Polynomials over Power Series and their Applications to Limit Computations (tutorial version)

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CASC 2018 Tutorial
Université de Lille
Bâtiment M^3
September 17, 2018

Plan

1. Motivating Examples
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   - The Ring of Puiseux Series
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Acknowledgements and materials

Acknowledgements

- Many thanks to INRIA, CRISTAL and CASC for their invitation
- Special thanks to those who help me developing those materials: Changbo Chen, Parisa Alvandi, Mahsa Kazemi, Masoud Ataei
- Very special thanks to François Lemaire who initiated the `Regularchains` library 20 years ago and François Boulier for 23 years of collaboration.

Materials

- This tutorial
  `http://www.csd.uwo.ca/~moreno/Publications/Polynomials_over_power_series_and_their_applications_tutorial.PDF`
- The supporting lecture
  `http://www.csd.uwo.ca/~moreno/Publications/Polynomials_over_power_series_and_their_applications_lecture.PDF`
- The `Regularchains` library web site
  `http://regularchains.org/index.html`
- The `PowerSeries` Maple worksheet
  `http://regularchains.org/Documentation/PowerSeries.mw`
- The `Regularchains` Maple worksheet
  `http://regularchains.org/Documentation/RegularChains.mw`
- The `Basic Polynomial Algebra Subprogram` (BPAS) web site
  `http://bpaslib.org/`

Plan

Does the parametrization reach all points of the surface? (1/8)

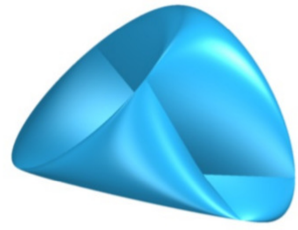


Figure: Steiner's Roman surface

https://upload.wikimedia.org/wikipedia/commons/e/ea/Steiner%27s_Roman_Surface.gif

An implicit formula of Steiner's Roman surface S is $f = 0$, where:

$$\begin{array}{l} f := 4\,x^4 - 8\,yx^3 + 9\,x^2y^2 - 8\,yzx^2 - 5\,y^3x + 8\,y^2zx + y^4 \\ -2\,y^3z + 3\,y^2z^2 - 2\,yz^3 + z^4 - 8\,yx^2 + 8\,zx^2 + 8\,y^2x \\ -8\,xyz - 2\,y^3 + 2\,y^2z - 2\,yz^2 + 4\,x^2 - 4\,yx + y^2. \end{array} \quad (1)$$

Does the parametrization reach all points of the surface? (2/8)

- With $q(s,t) := s^2 + t^2 + s - t + 1$, consider also the following map

$$\begin{array}{rccc} \vec{r}: & \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (s,t) & \mapsto & \left(\frac{s^2}{q(s,t)}, \frac{s^2+t^2}{q(s,t)}, \frac{s^2+s\,t+s+t}{q(s,t)} \right), \end{array} \quad (2)$$

- Do we have $\mathrm{Image}(\vec{r}) = S$?
- A preliminary question is whether $q(s,t)$ vanishes or not.

```
> R := PolynomialRing([s, t, x, y, z]) : q := s^2 + t^2 + s - t + 1 :
  RealTriangularize([q], R);

                                   [ ]
```

Figure: `RegularChains:-RealTriangularize` proves $q(s,t)$ has no real points.

Does the parametrization reach all points of the surface? (3/8)

Let us verify that the image of the map $\vec{r}$ is contained in the surface S.

```
> f := 4·x^4 − 8·y·x^3 + 9·x^2·y^2 − 8·y·z·x^2 − 5·y^3·x + 8·
  y^2·z·x + y^4 − 2·y^3·z + 3·y^2·z^2 − 2·y·z^3 + z^4 − 8·y·x^2 + 8·z·x^2 + 8·y^2·x − 8·x·y·z − 2
      ·y^3 + 2·y^2·z − 2·y·z^2 + 4·x^2 − 4·y·x + y^2 :
  R := PolynomialRing([s, t, x, y, z]) :
  dec1 := Triangularize([f], R); S := GeneralConstruct(dec1[1], map(Initial,
      Equations(dec1[1], R), R), R);
                              dec1 := [regular_chain]
                               S := constructible_set
> q := s^2 + t^2 + s − t + 1 :
  F := [q·x − s^2, q·y − (s^2 + t^2), q·z − (s^2 + s·t + s + t)] :
  dec2 := Triangularize(F, R); ImageR := GeneralConstruct(dec2[1], map(Initial, F, R), R);
                              dec2 := [regular_chain]
                            ImageR := constructible_set
> LM1 := Difference(ImageR, S, R); IsEmpty(LM1, R);
                              LM1 := constructible_set
                                       true
```

Figure: The command `Difference` computes the points in the image of $\vec{r}$ that do not belong to surface S, which is empty.

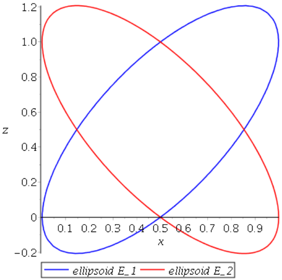
Does the parametrization reach all points of the surface? (4/8)

- Disproving $\mathrm{Image}(\vec{r}) = S$ can be done by specialization
- Computing $\mathrm{Image}(\vec{r}) \cap \{y = 1\}$ yields

$$2\,x^2 + 2\,x\,z + z^2 - 3\,x - 2\,z + 1 = 0$$

- While computing $S \cap \{y = 1\}$ brings more:

$$\left(2\,x^2 - 2\,x\,z + z^2 - x\right)\left(2\,x^2 + 2\,x\,z + z^2 - 3\,x - 2\,z + 1\right) = 0$$



Does the parametrization reach all points of the surface? (5/8)

```
> R := PolynomialRing([s, t, x, y, z]):
  q := s^2 + t^2 + s - t + 1:
  F := [x*q - s^2, y*q - (s^2 + t^2), z*q-(s^2 + s*t + s + t)]:
  dec2 := Projection([op(F), y - 1], [ ], [ ], [ ], 3, R) : Display(%, R)
```

$$\left[\left\{\begin{array}{c}4x+2z-3=0\\ y-1=0\\ 4z^2-4z-1=0\end{array}\right., \left\{\begin{array}{c}x=0\\ y-1=0\\ z-1=0\end{array}\right., \left\{\begin{array}{c}2x^2+(2z-3)x+z^2-2z+1=0\\ y-1=0\\ 4z^2-4z<1 \textbf{ and } z-1\neq 0\end{array}\right.\right]$$

```
> f := 4 x^4 - 8 y x^3 + (9 y^2 + (-8 z - 8) y + 8 z + 4) x^2 + (-5 y^3 + (8 z + 8) y^2 + (-8 z - 4) y) x
     + y^4 + y^3 (-2 z - 2) + (3 z^2 + 2 z + 1) y^2 + (-2 z^3 - 2 z^2) y + z^4:
  R := PolynomialRing([s, t, x, y, z]):
  dec1 := RealTriangularize([f, y - 1], R) : Display(dec1, R);
```

$$\left[\left\{\begin{array}{c}2x^2+(2z-3)x+z^2-2z+1=0\\ y-1=0\\ 4z^2-4z<1\end{array}\right., \left\{\begin{array}{c}4x+2z-3=0\\ y-1=0\\ 4z^2-4z-1=0\end{array}\right.,\right.$$

$$\left.\left\{\begin{array}{c}2x^2+(-2z-1)x+z^2=0\\ y-1=0\\ 4z^2-4z<1\end{array}\right., \left\{\begin{array}{c}4x-2z-1=0\\ y-1=0\\ 4z^2-4z-1=0\end{array}\right.\right]$$

Does the parametrization reach all points of the surface? (6/8)

> Difference(dec1, dec2, R) : Display(%, R);

$$\left[\begin{cases} x - 1 = 0 \\ y - 1 = 0 \\ z - 1 = 0 \end{cases}, \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}, \begin{cases} 2x - 1 = 0 \\ y - 1 = 0 \\ z - 1 = 0 \end{cases}, \begin{cases} 4x - 2z - 1 = 0 \\ y - 1 = 0 \\ 4z^2 - 4z - 1 = 0 \end{cases}, \begin{cases} 2x^2 + (-1 - 2z)x + z^2 = 0 \\ y - 1 = 0 \\ 4z^2 - 4z < 1 \textbf{ and } z \neq 0 \textbf{ and } z - 1 \neq 0 \textbf{ and } 2z - 1 \neq 0 \end{cases} \right]$$

Figure: The points on Steiner surface S and the plane $y = 1$ which do not belong to the intersection of the image of the parametrization $\vec{r}$ and the plane $y = 1$.

Observe that these calculations are done over the `reals`!

Does the parametrization reach all points of the surface? (7/8)

The next question

1. Therefore, $\mathrm{Image}(\vec{r}) = S$ does not hold!
2. Next question: can we recover from S what $\mathrm{Image}(\vec{r})$ is missing?
3. if the missing point are $\overline{\mathrm{Image}(\vec{r})} \setminus \mathrm{Image}(\vec{r})$, then the answer is yes.

The closure of a constructible set

1. Denote by $\overline{\mathrm{Image}(\vec{r})}$ the closure of $\mathrm{Image}(\vec{r})$ in the Euclidean topology (over $\mathbb{C}$).
2. Thanks to a theorem of David Mumford, $\overline{\mathrm{Image}(\vec{r})}$ is also the closure of $\mathrm{Image}(\vec{r})$ in Zariski topology.
3. Thus $\overline{\mathrm{Image}(\vec{r})}$ is the intersection of all algebraic sets containing $\mathrm{Image}(\vec{r})$.
4. By the way, Gröbner basis techniques can capture Zariski closures over algebraically closed fields.

Does the parametrization reach all points of the surface? (8/8)

```
> q := s^2 + t^2 + s - t + 1 :
  R := [x*q - s^2, y*q - (s^2 + t^2), z*q-(s^2 + s*t + s + t)] :
  with(PolynomialIdeals) :
  sat := Saturate(⟨op(R)⟩, q) :
  closure_of_Image_of_r := EliminationIdeal(sat, {x, y, z});
```

$$closure\_of\_Image\_of\_r := \langle 4x^4 - 8x^3y + 9x^2y^2 - 8x^2yz - 5xy^3 + 8xy^2z + y^4 - 2y^3z + 3y^2z^2 - 2yz^3 + z^4 - 8x^2y + 8x^2z + 8xy^2 - 8xyz - 2y^3 + 2y^2z - 2yz^2 + 4x^2 - 4xy + y^2 \rangle$$

Figure: Closure of $\mathrm{Image}(\vec{r})$.

- We retrieve the polynomial defining the implicit representation of S
- According to the so-called *Elimination Theorem* (see the book *Ideals, varieties and Algorithms*) the algebraic set of the elimination ideal $\mathcal{I} \subset \mathbb{K}[x_1 < \cdots < x_n]$ w.r.t. $x_1, \ldots, x_k$ (for some $1 \leq k < n$) is equal to the Zariski closure of the projection of $V(\mathcal{I})$ onto $x_1, \ldots, x_k$.

Summary 1

- Computing Zariski closures of constructible sets (that is, systems of polynomial equations and inequation) and semi-algebraic sets (that is, systems of polynomial equations and inequalities) appear naturally in practice: reachable sets, projection of constructible sets and semi-algebraic sets.
- Gröbner basis techniques can deal with the case of constructible sets.
- We are mainly interested here with the real case, that is, semi-algebraic sets .

Topological closure and limit points

Let (X, τ) be a topological space and $S \subseteq X$ be a subset.

Topological closure

The *closure* of S, denoted $\overline{S}$, is the intersection of all closed sets containing S.

Limit point

- A point $p \in X$ is a *limit point* of S if every neighbourhood of p contains at least one point of S different from p itself.
- The limit points of S which do not belong to S are called non-trivial, denoted by $\lim(S)$.

Properties

- If X is a metric space, the point p is a limit point of S if and only if there exists a sequence $(x_n, n \in \mathbb{N})$ of points of $S \setminus \{p\}$ such that $\lim_{n\to\infty} x_n = p$.
- We have $\overline{S} = S \cup \lim(S)$.

Zariski topology and the Euclidean topology

The relation between the two topologies

- With $\mathbb{K} = \mathbb{C}$, the affine space $\mathbb{A}^s$ is endowed with both topologies.
- The basic open sets of the Euclidean topology are the open balls.
- The basic open sets of Zariski topology are the complements of hypersurfaces.
- Thus, a Zariski closed (resp. open) set is closed (resp. open) in the Euclidean topology on $\mathbb{A}^s$.
- That is, Zariski topology is coarser than the Euclidean topology.

The relation between the two closures (D. Mumford)

- Let $V \subseteq \mathbb{A}^s$ be an irreducible affine variety.
- Let $U \subseteq V$ be nonempty and open in Zariski topology induced on V.

