $q(\underline{x})$ is bounded on $B(\underline{o}, r)$.
- From the hypothesis, for all $r' > 0$, we can choose $y \in B(\underline{o}, r) \cap \{g = 0, f > 0\}$.
- Using the continuity of f and making r' small enough, we have $B(\underline{y}, r') \cap C_{f,\underline{o}} = \emptyset$ as well as $B(\underline{y}, r') \subseteq B(\underline{o}, r)$.
- Observe that $1/(g(\underline{x}))$ is arbitrary large (in absolute value) on $B(\underline{y}, r')$ while $f(\underline{x})$ remains bounded on $B(\underline{y}, r')$.
- This contradicts the fact that $q(\underline{x})$ is bounded on $B(\underline{o}, r)$.

Sufficient condition for $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1,\ldots,x_n)$ not to exist

Proposition 2

Assume that $\underline{o}$ belongs to the closure of $\{f=0, g>0\}$ as well as the closure of $\{g=0, f>0\}$ Then, $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1,\ldots,x_n)$ does not exist.

- From the first assumption and the curve selection lemma, there exists a path to the origin along which q is identically zero. Hence, $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1,\ldots,x_n)$ must be null, if it exists.
- From the second assumption and the previous proposition, $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1,\ldots,x_n)$ cannot be finite.

What to do in general?

1. Let again $f, g \in \mathbb{Q}[X_1, \ldots, X_n]$ such that the fraction $q := f/g$ is irreducible and not constant.
2. Compute $Z_{\mathbb{R}}(g)$. If $\underline{o}$ is isolated in $Z_{\mathbb{R}}(g)$ then use our ISSAC 2016 algorithm (recalled above).
3. If $\underline{o} \in \overline{\{g = 0, f \neq 0\}}$, no finite limit exists
4. Assume from now on that $C_{g,\underline{o}} \subseteq C_{f,\underline{o}}$ holds. Let E be a connected component of $\mathbb{R}^n \setminus C_{f,\underline{o}}$.
5. Apply a multivariate version of L'Hospital's Rule to compute

$$\lim_{\substack{\underline{x} \to \underline{o} \\ \underline{x} \in E}} \frac{f(\underline{x})}{g(\underline{x})}$$

   which might a recursive call to this procedure.
6. Note that multivariate versions of L'Hospital's Rule have assumptions. What to do when these assumptions are not met is work in progress.

L'Hospital's Rule: a version for bivariate differentiable functions

A theorem of W.H. Young (1909)

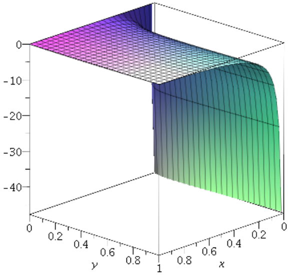
- Let $U \subseteq \{(x,y) \in \mathbb{R}^2 \ \mid \ 0 \leq x \ \text{ and } \ 0 \leq y\}$ be a neighbourhood of $\underline{o}$ such that $[0,\varepsilon] \times [0,\varepsilon] \subseteq U$ holds for ε small enough.
- Let $f, g : U \to \mathbb{R}$ such that the partial derivatives f_{xy} and g_{xy} exist on $]0,\varepsilon] \times]0,\varepsilon]$
- Assume $(\lim_{\underline{o}} f_{xy}, \lim_{\underline{o}} g_{xy}) \notin \{(0,0), (\pm\infty, \pm\infty)\}$, that is, no indeterminate forms.