Then, U has the same closure in both topologies. In fact, we have

$$V = \overline{U}^Z = \overline{U}^E.$$

Limit points: a first example

- Let S be the zero-set of a polynomial system and $\overline{S}$ be the topological closure S in the Euclidean topology.
- It can be proved that the set-theoretic difference $\overline{S} \setminus S$ can be obtained via a *limit computation process* illustrated below

Consider S below together with a Puiseux series expansion around $z = 0$:

$$S := \begin{cases} z\,x - y^2 = 0 \\ y^5 - z^4 = 0 \\ z \neq 0 \end{cases} \quad \text{and} \quad \begin{cases} x = \frac{t^{8/5}}{t} \\ y = t^{4/5} \\ z = t \\ t \neq 0 \end{cases}$$

Then we have:

$$\lim_{t \to 0} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{S} \setminus S = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Limit points: a second example

Consider S below together with a Puiseux series expansion around $z = 0$:

$$S := \begin{cases} z\,x - y^2 = 0 \\ y^5 - z^2 = 0 \\ z \neq 0 \end{cases} \quad \text{and} \quad \begin{cases} x = t^{-1/5} \\ y = t^{2/5} \\ z = t \\ t \neq 0 \end{cases}$$

Then we have:

$$\lim_{t \to 0} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \pm\infty \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{S} \setminus S = \emptyset$$

The Puiseux series solutions of a regular chain (1/2)

Regular chains in a nutshell

- Regular chains generalize the concept of *triangular system* from linear algebra to polynomial algebra.
- Thus, they are polynomial systems with a triangular shape and additional algebraic properties which support a back substitution process.
- Every (non-constant) bivariate polynomial forms a regular chain.

The solutions of a regular chain

- Like Gröbner bases, regular chains can be used to compute and describe the solutions of polynomial systems over algebraically closed fields, say $\mathbb{C}$.
- Regular chains can also be used to solve over real closed fields, say $\mathbb{R}$ but also Puiseux series.

The Puiseux series solutions of a regular chain (2/2)

```
> with(AlgebraicGeometryTools):
> R := PolynomialRing([x, y, z]):
> rc := Chain([-z^2+y, x*z-y^2], Empty(R), R):
> br := RegularChainBranches(rc, R, [z]);
                                                   2       3
                              br := [[z = T, y = T , x = T ]]
> rc := Chain([y^2*z+y+1, (z+2)*z*x^2+(y+1)*(x+1)], Empty(R),R):
> RegularChainBranches(rc, R, [z]);
                                     2        2
                          (T - 2) (T  + 4) (T  - 9 T - 54)
[[z = T, y = -T - 1, x = --------------------------------],
                                        432

                                      5    11   4          3          2
    [z = T, y = -T - 1, x = -1/432 T  + --- T  + 5/432 T  - 5/216 T  + 1/12 T - 1/2]]
                                           432
```

Limit points: yet another example

$R := PolynomialRing([x, y, z]) : rc := Chain([-y^3 + y^2 + z^5, z^4 * x + y^3 - y^2], Empty(R), R) :$
$Display(rc, R);$

$$\begin{cases} z^4 x + y^3 - y^2 = 0 \\ -y^3 + y^2 + z^5 = 0 \\ z^4 \neq 0 \end{cases}$$

$RegularChainBranches(rc, R, [z]);$

$$\Big[\Big[z = T^2, y = \frac{1}{2} T^5 \left(-T^5 + 2\, RootOf(\_Z^2 + 1)\right), x = -\frac{1}{8} T^2 \left(-T^{20} + 6\, T^{15}\, RootOf(\_Z^2 + 1) + 10\, T^{10} + 8\right)\Big], \Big[z = T^2, y = -\frac{1}{2} T^5 \left(T^5 + 2\, RootOf(\_Z^2 + 1)\right), x = \frac{1}{8} T^2 \left(T^{20} + 6\, T^{15}\, RootOf(\_Z^2 + 1) - 10\, T^{10} - 8\right)\Big], \left[z = T, y = T^5 + 1, x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\Big]$$

$lp := LimitPoints(rc, R) : Display(lp, R);$

$$\left[\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}, \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}\right]$$

Figure: Computation of (non-trivial) limit points with the RegularChains library

Limit points: statement of our quest

- Let $R := \{t_2(x_1, x_2), \ldots, t_n(x_1, \ldots, x_n)\}$
- We regard t_i as a univariate polynomial w.r.t. x_i, for $i = 2, \ldots, n$:
- We denote by h_i the leading coefficient (also called initial) of t_i w.r.t. x_i, and assume that h_i depends on x_1 only; hence we have $t_i \;=\; h_i(x_1)x_i^{d_i} + c_{d_i-1}(x_1, \ldots, x_{i-1})x_i^{d_i-1} + \cdots + c_0(x_1, \ldots, x_{i-1})$
- Consider the system

$$W(R) := \begin{cases} t_n(x_1, \ldots, x_n) = 0 \\ \vdots \\ t_2(x_1, x_2) = 0 \\ (h_2 \cdots h_n)(x_1) \neq 0 \end{cases}$$

Main Goal

- Where do the points of $W(R)$ go when x_1 approaches a root of $h_2 \cdots h_n$?
- In other words, we want to compute the points which belong to the topological closure of $W(R)$ but to $W(R)$ itself.

Limit points: yet again another example

```
> R := PolynomialRing([x, y, z]) :
  rc := Chain([y^(3)-2* y^(3) + y^(2) + z^(5), z^(4)* x + y^(3)-y^(2)], Empty(R), R) : Display(rc, R);
  br := RegularChainBranches(rc, R,  [z], coefficient = complex);
```

$$\begin{cases} z^4\, x + y^3 - y^2 = 0 \\ -y^3 + y^2 + z^5 = 0 \\ z^4 \neq 0 \end{cases}$$

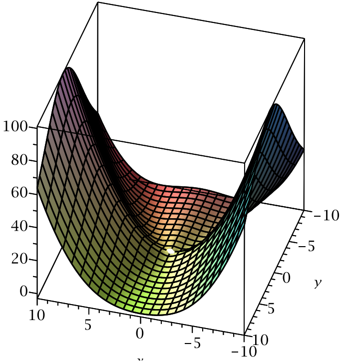
$$br := \left[\left[z = T^2,\ y = \frac{1}{2}\, T^5 \left(-T^5 + 2\, RootOf\left(\_Z^2 + 1\right)\right),\ x = -\frac{1}{8}\, T^2 \left(-T^{20} + 6\, T^{15}\, RootOf\left(\_Z^2 + 1\right) + 10\, T^{10} + 8\right)\right],\right.$$
$$\left[z = T^2,\ y = -\frac{1}{2}\, T^5 \left(T^5 + 2\, RootOf\left(\_Z^2 + 1\right)\right),\ x = \frac{1}{8}\, T^2 \left(T^{20} + 6\, T^{15}\, RootOf\left(\_Z^2 + 1\right) - 10\, T^{10} - 8\right)\right], \left[z\right.$$
$$\left.\left. = T,\ y = T^5 + 1,\ x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

```
> br := RegularChainBranches(rc, R, [z], coefficient = real);
```

$$br := \left[\left[z = T,\ y = T^5 + 1,\ x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

Figure: The command `RegularChainBranches` computes a parametrization for the complex and real paths of the quasi-component defined by rc. When coefficient argument is set as real, then the command `RegularChainBranches` computes the real branches.

Application 1: limit of multivariate rational functions



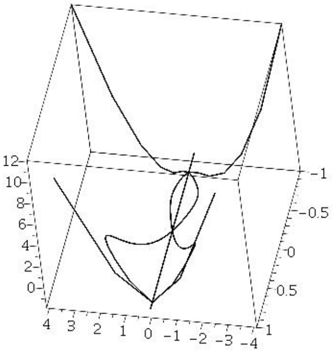


Figure: On the left: the surface defined by $q := \frac{x^4+3\,x^2\,y-x^2-y^2}{x^2+y^2} = z$ around the origin. On the right: the three paths of discriminant variety of q going through the point (0,0,-1).

Application 2: tangent cone computations

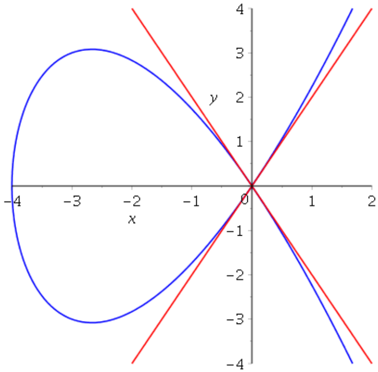
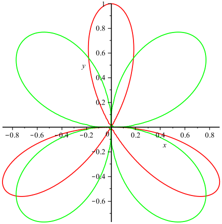


Figure: The tangent cone of the "fish" given by $f := y^2 - x^2\,(x+4) = 0$ at the origin consists of two tangent lines: $y = 2\,x$ and $y = -2\,x$.

Application 3: computing intersection multiplicities

$> F := \left[(x^2+y^2)^2 + 3x^2y - y^3, (x^2+y^2)^3 - 4x^2y^2\right]:$

$> \text{plots[implicitplot]}(Fs, x = -2..2, y = -2..2):$



(3)

$> R := \text{PolynomialRing}([x, y], 101):$

$> \text{TriangularizeWithMultiplicity}(F, R);$

$$\left[\left[1, \left\{\begin{array}{c} x-1=0 \\ y+14=0 \end{array}\right.\right]\right], \left[\left[1, \left\{\begin{array}{c} x+1=0 \\ y+14=0 \end{array}\right.\right]\right], \left[\left[1, \left\{\begin{array}{c} x-47=0 \\ y-14=0 \end{array}\right.\right]\right], \left[\left[1, \left\{\begin{array}{c} x+47=0 \\ y-14=0 \end{array}\right.\right]\right], \left[\left[14, \left\{\begin{array}{c} x=0 \\ y=0 \end{array}\right.\right]\right]$$

The command RegularChains:-TriangularizeWithMultiplicity computes the intersection multiplicities for each point of $V(F)$. In the above Maple session, computations are performed modulo a prime number for the only reason of keeping output expressions small. The same calculations can be performed with the TriangularizeWithMultiplicity command over the reals.

Summary 2

- The theory of regular chains allows us to reduce the question of computing limit points of constructible sets and semi-algebraic sets to that of computing limit points of zero sets of regular chains.
- We will restrict ourselves here to regular chains in dimension 1, that is, where only one variable is free.
- Then, the above question can be solved by computing the Puiseux series solutions of regular chains.

Plan

Plan

Definition

- For $m \geq 1$, there is an injective homomorphism
$$\mathbb{C}[[X]] \to \mathbb{C}[[T]],\ X \mapsto T^m.$$
- We regard this as a ring extension
$$\mathbb{C}[[X]] \subseteq \mathbb{C}[[T]] \equiv \mathbb{C}[[X^{\frac{1}{m}}]]$$
- If $m = kn$, there are injections
$$\begin{aligned} &\mathbb{C}[[X]] \to \mathbb{C}[[T]] \to \mathbb{C}[[S]], \\ &X \mapsto T^n.\ T \mapsto S^k, \\ &X \mapsto (S^k)^n = S^m. \end{aligned}$$
which can be regarded as inclusions
$$\mathbb{C}[[X]] \subseteq \mathbb{C}[[X^{\frac{1}{n}}]] \subseteq \mathbb{C}[[X^{\frac{1}{m}}]].$$
- Continuing in this way, we define
$$\mathbb{C}[[X^*]] \ = \ \bigcup_{n=1}^{\infty} \mathbb{C}[[X^{\frac{1}{n}}]].$$
- This is an integral domain that contains all *formal Puiseux series*.

The ring of Puiseux series (2/9)

Definition

For a fixed $\varphi \in \mathbb{C}[[X^*]]$, there is an $n \in \mathbb{N}$ such that $\varphi \in \mathbb{C}[[X^{\frac{1}{n}}]]$. Hence

$$\varphi = \sum_{m=0}^{\infty} a_m X^{\frac{m}{n}}, \quad \text{where} \quad a_m \in \mathbb{C}.$$

and we call *order of* φ the rational number defined by

$$\mathrm{ord}(\varphi) \; = \; \min\{\tfrac{m}{n} \;\mid\; a_m \neq 0\} \geq 0.$$

Lemma

Every monic polynomial of $\mathbb{C}\langle X\rangle[Y]$ splits into linear factors in $\mathbb{C}[[X^*]][Y]$.

Proof of the lemma (1/3)

- Let $f \in \mathbb{C}\langle X\rangle[Y]$ be monic and $k := \deg(f)$. There exist $k_1, \ldots, k_r \in \mathbb{N}_{>0}$ and pairwise distinct $c_1, \ldots, c_r \in \mathbb{C}$ s, t. we have
$$f(0, Y) = (Y - c_1)^{k_1} \cdots (Y - c_r)^{k_r}.$$

Proof of the lemma (2/3)

- By Hensel's Lemma, there exist monic polynomials $f_1, \ldots, f_r \in \mathbb{C}\langle X \rangle[Y]$ such that $f_i(0, Y) = (Y - c_i)^{k_i}$ and
$$f = f_1 \cdots f_r.$$
- If some i, we have $c_i = 0$, then the Weierstrass preparation theorem can be applied to f_i, so $f_i = \alpha_i p_i$ where p_i is a Weierstrass polynomial of degree k_i and α_i is a unit.
- If q is an irreducible factor of p_i, say of degree ℓ, then q is itself a Weierstrass polynomial. Moreover, the geometric version of Puiseux's theorem implies the existence of Puiseux series $\phi_1, \ldots, \phi_\ell \in \mathbb{C}[[X^*]]$ of positive order such that we have
$$q(X, Y) \ = \ (Y - \phi_1(X)) \cdots (Y - \phi_\ell(X)).$$
- Thus, there exist Puiseux series $\varphi_{i,1}, \ldots, \varphi_{i,k_i} \in \mathbb{C}[[X^*]]$ s. t. we have
$$p_i \ = \ (Y - \varphi_{i,1}(X)) \cdots (Y - \varphi_{i,k_i}(X)).$$
and $\operatorname{ord}(\varphi_{i,j}) > 0$ for all $1 \leq j \leq k_i$.

Proof of the lemma (2/3)

- For each i, such that $c_i \neq 0$ holds, we apply the change of coordinates $\widetilde{Y} = Y + c_i$ and set $\widetilde{f_i}(Y) = f_i(\widetilde{Y})$. After returning to the original coordinate system, this gives a factorization of p_i similar to the one in the previous case (that is, the case $c_i = 0$) up to the fact that $\varphi_{i,j} = c_i + \cdots$, that is, $\mathrm{ord}(\varphi_{i,j}) = 0$ for all $1 \leq j \leq k_i$.
- Putting things together, we define $p := p_1 \cdots p_r$ and we have

$$p = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq k_i}} (Y - \varphi_{i,k_i}(X).$$

- Since f and p have the same roots (counted with multiplicities) in $\mathbb{C}[[X^*]]$ and are both normalized, we conclude $f = p$.

The ring of Puiseux series (5/9)

Notation

We denote by $\mathbb{C}((X^*))$ the quotient field of $\mathbb{C}[[X^*]]$.

Remark

In the previous lemma, the hypothesis f monic is essential. Consider $f = XY^2 + Y + 1$. We write $f = Xg(1/X, Y)$ with $g(T, Y) = Y^2 + TY + T$. The previous lemma applies to g which yields a factorization of f into linear factors of $\mathbb{C}((X^*))[Y]$.

Definition

Let $\varphi \in \mathbb{C}[[X^*]]$ and $n \in \mathbb{N}$ minimum with the property that $\varphi \in \mathbb{C}[[X^{\frac{1}{n}}]]$ holds. We say that the Puiseux series φ is *convergent* if we have $\varphi \in \mathbb{C}\langle X \rangle$. Convergent Puiseux series form an integral domain denoted by $\mathbb{C}\langle X^* \rangle$ and whose quotient field is denoted by $\mathbb{C}(\langle X^* \rangle)$.

Proposition

For every element $\varphi \in ((X^*))$, there exist $n \in \mathbb{Z}$, $r \in \mathbb{N}_{>0}$ and a sequence of complex numbers $a_n, a_{n+1}, a_{n+2}, \ldots$ such that

$$\varphi \ = \ \sum_{m=n}^{\infty} a_m X^{\frac{m}{r}} \ \text{ and } \ a_n \neq 0.$$

and we define $\operatorname{ord}(\varphi) = \frac{n}{r}$. The proof, easy, uses power series inversion.

Remark

Formal Puiseux series can be defined over an arbitrary field $\mathbb{K}$. One essential property of Puiseux series is expressed by the following theorem, attributed to Puiseux for $\mathbb{K} = \mathbb{C}$ but which was implicit in Newton's use of the Newton polygon as early as 1671 and therefore known either as Puiseux's theorem or as the Newton–Puiseux theorem. In its modern version, this theorem requires only $\mathbb{K}$ to be algebraically closed and of characteristic zero. See corollary 13.15 in D. Eisenbud's *Commutative Algebra with a View Toward Algebraic Geometry*.

Theorem

If $\mathbb{K}$ is an algebraically closed field of characteristic zero, then the field $\mathbb{K}((X^*))$ of formal Puiseux series over $\mathbb{K}$ is the algebraic closure of the field of formal Laurent series over $\mathbb{K}$. Moreover, if $\mathbb{K} = \mathbb{C}$, then the field $\mathbb{C}(\langle X^* \rangle)$ of convergent Puiseux series over $\mathbb{C}$ is algebraically closed as well.

Proof of the Theorem (1/3)

- We restrict the proof to the case $\mathbb{K} = \mathbb{C}$. Hence, we prove that $\mathbb{C}((X^*))$ and $\mathbb{C}(\langle X^* \rangle)$ are algebraically closed. We follow the elegant and short proof of K. J. Nowak which relies only on Hensel's lemma.
- It suffices to prove that any monic polynomial $f \in \mathbb{C}((X^*))[Y]$ (resp. $f \in \mathbb{C}(\langle X^* \rangle)[Y]$)

$$f(X,Y) = Y^n + a_1(X)Y^{n-1} + \cdots + a_n(X)$$

of degree $n > 1$ is reducible.

Proof of the Theorem (2/3)

- Making use of the Tschirnhausen transformation of variables $\widetilde{Y} = Y + \frac{1}{n} a_1(X)$, we can assume that the coefficient $a_1(X)$ is identically zero. W.l.o.g., we assume $a_n(X)$ not identically zero.
- For each $k = 1, \ldots, n$, define $r_k = \mathrm{ord}(a_k(X)) \in \mathbb{Q}$, unless a_k is identically zero.
- Define $r := \min\{r_k/k\}$. Obviously, we have $r_k/k - r \geq 0$, with equality for at least one k.
- Take a positive integer q so large that all Puiseux series $a_k(X)$ are of the form $f_k(X^{1/q})$ for $f_k \in \mathbb{C}[[Z]]$ (resp. $f_k \in \mathbb{C}\langle Z \rangle$). Let $r := p/q$ for an appropriate $p \in \mathbb{Z}$.
- After the transformation of variables $X = w^q, \ Y = U \cdot w^p$, we get

$$f(X, Y) = w^{np} \cdot Q(w, U), \quad \text{where}$$

$$Q(w, U) = U^n + b_2(w)U^{n-2} + \cdots + b_n(w) \ \text{ and } \ b_k(w) = a_k(w^q)w^{-kp}.$$

Proof of the Theorem (3/3)

- Observe that $\operatorname{ord}(b_k(w)) \in \mathbb{Z}$ and satisfies in fact
$$\operatorname{ord}(b_k(w)) = q \cdot r_k - k \cdot p = q \cdot k(r_k \cdot k - r) \geq 0.$$
- Therefore $Q(w, U)$ is a polynomial in $\mathbb{C}[[w]][U]$ (resp. $\mathbb{C}\langle w\rangle[U]$).
- Furthermore we have $\operatorname{ord}(b_k(w)) = 0$ for at least one k. Thus, for every such k, we have $b_k(0) \neq 0$.
- Therefore, the complex polynomial
$$Q(0, U) = U^n + b_2(0)U^{n-2} + \cdots + b_n(0) \not\equiv (U - c)^n$$
for any $c \in \mathbb{C}$.
- Hence, $Q(0, U)$ is the product of two coprime polynomials in $\mathbb{C}[U]$.
- By Hensel's lemma, $Q(w, U)$ is the product of two polynomials $Q_1(w, U)$ and $Q_2(w, U)$ in $\mathbb{C}[[w]][U]$ (resp. $\mathbb{C}\langle w\rangle[U]$).
- Finally, we have
$$f(X, Y) = X^{nr} \cdot Q_1(X^{1/q}, X^{-r}Y) \cdot Q_2(X^{1/q}, X^{-r}Y).$$

Plan

The extended Hensel construction (EHC)

Goal

- Factorize $F(X,Y) \in \mathbb{C}[X,Y]$ into linear factors in X over $\mathbb{C}(\langle Y^* \rangle)$:
$$F(X,Y) = (X - \chi_1(Y))(X - \chi_2(Y)) \cdots (X - \chi_d(Y))$$
where each $\chi_i(Y)$ is a *Puiseux series*.
- Thus offers an alternative algorithm to that of Newton-Puiseux.

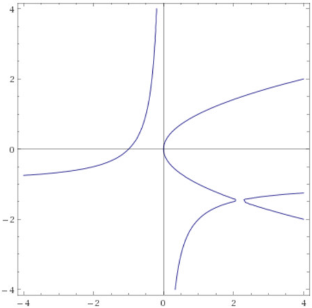
Remarks

- The EHC generalizes to factorize polynomials over multivariate power series rings
- Hence, the EHC has similar goal to Abhyankar-Jung theorem
- However, it is a weaker form:
  - less demanding hypotheses, and
  - weaker output format, making it easier to compute.

An example with the PowerSeries library

```
> P := PowerSeries([y]):
> U := UnivariatePolynomialOverPowerSeries([y], x):
> poly := y^2 *x + y^2 - y*x^3 - y*x^2 + y -x^2;
                              3      2         2    2    2
                  poly := -x   y - x   y + x y   - x   + y   + y

U:-ExtendedHenselConstruction(poly,[0],3);
                         -T - 1          2         2          2       2
           [[y = T, x = ------], [y = T , x = -T ], [y = T , x = T ]]
                           T
```



Another example

```
> P := PowerSeries([y, z]) :
  U := UnivariatePolynomialOverPowerSeries([y, z], x) :
  poly := y·x^3 + (-2 · y + z + 1)·x + y :
  U:-ExtendedHenselConstruction(poly, [0, 0], 3);
```

$$\left[\left[x = \frac{-RootOf(\_Z^2 + y) + RootOf(\_Z^2 + y)\, y - \frac{1}{2}\, RootOf(\_Z^2 + y)\, z + \frac{1}{2}\, y^2}{y}\right],\right.$$

$$\left[x = \frac{RootOf(\_Z^2 + y) - RootOf(\_Z^2 + y)\, y + \frac{1}{2}\, RootOf(\_Z^2 + y)\, z + \frac{1}{2}\, y^2}{y}\right],$$

$$\left.[x = -y]\right]$$

Related works (1/2)

1. Extended Hensel Construction (EHC):
   - Introduction: F. Kako and T. Sasaki, 1999
   - Extensions:
     - M. Iwami, 2003,
     - D. Inaba, 2005,
     - D. Inaba and T. Sasaki 2007,
     - D. Inaba and T. Sasaki 2016.
2. Newton-Puiseux:
   - H. T. Kung and J. F. Traub, 1978,
   - D. V. Chudnovsky and G. V. Chudnovsky, 1986
   - A. Poteaux and M. Rybowicz, 2015.

Related works (2/2)

- The Extended Hensel Construction (EHC) compute all branches concurrently
- while approaches based on Newton-Puiseux computes one branch after another.

For $F(X,Y) := -X^3 + Y\,X + Y$:

1. the EHC produces
   1. $\chi_1(Y) := Y^{\frac{1}{3}} + \frac{1}{3}\,Y^{\frac{2}{3}} + O(Y)$,
   2. $\chi_2(Y) := \frac{-1+\sqrt{-3}}{2}\,Y^{\frac{1}{3}} + \frac{1}{3}\,(\frac{-1-\sqrt{-3}}{2})\,Y^{\frac{2}{3}} + O(Y)$,
   3. $\chi_3(Y) := (\frac{-1-\sqrt{-3}}{2})\,Y^{\frac{1}{3}} + \frac{1}{3}\,(\frac{-1+\sqrt{-3}}{2})\,Y^{\frac{2}{3}} + O(Y)$.
2. Whereas Kung-Traub's method (based on Newton-Puiseux) computes
   1. $\chi_1(Y) := Y^{\frac{1}{3}} + \frac{1}{3}\,Y^{\frac{2}{3}} + O(Y)$,
   2. $\chi_2(Y) := \theta\,Y^{\frac{1}{3}} + \frac{\theta^2}{3}\,Y^{\frac{2}{3}} + O(Y)$,
   3. $\chi_3(Y) := \theta^2\,Y^{\frac{1}{3}} + \frac{\theta}{3}\,Y^{\frac{2}{3}} + O(Y)$,

   for $\theta \in \mathbb{C}$ such that $\theta^3 = 1, \theta^2 \neq 1, \theta \neq 1$, since $F(X,Y)$ is a Weierstrass polynomial.

Related works (2/2)

- The Extended Hensel Construction (EHC) compute all branches concurrently
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    3. $\chi_3(Y) := \theta^2\,Y^{\frac{1}{3}} + \frac{\theta}{3}\,Y^{\frac{2}{3}} + O(Y)$,

    for $\theta \in \mathbb{C}$ such that $\theta^3 = 1, \theta^2 \neq 1, \theta \neq 1$, since $F(X,Y)$ is a Weierstrass polynomial.