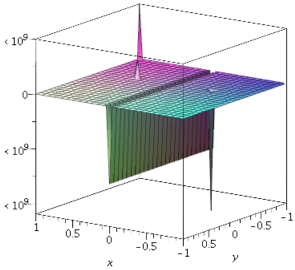
Then we have

$$\lim_{\substack{(x,y) \to (0,0) \\ (x,y) \in U}} \frac{f(x,y)}{g(x,y)} = \lim_{\substack{(x,y) \to (0,0) \\ (x,y) \in U}} \frac{f_{xy}(x,y)}{g_{xy}(x,y)}.$$

L'Hospital's Rule: a version for bivariate differentiable functions

- Consider $f(x,y) = xy^2 - y$ and $g(x,y) = xy - x^3$
- $Z_{\mathbb{R}}(f) = \{xy - 1\} \cup \{y = 0\}$ and
  $Z_{\mathbb{R}}(g) = \{x^2 + y = 0, y < 0\} \cup \{x = 0\}$.
- $\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in U}} \frac{f(x,y)}{g(x,y)} = \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in U}} \frac{f_{xy}(x,y)}{g_{xy}(x,y)} =$
  $\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in U}} \frac{2y}{1} = 0.$
- 0 is the limit over the quadrant $x > 0, y > 0$ and the global limit.





L'Hospital's Rule: a version for multivariate differentiable functions

A theorem of G.R. Lawlor (2012)

- Let $U \subseteq \mathbb{R}^n$ be a neighbourhood of $\underline{o}$.
- Let $f, g : U \to \mathbb{R}$ be differentiable on U and vanishing at $\underline{o}$.
- Let $\mathcal{C}$ be the connected component of $\{\underline{x} \in \mathbb{R}^n \mid f(\underline{x}) = g(\underline{x}) = 0\}$ through $\underline{o}$; assume $\mathcal{C}$ is smooth at $\underline{o}$.
- Let E be a connected component of $\mathbb{R}^n \setminus \mathcal{C}$ (or a CAD cell of $\mathbb{R}^n \setminus \mathcal{C}$ s.t. $\underline{o} \in \overline{E}$).
- Let $\vec{v} \in \mathbb{R}^n$ be not tangent to $\mathcal{C}$ ar $\underline{o}$ s. t. the directional derivative $D_{\vec{v}}\, g := \vec{\nabla} g \cdot \vec{v}$ does not vanish on $V := B(\underline{o}, \varepsilon) \cap E$ for some $\varepsilon > 0$.

Then we have

$$\lim_{\substack{\underline{x} \to \underline{o} \\ \underline{x} \in V}} \frac{f(\underline{x})}{g(\underline{x})} = \lim_{\substack{\underline{x} \to \underline{o} \\ \underline{x} \in V}} \frac{D_{\vec{v}}\, f(\underline{x})}{D_{\vec{v}}\, g(\underline{x})}.$$

Example

- Consider $f(x, y) = x^2 - y^2$ and $g(x, y) = (x - y)^2 + z^2$
- $Z_{\mathbb{R}}(f) = \{x = y\} \cup \{x = -y\}$ and $Z_{\mathbb{R}}(g) = \{x - y, z = 0\}$.
- Within $\mathbb{R}^3 \setminus Z_{\mathbb{R}}(g)$ consider the CAD cell $E := \{x \neq y\}$.
- Choose $\vec{v} = (-1, 1, 0)$ thus $D_{\vec{v}}\, g = \vec{\nabla} g \cdot \vec{v} = -4x + 4y$
- Observe that $D_{\vec{v}}\, g$ does not vanish within E.
- Then we have

$$\lim_{\substack{\underline{x} \to \underline{o} \\ \underline{x} \in V}} \frac{f(\underline{x})}{g(\underline{x})} = \lim_{\substack{\underline{x} \to \underline{o} \\ \underline{x} \in V}} \frac{D_{\vec{v}}\, f(\underline{x})}{D_{\vec{v}}\, g(\underline{x})} = \lim_{\substack{\underline{x} \to \underline{o} \\ \underline{x} \in V}} \frac{-2x - 2y}{-4x + 4y}$$

which does not exist.

Conclusion and future works

- We presented an algorithm for determining the real branches of a space curve about one of its point.
- This is a core routine for computing limits of real multivariate rational functions as well as for addressing topological questions like whether a point belongs to the closure of a CAD cell.
- To this end, we revisited the Hensel-Sasaki construction and established properties of the Yun-Moses polynomials.
- We sketched a general algorithm for computing limits of real multivariate rational functions, which is work in progress.