Overview

Notations

- Let $F(x, y) \in \mathbb{C}[x, y]$ be square-free, monic in x and let $d := \deg_x(F)$.
- Note that assuming $F(x, y)$ is general in x of order $d = \deg_x(F)$ (thus meaning $F(x, 0) = x^d$ and $F(x, y)$ is a Weierstrass polynomial) is a stronger condition, which is not required here.
- On can easily reduce to the case where F is monic in x as long as the leading coefficient of F in x can be seen an invertible power series in $\mathbb{C}\langle y \rangle$.

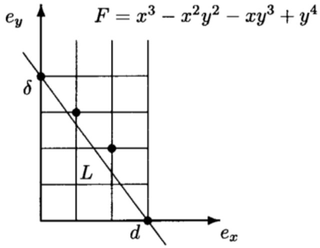
Objectives

- The final goal is to to factorize F over the field $\mathbb{C}(\langle y^* \rangle)$ of convergent Puiseux series over $\mathbb{C}$.
- This follows the ideas of Hensel lemma: lifting the factors of an initial factorization.
- If the initial factorization has no multiple roots, then we are able to generate the homogeneous parts (one degree after another) of the coefficients of the factors predicted by Puiseux's theorem.

Newton line (1/2)

Definition

- We consider a 2D grid G where the Cartesian coordinates (e_x, e_y) of a point are integers.
- Each nonzero term $c\, x^{e_x} y^{e_y}$ of $F(x, y)$, with $c \in \mathbb{C}$ is mapped to the point of coordinates (e_x, e_y) on the grid.
- Let L be the straight line passing through the point $(d, 0)$ as well as another point of the plot of F such that no points in the plot of F lye below L; The line L is called the *Newton line* of F.



Newton line (2/2)

```
> F := x^3  - x^2 * y^2  -x*y^3 + y^4;
                                  2  2       3    4    3
                           F := -x  y  - x y  + y  + x
> U := UnivariatePolynomialOverPowerSeries([y], x):
> U:-ExtendedHenselConstruction(F,[0],2);
                                       5      6
       3       4            5        T      T
[[y = T , x = T  %1 - 1/3 T  %1 + ---- + ----],
                                      3      3
          3        4        5        6
    [y = T , x = -T  - 1/3 T  + 1/3 T ],
                                              6
          3        4       4        5       T
    [y = T , x = -T  %1 + T  + 1/3 T  %1 + ----]]
                                             3
                 2
%1 := RootOf(_Z  - _Z + 1)
```

Newton polynomial

Definition

The sum of all the terms of $F(x, y)$, which are plotted on the Newton line of F is called the *Newton polynomial* of F and is denoted by $F^{(0)}(x, y)$.

Remarks

- The Newton polynomial is uniquely determined and has at least two terms.
- Let $\delta \in \mathbb{Q}$ such that the equating of the Newton line is $e_x/d + e_y/\delta = 1$.
- Observe that $F^{(0)}(x, y)$ is homogeneous in $(x, y^{\delta/d})$ of degree d.
- That is, $F^{(0)}(x, y)$ consists of monomials included in the set $\{x^d, x^{d-1}y^{\delta/d}, x^{d-2}y^{2\delta/d}, \ldots, y^{d\delta/d}\}$.

Factorizing Newton polynomial (1/2)

Notations

Let $r \geq 1$ be an integer, let $\zeta_1, \ldots, \zeta_r \in \mathbb{C}$, with $\zeta_i \neq \zeta_j$ for any $i \neq j$ and let $m_1, \ldots, m_r \in \mathbb{N}$ be positive such that we have

$$F^{(0)}(x,1) = (x - \zeta_1)^{m_1} \cdots (x - \zeta_r)^{m_r}.$$

Recall that $F^{(0)}(x,y)$ is homogeneous in $(x, y^{\delta/d})$ of degree d.

Lemma

We have:

$$F^{(0)}(x,y) = (x - \zeta_1 y^{\delta/d})^{m_1} \cdots (x - \zeta_r y^{\delta/d})^{m_r}.$$

Proof of the lemma

- It is enough to show that $(\zeta_i y^{\delta/d}, y)$ vanishes $F^{(0)}(x,y)$ for all y.
- Define $\hat{y} = y^{\delta/d}$ such that $F^{(0)}(x, \hat{y})$ is homogeneous of degree d in $(x, \hat{y})$.
- Since each monomial of $F^{(0)}(x, \hat{y})$ is of the form $x^{e_x} y^{e_y}$ where $e_x + e_y = d$, we have
$$F^{(0)}(\zeta_i \hat{y}, \hat{y}) = \hat{y}^d \underbrace{(\cdots)}_{\text{some constant terms}} = 0.$$
- The last equality is valid since $F^{(0)}(\zeta_i, 1) = 0$ clearly holds.

Factorizing Newton polynomial (2/2)

```
> F := x^3  - x^2 * y^2  -x*y^3 + y^4;
                                  2  2      3    4    3
                           F := -x  y  - x y  + y  + x

>  L := x^3 - y^4;
                                          4    3
                                   L := -y  + x

> PolynomialTools:-Split(eval(L,[y=1]), x);
                        2                              2
  (x - 1) (x - RootOf(_Z  + _Z + 1)) (x + 1 + RootOf(_Z  + _Z + 1))

> U:-ExtendedHenselConstruction(F,[0],1);
                                       5      6
       3       4            5         T      T
[[y = T , x = T  %1 - 1/3 T  %1 + ---- + ----],
                                      3      3
          3        4        5        6
    [y = T , x = -T  - 1/3 T  + 1/3 T ],
                                                6
          3        4       4        5         T
    [y = T , x = -T  %1 + T  + 1/3 T  %1 + ----]]
                                               3
                 2
%1 := RootOf(_Z  - _Z + 1)
```

The moduli of the Hensel-Sasaki construction (1/2)

Notations

Let $\hat{\delta}, \hat{d} \in \mathbb{Z}^{>0}$ such that:

$$\hat{\delta}/\hat{d} = \delta/d, \quad \gcd \hat{\delta}, \hat{d} = 1$$

Choosing such integers $\hat{\delta}, \hat{d}$ is possible since $\delta \in \mathbb{Q}$ and $d \in \mathbb{N}^{>0}$.

Lemma

Each non-constant monomial of $F(x, y)$ is contained in the set

$$\{x^d y^{(k+0)/\hat{d}}, x^{d-1} y^{(k+\hat{\delta})/\hat{d}}, x^{d-2} y^{(k+2\hat{\delta})/\hat{d}}, \ldots, x^0 y^{(k+d\hat{\delta})/\hat{d}} \mid k = 0, 1, 2, \ldots\}.$$

Proof of the lemma

- It is enough to show that for each exponent vector (e_x, e_y) which is not below the Newton's line, there exists i, k such that we have $x^{e_x} y^{e_y} = x^{d-i} y^{(k+i\hat{\delta})/\hat{d}}$.
- Given such an exponent vector (e_x, e_y), let us choose $i = d - e_x$ and $k = e_y \hat{d} - i\hat{\delta}$.
- One should check, of course, that $k \geq 0$ holds, which follows easily from $e_x/d + e_y/\delta \geq 1$.

The moduli of the Hensel-Sasaki construction (2/2)

Notations

The previous lemma leads us to define the following monomial ideals

$$\begin{array}{rcl} S_k & = & <x, y^{\hat{\delta}/\hat{d}}>^d \times <y^{1/\hat{d}}>^k \\ & = & <x^d, x^{d-1}y^{\hat{\delta}/\hat{d}}, x^{d-2}y^{2\hat{\delta}/\hat{d}}, \ldots, x^0 y^{d\hat{\delta}/\hat{d}}> \times <y^{1/\hat{d}}>^k \\ & = & <x^d y^{(k+0)/\hat{d}}, x^{d-1}y^{(k+\hat{\delta})/\hat{d}}, x^{d-2}y^{(k+2\hat{\delta})/\hat{d}}, \ldots, x^0 y^{(k+d\hat{\delta})/\hat{d}}> \end{array}$$

Remark

- The generators of $<x, y^{\hat{\delta}/\hat{d}}>^d$ are homogeneous monomials in $(x, y^{\hat{\delta}/\hat{d}})$ of degree d.
- We can view S_k as a polynomial ideal in the variables x and $y^{1/\hat{d}}$; note that the monomials generating S_k regarded in this way do not all have the same total degree.
- We shall use the ideals S_k, for $k = 1, 2, \ldots$, as moduli of the Hensel-Sasaki construction to be described hereafter.
- We have $F(x, y) \equiv F^{(0)}(x, y) \mod S^{(1)}$.

Algorithm 1: EHC_Lift(F, k)

begin

Compute the Newton polynomial $F^{(0)}$ and $\hat{\delta}, \hat{d}$;

Compute $G_i^{(0)} = (X - \zeta_i Y)^{m_i}$, with $1 \leq i \leq r$;

Compute the Yun-Moses polynomial $W_i^{(\ell)}$ for $i = 1, \cdots, r$ and $\ell = 0, \ldots, d-1$;

for $j = 1, \ldots, k$ **do**

  Compute $\Delta F^{(j)}(X,Y) := F(X,Y) - \prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$;

  Compute $\Delta G_i^{(j)} = \sum_{\ell=0}^{m-1} W_i^{(\ell)} f_\ell^{(j)}$, for $i = 1, \cdots, r$;

  Let $G_i^{(j)} = G_i^{(j-1)} + \Delta G_i^{(j)}$ for $i = 1, \cdots, r$;

return $G_1^{(k)}, \ldots, G_r^{(k)}$;

Algorithm 2: EHC_Lift(F, k)

begin

Compute the Newton polynomial $F^{(0)}$ and $\hat{\delta}, \hat{d}$;

Compute $G_i^{(0)} = (X - \zeta_i Y)^{m_i}$, with $1 \leq i \leq r$;

Compute the Yun-Moses polynomial $W_i^{(\ell)}$ for $i = 1, \cdots, r$ and $\ell = 0, \cdots, d-1$;

for $j = 1, \ldots, k$ **do**

Compute $\Delta F^{(j)}(X, Y) := F(X, Y) - \prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$;

Compute $\Delta G_i^{(j)} = \sum_{\ell=0}^{m-1} W_i^{(\ell)} f_\ell^{(j)}$, for $i = 1, \cdots, r$;

Let $G_i^{(j)} = G_i^{(j-1)} + \Delta G_i^{(j)}$ for $i = 1, \cdots, r$;

return $G_1^{(k)}, \ldots, G_r^{(k)}$;

Algorithm 3: EHC_LiftF, k

begin

Compute the Newton polynomial $F^{(0)}$ and $\hat{\delta}, \hat{d}$;

Compute $G_i^{(0)} = (X - \zeta_i Y)^{m_i}$, with $1 \leq i \leq r$;

Compute the Yun-Moses polynomial $W_i^{(\ell)}$ for $i = 1, \cdots, r$ and $\ell = 0, \cdots, d-1$;

for $j = 1, \ldots, k$ **do**

Compute $\Delta F^{(j)}(X,Y) := F(X,Y) - \prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$;

Compute $\Delta G_i^{(j)} = \sum_{\ell=0}^{m-1} W_i^{(\ell)} f_\ell^{(j)}$, for $i = 1, \cdots, r$;

Let $G_i^{(j)} = G_i^{(j-1)} + \Delta G_i^{(j)}$ for $i = 1, \cdots, r$;

return $G_1^{(k)}, \ldots, G_r^{(k)}$;

Example of Extended Hensel Construction

Consider

$$F(x,y) = x^5 + x^4\,y - 2\,x^3\,y - 2\,x^2\,y^2 + x\,(y^2 - y^3) + y^3. \tag{4}$$

Then, we have

- $d = \deg_x(F(x,y)) = 5$,
- Newton line: $e_x/5 + e_y/2.5 = 1$
- $\delta/d = 1/2 = \hat{\delta}/\hat{d}$
- $S_0 = <x^5, x^4\,y^{1/2}, x^3\,y, x^2\,y^{3/2}, x\,y^2, y^{5/2}>$
- $F^{(0)}(x,y) = x^5 - 2\,x^3\,y + x\,y^2 = x\,(x + y^{1/2})^2\,(x - y^{1/2})^2$

Note that

$$F^{(0)}(x,1) = x\,(x+1)^2\,(x-1)^2 \tag{5}$$

Example of Extended Hensel Construction

Hence, we can put

$$G_1^{(0)} = x, G_2^{(0)} = (x + y^{1/2})^2, G_3^{(0)} = (x - y^{1/2})^2.$$

Yun-Moses polynomials are calculated as,

$$\begin{array}{lll}
W_1^{(0)} = y^{1/2} & W_2^{(0)} = -\frac{1}{2}x\,y^{1/2} - \frac{3}{4}y & W_3^{(0)} = -\frac{1}{2}xy^{1/2} + \frac{3}{4}y \\
W_1^{(1)} = 0 & W_2^{(1)} = \frac{1}{4}x\,y^{1/2} + \frac{1}{2}y & W_3^{(1)} = -\frac{1}{4}x\,y^{1/2} + \frac{1}{2}y \\
W_1^{(2)} = 0 & W_2^{(2)} = -\frac{1}{4}y & W_3^{(2)} = \frac{1}{4}y \\
W_1^{(3)} = 0 & W_2^{(3)} = -\frac{1}{4}x\,y^{1/2} & W_3^{(3)} = \frac{1}{4}x\,y^{1/2} \\
W_1^{(4)} = 0 & W_2^{(4)} = \frac{1}{2}x\,y^{1/2} + \frac{1}{4}y & W_3^{(4)} = \frac{1}{2}x\,y^{1/2} - \frac{1}{4}y
\end{array}$$

Example of Extended Hensel Construction

For

$$S_2 = < x^5\, y, x^4 y^{3/2}, x^3 y^2, x^2 y^{5/2}, x y^3, y^{7/2} >$$

We have,

$$\begin{aligned} \Delta F^{(1)} &\equiv F - G_1^{(0)} G_2^{(0)} G_3^{(0)} \mod S_2 \\ &= x^4 y - 2x^2 y^2 - x y^3 + y^3 \\ &= y^{1/2} \cdot x^4 y^{1/2} - 2y^{1/2} \cdot x^2 y^{3/2} + y^{1/2} y^{5/2} \end{aligned}$$

The last representation of $\Delta F^{(1)}$ in the last equation is for the purpose of computing $f_\ell^{(1)}$ for $\ell = 0, \ldots, d-1$ in

$$\Delta F^{(k)} = \sum_{\ell=0}^{5-1} f_\ell^{(k)} \hat{y}^{d-\ell} x^\ell \quad \text{when } k = 1$$

Example of Extended Hensel Construction

Therefore,

$$f_4^{(1)} = y^{1/2}, f_2^{(1)} = -2y^{1/2}, f_0^{(1)} = y^{1/2}, f_3^{(1)} = f_1^{(1)} = 0$$

Considering the above polynomials and also the Lagrange's interpolation polynomials, we obtain:

- $G_1^{(1)} = G_1^{(0)} + W_1^{(0)} f_0^{(1)} = x + y$
- $G_2^{(1)} = G_2^{(0)} + W_2^{(4)} f_4^{(1)} + W_2^{(0)} f_0^{(1)} + W_2^{(2)} f_2^{(1)} = (x + y^{1/2})^2$
- $G_3^{(1)} = G_3^{(0)} + W_3^{(4)} f_4^{(1)} + W_3^{(0)} f_0^{(1)} + W_3^{(2)} f_2^{(1)} = (x - y^{1/2})^2$

Example of Extended Hensel Construction

Now for $S_3 = <x^5y^{3/2}, x^4y^2, x^3y^{5/2}, x^2y^3, xy^{7/2}, y^4>$, we have

$$\begin{array}{rcl} \Delta F^{(2)} & \equiv & F - G_1^{(1)} G_2^{(1)} G_3^{(1)} \mod S_3 \\ & = & -y \cdot xy^2 \end{array} .$$

Hence,

$$f_1^{(2)} = -y, f_0^{(2)} = f_2^{(2)} = f_3^{(2)} = f_4^{(2)} = 0.$$

And then we obtain,

- $G_1^{(2)} = G_1^{(1)} + 0 = x + y$
- $G_2^{(2)} = G_2^{(1)} + W_2^{(1)} f_1^{(2)} = (x + y^{1/2})^2 - (\frac{1}{4}x\, y^{3/2} + \frac{1}{2}y^2)$
- $G_3^{(2)} = G_3^{(1)} + W_3^{(1)} f_1^{(2)} = (x - y^{1/2})^2 + (\frac{1}{4}x\, y^{3/2} - \frac{1}{2}y^2)$

Example of Extended Hensel Construction

Continuing two more iterations, we have

- $G_1^{(4)} = x + y + y^2$
- $G_2^{(4)} = (x + y^{\frac{1}{2}})^2 - (\frac{1}{4}x\,y^{\frac{3}{2}} + \frac{1}{2}y^2) - (\frac{1}{2}xy^2 + \frac{3}{4}y^{\frac{5}{2}}) - (\frac{53}{64}xy^{\frac{5}{2}} + \frac{9}{8}y^3)$
- $G_3^{(4)} = (x - y^{\frac{1}{2}})^2 + (\frac{1}{4}x\,y^{\frac{3}{2}} - \frac{1}{2}y^2) - (\frac{1}{2}xy^2 + \frac{3}{4}y^{\frac{5}{2}}) + (\frac{53}{64}xy^{\frac{5}{2}} - \frac{9}{8}y^3)$

We note that $G_2^{(4)}$ and $G_3^{(4)}$ can be written as:

- $G_2^{(4)} = G_P^{(4)} + y^{1/2}G_A^{(4)}$
- $G_3^{(4)} = G_P^{(4)} - y^{1/2}G_A^{(4)}$

where

- $G_P^{(4)} = x^2 + y - \frac{1}{2}y^2 - \frac{1}{2}x\,y^2 - \frac{9}{8}y^3$
- $G_A^{(4)} = 2x - \frac{1}{4}x\,y - \frac{3}{4}y^2 - \frac{53}{64}xy^2$

Note: $G_1^{(\infty)} \in \mathbb{C}[x, y]$, since $F^{(0)}(x, y) = x(x^4 - 2x^2y + y^2)$

Yun-Moses Polynomials (1/3)

Assume $G_1(X,Y),\ldots,G_r(X,Y)$ are homogeneous polynomials. Regarding them as polynomials of $\mathbb{C}\langle Y\rangle[X]$, further assume

$$\gcd(\hat{G}_i,\hat{G}_j)=1 \text{ for } i\neq j,$$

Let $d:=\deg(G_1(X,Y)\,\ldots\,G_r(X,Y))$. Then, for each $\ell\in\{0,\ldots,d-1\}$, there exists a unique set of polynomials $\{W_i^{(\ell)}(X,Y)\in\mathbb{C}\langle Y\rangle[X]\mid i=1,\ldots,r\}$ satisfying

$$W_1^{(\ell)}\left(\frac{G_1\cdots G_r}{G_1}\right)+\cdots+W_r^{(\ell)}\left(\frac{G_1\cdots G_r}{G_r}\right)=X^\ell Y^{d-\ell},$$

where $\deg_X(W_i^{(\ell)}(X,Y))<\deg_X(G_i(X,Y)),\quad i=1,\ldots,r.$

Yun-Moses Polynomials (2/3)

Key observation

Let us fix $i := \lambda$. Writing $W_\lambda^{(\ell)} = \sum_{j=0}^{m_\lambda - 1} w_{\lambda,j}(\hat{Y}) X^j$, we have

$$\sum_{j=0}^{m_\lambda - 1} \frac{\partial^\mu}{\partial X^\mu} \left(X^j \frac{F^{(0)}}{G_\lambda^{(0)}} \right) \Big|_{X=\zeta_\lambda \hat{Y}} w_{\lambda,j}^{(\ell)} = \frac{\partial^\mu}{\partial X^\mu} (X^\ell \hat{Y}^{d-\ell}) \Big|_{X=\zeta_\lambda \hat{Y}} .$$

where ζ_λ is a root of $F^{(0)}(X,1)$ and m_λ is its multiplicity

Consequences

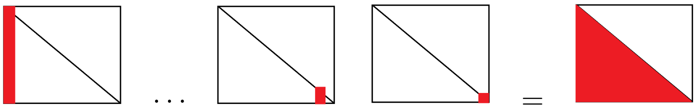
- This is a system of linear equations $\mathcal{W}_\lambda \mathcal{X}_\lambda^{(\ell)} = \mathcal{B}_\lambda^{(\ell)}$.
- The matrix $\mathcal{W}_\lambda$ is a Wronskian matrix.

The inverse of $\mathcal{W}_\lambda$ is $\mathcal{W}_\lambda^{-1} = M_2 M_1$ where M_1 and M_2 are square matrices of order m_λ, defined as follows. The matrix M_1 writes
$M_1 = M_{1(m_\lambda-1)} \cdots M_{11} M_{10}$ such that, for $j = 0, \cdots, m_\lambda - 1$, we have

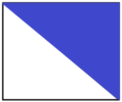
$$M_{1j} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{j!f} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \binom{j+1}{j}\frac{-f'}{f} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \binom{m_\lambda-1}{j}\frac{-f^{(m_\lambda-1-j)}}{f} & 0 & \cdots & 1 \end{bmatrix}.$$

Hence, the matrix M_{1j} differs from the identity matrix only in its $(j+1)$-th column. The matrix M_2 is an upper triangular matrix $M_2 = [\gamma_{j,k}]$ with
$\gamma_{j,k} = (-1)^{j+k}\binom{k}{k-j}\zeta_\lambda^{k-j}\hat{Y}^{k-j}$ if $j \leq k$ and $\gamma_{j,k} = 0$ if $j > k$, for
$j, k \in \{0, 1, \ldots, m_\lambda - 1\}$.

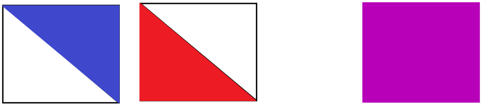
Matrix M_1



Matrix M_2



Matrix $\mathcal{W}_i^{-1} = M_2 M_1$



Complexity Result:

Theorem 1:

One can compute all the Yun-Moses polynomials $W_i^{(\ell)}$ $(0 \leq \ell \leq d-1,$ $1 \leq i \leq r)$, within

- $\mathcal{O}(d^3)$ operations in $\mathbb{C}$, or
- $\mathcal{O}(d^3\, \mathsf{M}(d))$ operations in the field of coefficients of $F(X, Y)$.

Algorithm 4: EHC_LiftF, k

begin

Compute the Newton polynomial $F^{(0)}$ and $\hat{\delta}, \hat{d}$;

Compute $G_i^{(0)} = (X - \zeta_i Y)^{m_i}$, with $1 \leq i \leq r$;

Compute the Yun-Moses polynomial $W_i^{(\ell)}$ for $i = 1, \cdots, r$ and $\ell = 0, \cdots, d-1$;

for $j = 1, \ldots, k$ **do**

Compute

$\Delta F^{(j)}(X,Y) := F(X,Y) - \prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$;

Compute $\Delta G_i^{(j)} = \sum_{\ell=0}^{m-1} W_i^{(\ell)} f_\ell^{(j)}$, for $i = 1, \cdots, r$;

Let $G_i^{(j)} = G_i^{(j-1)} + \Delta G_i^{(j)}$ for $i = 1, \cdots, r$;

return $G_1^{(k)}, \ldots, G_r^{(k)}$;

Computing $\Delta F^{(j)}(X,Y)$

Goal

$$\Delta F^{(j)}(X,Y) := F(X,Y) - \prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$$

Oobservation

- $G_i^{(j-2)} := G_i^{(0)} + \Delta G_i^{(1)} + \cdots + \Delta G_i^{(j-2)}$
- $G_i^{(j-1)} := G_i^{(0)} + \Delta G_i^{(1)} + \cdots + \Delta G_i^{(j-2)} + \Delta G_i^{(j-1)}$

Hence, we aim at recycling terms in the product $\prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$ computed from previous iterations.

Notations

1. $P_2^{k+1} := \prod_{i=1}^{2} G_i^{(k)} \mod \overline{S}_{k+1}$
2. $P_j^{k+1} := \prod_{i=1}^{j} G_i^{(k)} \mod \overline{S}_{k+1}$, for $j = 3, \ldots, r$.

We want

$$P_r^{k+1} = \prod_{i=1}^{r} G_i^{(k)} \mod \overline{S}_{k+2}$$

Computing $\Delta F^{(j)}(X,Y)$

Initially define: $P_j^1 \equiv G_1^{(0)} \cdots G_j^{(0)} \mod S_2$, for $j = 2, \cdots, r$. and recursively compute:

$$P_2^{k+1} = P_2^k + (\Delta_1^0 \Delta_2^k + \Delta_1^k \Delta_2^0)\tilde{Y}^k + (\Delta_1^1 \Delta_2^k + \cdots + \Delta_1^k \Delta_2^1)\tilde{Y}^{k+1} = \prod_{i=1}^{2} G_i^{(k)}$$

Now for $j = 3, \ldots, r$, define

$$P_j^k \equiv P_{j-1}^k G_j^{(k-1)} \mod S_{k+1}$$

and assume q_j^{k+1} is recursively given by

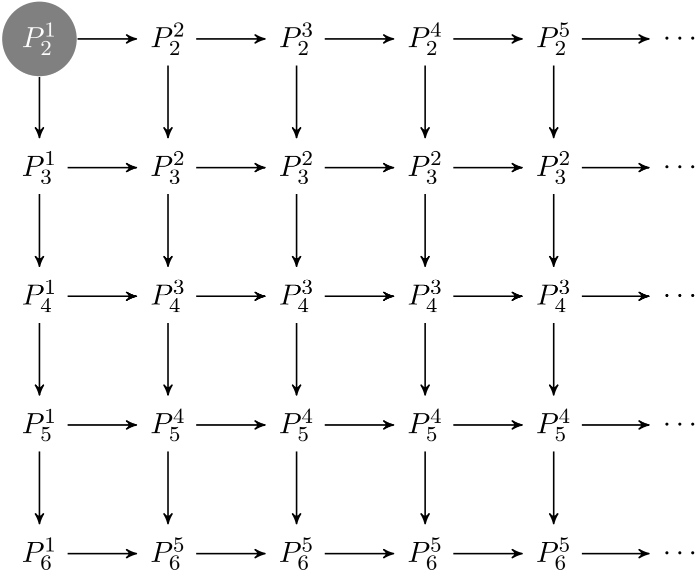
$$q_j^{k+1} = p_{j-1}^{k+1,0} \Delta_j^k + q_{j-1}^{k+1} \Delta_j^0 \quad \text{with} \quad q_2^{k+1} = \Delta_2^k \Delta_1^0 + \Delta_2^0 \Delta_1^k. \tag{6}$$

where $p_{j-1}^{k+1,0}$ is the coefficient of $\tilde{Y}^0$ in P_{j-1}^{k+1}. We can compute

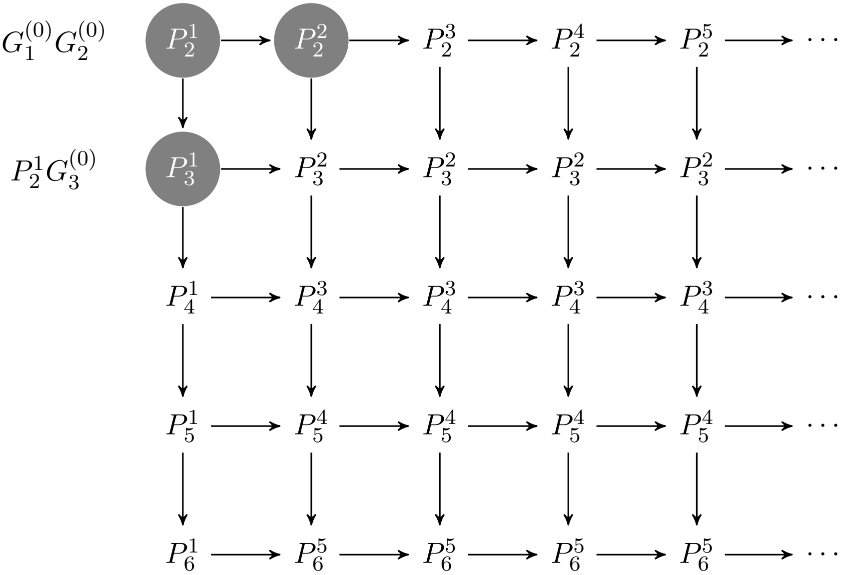
$$P_j^{k+1} = P_j^k + q_j^{k+1}\tilde{Y}^k + \left(p_{j-1}^{k+1,1}\Delta_j^k + \cdots + p_{j-1}^{k+1,k+1}\Delta_j^0\right)\tilde{Y}^{k+1} = \prod_{i=1}^{j} G_i^{(k)}$$

Computing $\Delta F^{(j)}(X, Y)$

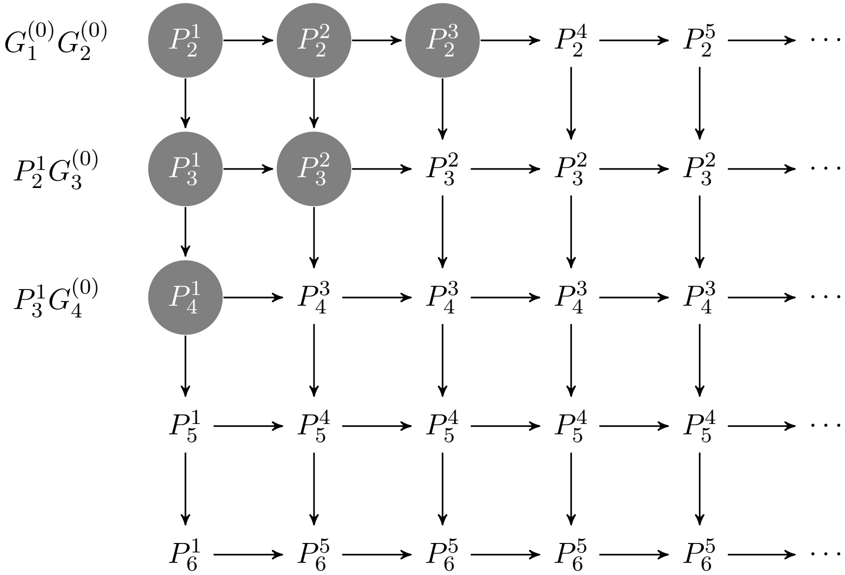
$G_1^{(0)}G_2^{(0)}$



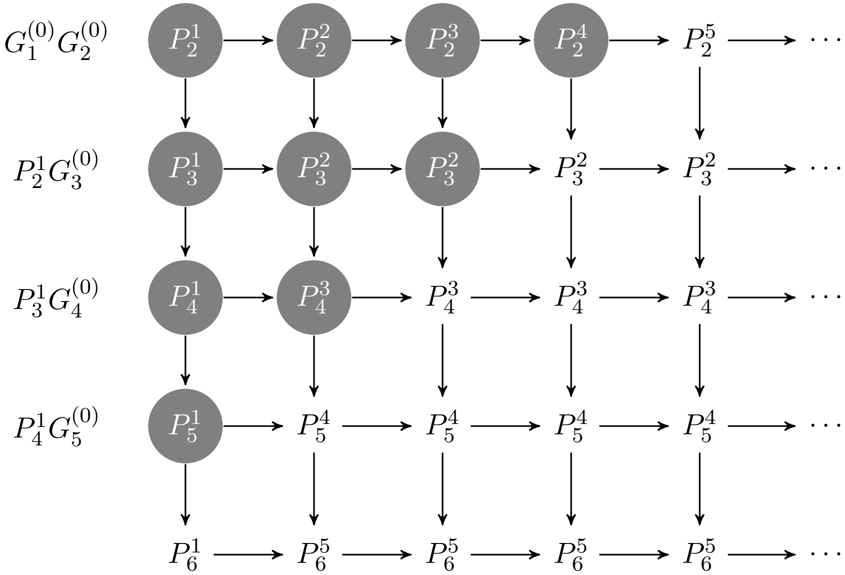
Computing $\Delta F^{(j)}(X, Y)$



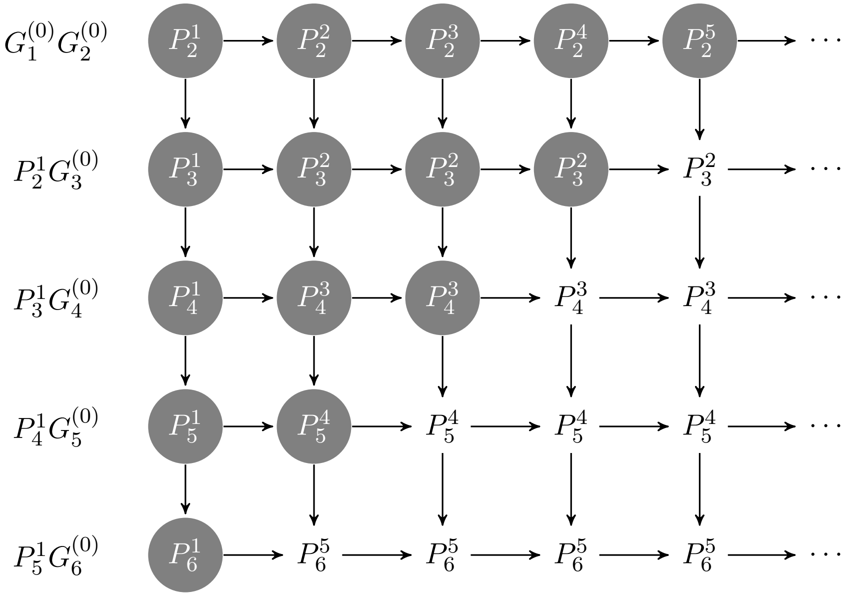
Computing $\Delta F^{(j)}(X,Y)$



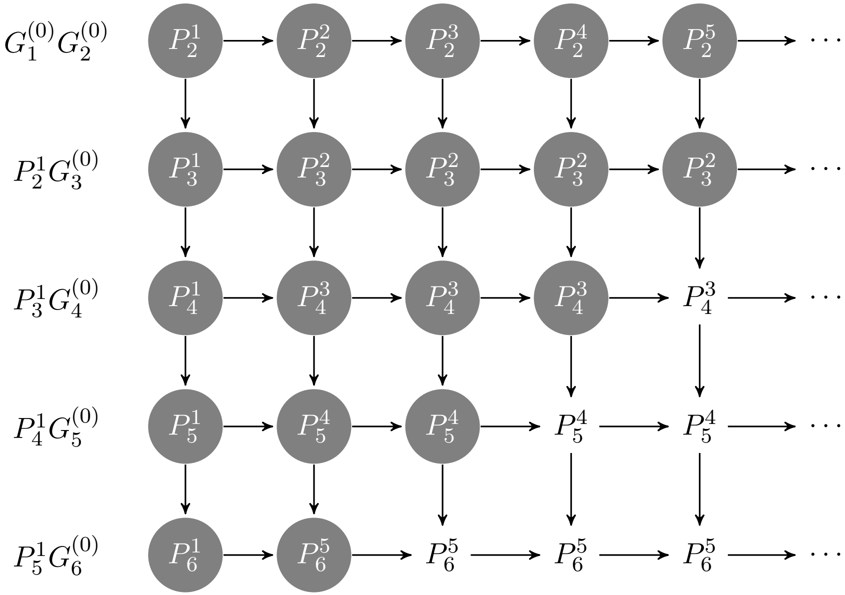
Computing $\Delta F^{(j)}(X,Y)$



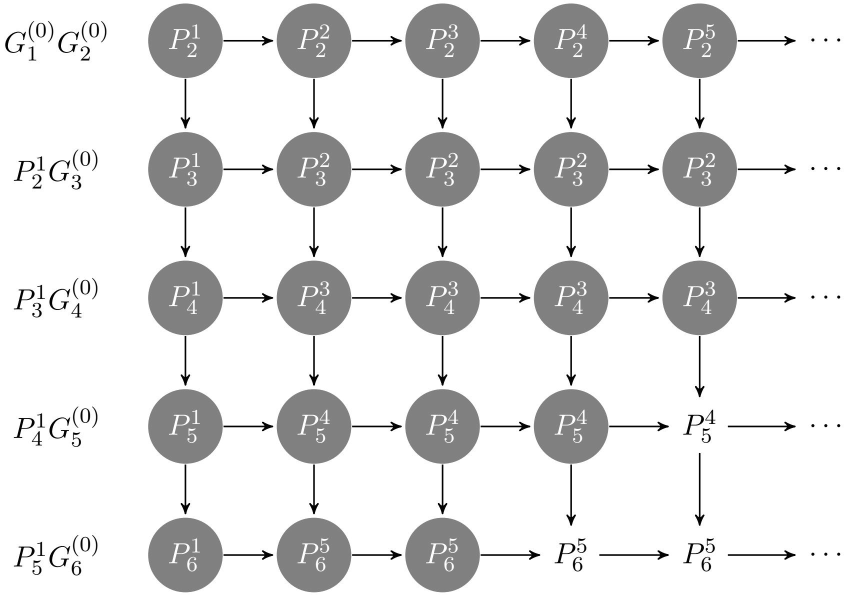
Computing $\Delta F^{(j)}(X,Y)$



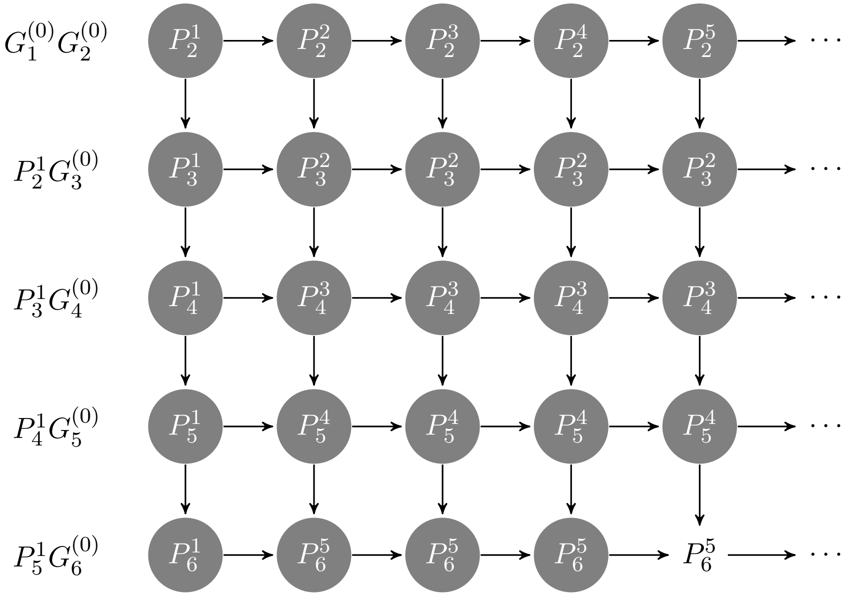
Computing $\Delta F^{(j)}(X, Y)$



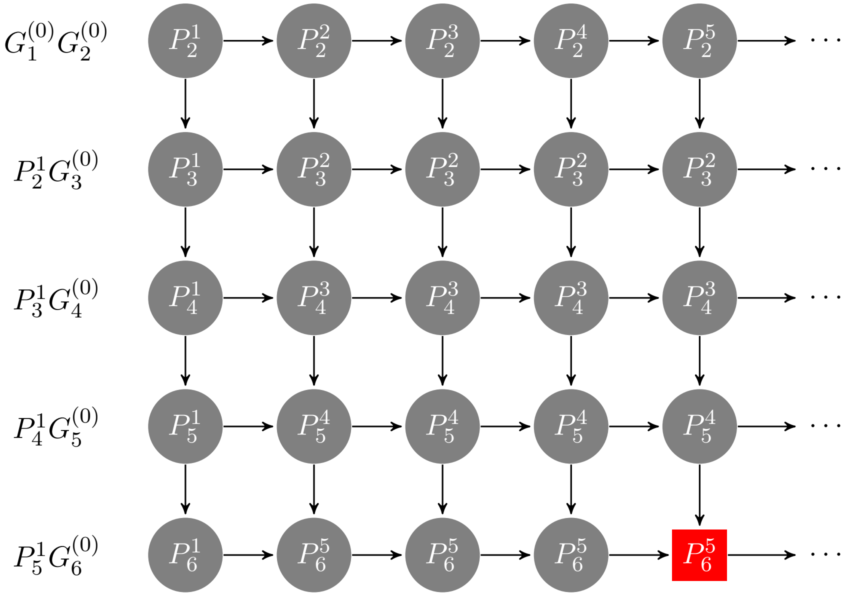
Computing $\Delta F^{(j)}(X,Y)$



Computing $\Delta F^{(j)}(X,Y)$



Computing $\Delta F^{(j)}(X,Y)$



Complexity result:

Theorem 2:

he k-th iteration of Step 9 in the Algorithm 4 runs within

- $\mathcal{O}(k\, d\mathsf{M}(d))$ operations in $\mathbb{C}$,
- $\mathcal{O}(k\, d\mathsf{M}(d)^2)$ operations in the field of coefficients of $F(X,Y)$.

Comparative complexity results

Theorem 3:

Our enhancement of the EHC computes all the branches in $\mathcal{O}(k^2\, d\, \mathsf{M}(d))$ operations in $\mathbb{C}$, using a *linear lifting scheme*.

Kung-Traub, 1987

The first k iterations of Newton-Puiseux on an input bivariate polynomial of degree d computes all branches within

- $\mathcal{O}(d^2\, k\, \mathsf{M}(k))$ operations in $\mathbb{C}$ using a *linear lifting scheme* (Theorem 5.2 in their paper)
- $\mathcal{O}(d^2\, \mathsf{M}(k))$ operations in $\mathbb{C}$ using a *quadratic lifting scheme* (Corollary 5.1 in their paper)

D. V. Chudnovsky and G. V. Chudnovsky, 2015

The latter estimate reported by Kung and Traub is improved to $\mathcal{O}(d^2\, k)$ operations in $\mathbb{C}$ for computing all the branches.

Remark

A quadratic lifting scheme for the EHC is work in progress.

Plan

1. Motivating Examples
2. Polynomials over Power Series
   - The Ring of Puiseux Series
   - The Hensel-Sasaki Construction: Bivariate Case
   - Limit Points: Review and Complement
3. Applications
   - Limits of Multivariate Real Analytic Functions
   - Tangent Cones
   - Intersection Multiplicities

Limit points of (the quasi-component of) a regular chain

- Let $R := \{t_2(x_1, x_2), \ldots, t_n(x_1, \ldots, x_n)\}$ where t_i has its coefficients in $\mathbb{C}$.
- We regard t_i as a univariate polynomial w.r.t. x_i, for $i = 2, \ldots, n$:
- We denote by h_i the leading coefficient (also called initial) of t_i w.r.t. x_i, and assume that h_i depends on x_1 only; hence we have
  $t_i \; = \; h_i(x_1) x_i^{d_i} + c_{d_i - 1}(x_1, \ldots, x_{i-1}) x_i^{d_i - 1} + \cdots + c_0(x_1, \ldots, x_{i-1})$
- Consider the system

$$W(R) := \begin{cases} t_n(x_1, \ldots, x_n) = 0 \\ \vdots \\ t_2(x_1, x_2) = 0 \\ (h_2 \cdots h_n)(x_1) \neq 0 \end{cases}$$

- We want to compute the non-trivial limit points of $W(R)$, that is

$$\lim(W(R)) \; := \; \overline{W(R)}^{Z} \setminus W(R).$$

Puiseux expansions of a regular chain (1/2)

Notation

- Let R be as before. Assume R is strongly normalized, that is, every initial is a univariate polynomial in x_1
- Let $\mathbb{K} = \mathbb{C}(\langle x_1^* \rangle)$.
- Then R generates a zero-dimensional ideal in $\mathbb{C}[x_2, \ldots, x_n]$.
- Let $V^*(R)$ be the zero set of R in $\mathbb{K}^{n-1}$.

Definition

We call *Puiseux expansions* of R the elements of $V^*(R)$.

Puiseux expansions of a regular chain (1/2)

A regular chain R

$$R := \begin{cases} X_1 X_3^2 + X_2 \\ X_1 X_2^2 + X_2 + X_1 \end{cases}$$

Puiseux expansions of R

$$\begin{cases} X_3 = 1 + O(X_1^2) \\ X_2 = -X_1 + O(X_1^2) \end{cases} \quad \begin{cases} X_3 = -1 + O(X_1^2) \\ X_2 = -X_1 + O(X_1^2) \end{cases}$$

$$\begin{cases} X_3 = {X_1}^{-1} - \frac{1}{2} X_1 + O(X_1^2) \\ X_2 = -{X_1}^{-1} + X_1 + O(X_1^2) \end{cases} \quad \begin{cases} X_3 = -{X_1}^{-1} + \frac{1}{2} X_1 + O(X_1^2) \\ X_2 = -{X_1}^{-1} + X_1 + O(X_1^2) \end{cases}$$

Relation between $\lim_0(W(R))$ and Puiseux expansions of R

Theorem

For $W \subseteq \mathbb{C}^s$, denote

$$\lim_0(W) := \{x = (x_1, \ldots, x_s) \in \mathbb{C}^s \mid x \in \lim(W) \text{ and } x_1 = 0\},$$

and define

$$V^*_{\geq 0}(R) := \{\Phi = (\Phi^1, \ldots, \Phi^{s-1}) \in V^*(R) \mid \operatorname{ord}(\Phi^j) \geq 0, j = 1, \ldots, s-1\}.$$

Then we have

$$\lim_0(W(R)) = \cup_{\Phi \in V^*_{\geq 0}(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}.$$

$$V^*_{\geq 0}(R) := \left\{ \begin{array}{lcl} X_3 & = & 1 + O(X_1^2) \\ X_2 & = & -X_1 + O(X_1^2) \end{array} \right. \cup \left\{ \begin{array}{lcl} X_3 & = & -1 + O(X_1^2) \\ X_2 & = & -X_1 + O(X_1^2) \end{array} \right.$$

Thus the limit ponts are $\lim_0(W(R)) = \{(0, 0, 1), (0, 0, -1)\}$.

Limit points: this example again

```
> R := PolynomialRing([x, y, z]) :
  rc := Chain([y^(3)-2*y^(3) + y^(2) + z^(5), z^(4)*x + y^(3)-y^(2)], Empty(R), R) : Display(rc, R);
  br := RegularChainBranches(rc, R, [z], coefficient = complex);
```

$$\begin{cases} z^4 x + y^3 - y^2 = 0 \\ -y^3 + y^2 + z^5 = 0 \\ z^4 \neq 0 \end{cases}$$

$$br := \left[\left[z = T^2, y = \frac{1}{2} T^5 \left(-T^5 + 2\, RootOf\left(\_Z^2 + 1\right)\right), x = -\frac{1}{8} T^2 \left(-T^{20} + 6\, T^{15}\, RootOf\left(\_Z^2 + 1\right) + 10\, T^{10} + 8\right)\right],\right.$$
$$\left[z = T^2, y = -\frac{1}{2} T^5 \left(T^5 + 2\, RootOf\left(\_Z^2 + 1\right)\right), x = \frac{1}{8} T^2 \left(T^{20} + 6\, T^{15}\, RootOf\left(\_Z^2 + 1\right) - 10\, T^{10} - 8\right)\right], \left[z\right.$$
$$\left.\left. = T, y = T^5 + 1, x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

```
> br := RegularChainBranches(rc, R, [z], coefficient = real);
```

$$br := \left[\left[z = T, y = T^5 + 1, x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

Figure: The command `RegularChainBranches` computes a parametrization for the complex and real paths of the quasi-component defined by rc. When coefficient argument is set as real, then the command `RegularChainBranches` computes the real branches.

Plan

Plan

Limits of multivariate real rational functions

Notations

Let $q \in \mathbb{Q}(X_1, \ldots, X_n)$ be a multivariate rational function.

The problem

We want to decide whether

$$\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1, \ldots, x_n)$$

exists, and if it does, whether it is finite.

Limits of rational functions: previous works (1/3)

Univariate functions (including transcendental ones)

D. Gruntz (1993, 1996), B. Salvy and J. Shackell (1999)

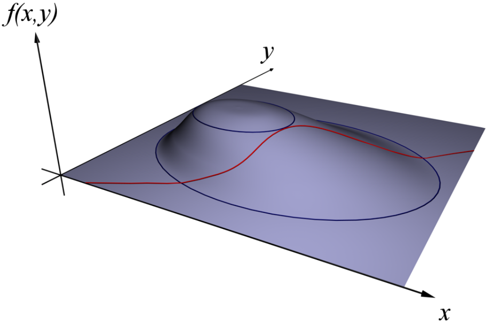
- Corresponding algorithms are available in popular computer algebra systems

Multivariate rational functions

S.J. Xiao and G.X. Zeng (2014)

- Given $q \in \mathbb{Q}(X_1, \ldots, X_n)$, they proposed an algorithm deciding whether or not: $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q$ *exists and is zero*.
- No assumptions on the input multivariate rational function
- Techniques used:
  - triangular decomposition of algebraic systems,
  - rational univariate representation,
  - adjoining infinitesimal elements to the base field.

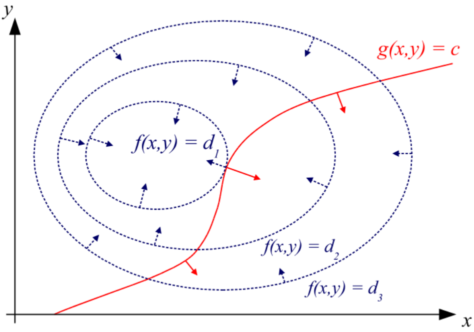
Interlude: the method of Lagrange multipliers (1/3)



- Let f and g be functions from $\mathbb{R}^n$ to $\mathbb{R}$ with continuous first partial derivatives.
- Consider the ooptimization problem

$$\max_{\text{subject to } g(x_1,\ldots,x_n)=0} f(x_1,\ldots,x_n)$$

Interlude: the method of Lagrange multipliers (2/3)



We are looking at points $(x_1, \ldots, x_n)$ where $f(x_1, \ldots, x_n)$ does not change much as we walk along $g(x_1, \ldots, x_n) = 0$. This can happen in two ways:

- either such a point is a optimizer (maximizer or minimizer),
- or we are following a level of f, that is, $f(x_1, \ldots, x_n) = d$ for some d.

Both cases are captured by imposing that the gradient vectors $\nabla_{x_1,\ldots,x_n} f$ and $\nabla_{x_1,\ldots,x_n} g$ are parallel.

Interlude: the method of Lagrange multipliers (3/3)

The previous observation translates into a system of equations that, in particular, maximizers and minimizers must satisfy.

$$
\begin{aligned}
g\left(x_1, x_2, \ldots, x_n\right) &= 0 \\
\frac{\partial f}{\partial x_1}\left(x_1, x_2, \ldots, x_n\right) - \lambda \frac{\partial g}{\partial x_1}\left(x_1, x_2, \ldots, x_n\right) &= 0 \\
\frac{\partial f}{\partial x_2}\left(x_1, x_2, \ldots, x_n\right) - \lambda \frac{\partial g}{\partial x_2}\left(x_1, x_2, \ldots, x_n\right) &= 0 \\
&\vdots \\
\frac{\partial f}{\partial x_n}\left(x_1, x_2, \ldots x_n\right) - \lambda \frac{\partial g}{\partial x_n}\left(x_1, x_2, \ldots, x_n\right) &= 0.
\end{aligned}
$$

where λ is an additional variable, called the Lagrange multiplier of the corresponding optimization problem.

Limits of rational functions: previous works (2/3)

C. Cadavid, S. Molina, and J. D. Vélez (2013):

- Assumes that the origin is an isolated zero of the denominator
- Maple built-in command `limit/multi`

Discriminant variety

$\chi(q) \; = \; \{(x,y) \in \mathbb{R}^2 \;\mid\; y\frac{\partial q}{\partial x} - x\frac{\partial q}{\partial y} = 0\}.$

Key observation

For determining the existence and possible value of

$$\lim_{(x,y)\to(0,0)} q(x,y),$$

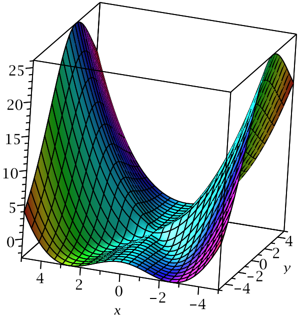
it is sufficient to compute

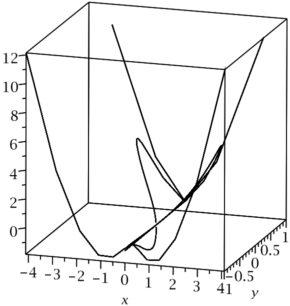
$$\lim_{\substack{(x,y)\,\to\,(0,0)\\(x,y)\,\in\,\chi(q)}} q(x,y).$$

Example

Let $q \in \mathbb{Q}(x,y)$ be a rational function defined by $q(x,y) = \frac{x^4+3x^2y-x^2-y^2}{x^2+y^2}$.

$$\chi(q) \quad = \quad \left\{ \begin{array}{rcl} x^4+2x^2y^2+3y^3 & = & 0 \\ y & < & 0 \end{array} \right. \quad \cup \quad \left\{ \begin{array}{rcl} x & = & 0 \end{array} \right.$$





The discriminant variety of Cadavid, Molina, Vélez (1/2)

Notations

- Let $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function with continuous first partial derivatives.
- For a postive real number ρ, let D_ρ^* be the punctured ball
$$D_\rho^* \; = \; \{(x_1, \ldots, x_n) \in \mathbb{R}^n \; \mid \; 0 < \sqrt{x_1^2 + \cdots + x_n^2} < \rho\}.$$
- Let $\chi(q)$ be the subset of $\mathbb{R}^n$ where the vectors $\nabla_{x_1, \ldots, x_n} q$ and $(x_1, \ldots, x_n)$ are parallel.
- For $n = 2$, we have
$$\chi(q) \; = \; \{(x, y) \in \mathbb{R}^2 \; \mid \; y \frac{\partial q}{\partial x} - x \frac{\partial q}{\partial y} = 0\}.$$

Theorem (Cadavid, Molina, Vélez)

For all $L \in \mathbb{R}$ the following assertions re equivalent:

1. $\lim_{(x_1, \ldots, x_n) \rightarrow (0, \ldots, 0)} q(x_1, \ldots, x_n)$ exists and equals L,
2. for all $\varepsilon > 0$, there exists $0 < \delta < \rho$ such that for all $(x_1, \ldots, x_n) \in \chi(q) \cap D_\rho^*$ the inequality $|q(x_1, \ldots, x_n) - L| < \varepsilon$ holds.

The discriminant variety of Cadavid, Molina, Vélez (2/2)

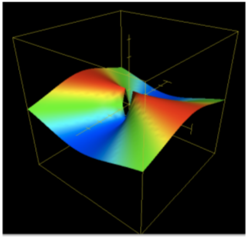
Proof

- Clearly the first assertion implies the second one.
- Next, we assume that the second one holds and we prove the first one.
- Hence, we assume that for all $\varepsilon > 0$, there exists $0 < \delta < \rho$ such that for all $(x_1, \ldots, x_n) \in \chi(q) \cap D_\rho^*$ the inequality $|q(x_1, \ldots, x_n) - L| < \varepsilon$ holds.
- We fix $\varepsilon > 0$. For every $r > 0$, we define
$$C_r = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ \mid \ \sqrt{x_1^2 + \cdots + x_n^2} = r\}.$$
- For all $r > 0$, we choose $t_1(r)$ (resp. $t_2(r)$) minimzing (resp. maximizing) q on C_r. Hence, for all $r > 0$, we have $t_1(r), t_2(r) \in \chi(q)$.
- For all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, we have
$$q(t_1(r)) - L \ \leq \ q(x_1, \ldots, x_n) - L \ \leq \ q(t_2(r)) - L,$$
where $r = \sqrt{x_1^2 + \cdots |x_n^2}$.
- From the assumption and the definitions of $t_1(r), t_2(r)$, there exists $0 < \delta < \rho$ such that for all $r < \rho$ we have
$$-\varepsilon < q(t_1(r)) - L \ \text{ and } \ q(t_2(r)) - L < \varepsilon.$$
- Therefore, there exists $0 < \delta < \rho$ such that for all $(x_1, \ldots, x_n) \in D_\rho^*$ the inequality $|q(x_1, \ldots, x_n) - L| < \varepsilon$ holds.

The method of Cadavid, Molina, Vélez (1/2)

- Their approach transforms the initial limit computation in $n = 2$ variables to one or more limit computations in $n - 1 = 1$ variable.
- One non-trivial part of the method is to find the *real branches* of the variety $\chi(q)$ around the origin.
- This requires tools like Newton-Puiseux theorem in order to *parametrize* $\chi(q)$ around the origin.

The method of Cadavid, Molina, Vélez (2/2)



- Consider $q(x,y) = \frac{f(x,y)}{g(x,y)}$ with $f(x,y) = x^2 - y^2$ and $g(x,y) = x^2 + y^2$.
- We have $\chi(q) = \{(x,y) \in \mathbb{R}^2 \ \mid \ xy\left(x^2+y^2\right) = 0\}$
- Hence, $\chi(q)$ consists of the planes $x = 0$ and $y = 0$.
- Thus, for computing $\lim_{(x,y)\to(0,0)} q(x,y)$, it is enough to consider $\lim_{x\to 0} q(x,0)$ and $\lim_{y\to 0} q(0,y)$ which are equal to 1 and -1 respectively.
- Therefore, $\lim_{(x,y)\to(0,0)} q(x,y)$ does not exist.

Overview of main algorithms

Top-level algorithm

1. computes the discriminant variety $\chi(q)$ of q
2. applies the previous lemma and reduces the whole process to computing limits of q along finitely many pathes (i.e. space curves)
3. as soon as either one path produces an infinite limit or two pathes produce two different finite limits, the procedure stops and returns `no_finite_limit`.

Core algorithm

- reduces computations of limits of q along branches of $\chi(q)$ to computing limits of q along pathes.

Base-case algorithm

- handles the computation of q along space curves by means of Puiseux series expansions

The algorithm `RationalFunctionLimit`

Input: a rational function $q \in \mathbb{Q}(X_1, \ldots, X_n)$ such that origin is an isolated zero of the denominator.

Output: $\lim_{(x_1,\ldots,x_n)\to(0,\ldots,0)} q(x_1, \ldots, x_n)$

1. Apply `RealTriangularize` on $\chi(q)$, obtaining rsas $R_1, \ldots, R_e$
2. Discard R_i if either $\dim(R_i) = 0$ or $\underline{o} \notin \overline{Z_{\mathbb{R}}(R_i)}$
   - QuantifierElimination checks whether $\underline{o} \in \overline{Z_{\mathbb{R}}(R_i)}$ or not.
3. Apply LimitInner(R) on each regular semi algebraic system of dimension higher than one.
   - main task: solving constrained optimization problems
4. Apply `LimitAlongCurve` on each one-dimensional regular semi algebraic system resulting from Step 3
   - main task: Puiseux series expansions

Principles of LimitInner

Input: a rational function q and a regular semi algebraic system $R := [Q, T, P_>]$ with $\dim(Z_{\mathbb{R}}(R)) \geq 1$ and $\underline{o} \in \overline{Z_{\mathbb{R}}(R)}$

Output: limit of q at the origin along $Z_{\mathbb{R}}(R)$

1. if $\dim(Z_{\mathbb{R}}(R)) = 1$ then return LimitAlongCurve(q, R)
2. otherwise build $\mathcal{M} := \begin{bmatrix} X_1 & \cdots & X_n \\ \multicolumn{3}{c}{\nabla t, t \in T} \end{bmatrix}$
3. For all $m \in$ Minors$(\mathcal{M})$ such that $Z_{\mathbb{R}}(R) \not\subseteq Z_{\mathbb{R}}(m)$ build
$$\mathcal{M}' := \begin{bmatrix} \frac{\partial E_r}{\partial X_1} & \cdots & \frac{\partial E_r}{\partial X_n} \\ X_1 & \cdots & X_n \\ \multicolumn{3}{c}{\nabla t, t \in T} \end{bmatrix} \text{ with } E_r := \sum_{i=1}^{n} A_i X_i^2 - r^2$$
4. For all $m' \in$ Minors$(\mathcal{M}')$ $\mathcal{C} :=$ `RealIntersect`$(R, m' = 0, m \neq 0)$
5. For all $C \in \mathcal{C}$ such that $\dim(Z_{\mathbb{R}}(C)) > 0$ and $\underline{o} \in \overline{Z_{\mathbb{R}}(C)}$
   1. compute $L :=$ LimitInner(q, C);
   2. if L is `no_finite_limit` or L is finite but different from a previously found finite L then return `no_finite_limit`
6. If the search completes then a unique finite was found and is returned.

Principles of LimitAlongCurve

Input: a rational function q and a curve C given by $[Q, T, P_>]$

Output: limit of q at the origin along C

1. Let f, g be the numerator and denominator of q
2. Let $T' := \{gX_{n+1} - f\} \cup T$ with X_{n+1} a new variable
3. Compute the real branches of $W_{\mathbb{R}}(T') := Z_{\mathbb{R}}(T') \setminus Z_{\mathbb{R}}(h_{T'})$ in $\mathbb{R}^n$ about the origin via Puiseux series expansions
4. If no branches escape to infinity and if $W_{\mathbb{R}}(T')$ has only one limit point $(x_1, \ldots, x_n, x_{n+1})$ with $x_1 = \cdots = x_n = 0$, then x_{n+1} is the desired limit of q
5. Otherwise return `no_finite_limit`

Example

Let $q(x, y, z, w) = \frac{z\,w + x^2 + y^2}{x^2 + y^2 + z^2 + w^2}$.
RealTriangularize $(\chi(q))$:

$$Z_{\mathbb{R}}(\chi(q)) = Z_{\mathbb{R}}(R_1) \cup Z_{\mathbb{R}}(R_2) \cup Z_{\mathbb{R}}(R_3) \cup Z_{\mathbb{R}}(R_4),$$

where

$$R_1 := \begin{cases} x = 0 \\ y = 0 \\ z = 0 \\ w = 0 \end{cases}, R_2 := \begin{cases} x = 0 \\ y = 0 \\ z + w = 0 \end{cases},$$

$$R_3 := \begin{cases} x = 0 \\ y = 0 \\ z - w = 0 \end{cases}, R_4 := \begin{cases} z = 0 \\ w = 0 \end{cases}.$$

Example

- $\dim(Z_{\mathbb{R}}(R_1)) = 0$
- $\dim(Z_{\mathbb{R}}(R_2)) = 1 \implies$ LimitAlongCurve $(q, R_2) = \frac{-1}{2}$
- $\dim(Z_{\mathbb{R}}(R_3)) = 1 \implies$ LimitAlongCurve $(q, R_3) = \frac{1}{2}$
- $\dim(Z_{\mathbb{R}}(R_4)) = 2 \implies$ LimitInner (q, R_4)
  -

$$R_5 := \begin{cases} z = 0 \\ w = 0 \\ x = 0 \\ y \neq 0 \end{cases}, R_6 := \begin{cases} z = 0 \\ w = 0 \\ y = 0 \\ x \neq 0 \end{cases}$$

- $\dim(Z_{\mathbb{R}}(R_5)) = 1 \implies$ LimitAlongCurve $(q, R_5) = 1$
- $\dim(Z_{\mathbb{R}}(R_6)) = 1 \implies$ LimitAlongCurve $(q, R_6) = 1$

$\implies$ the limit does not exists.

Plan

Tangent cones of space curves

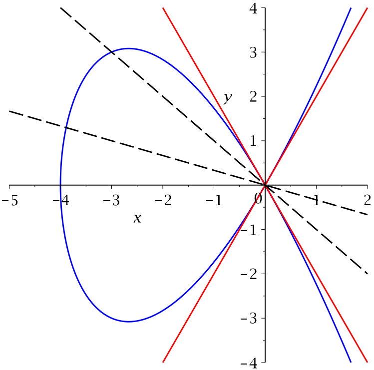
Previous Works

1. An algorithm to compute the equations of tangent cones (Mora 1982):
    - Based on Groebner basis (in fact Standard basis) computations

Our Contribution

1. A Standard Basis Free Algorithm for Computing the Tangent Cones of a Space Curve (P. Alvandi, M. Moreno Maza, É. Schost, P. Vrbik CASC 2015)
    - Based on computation of limit of secant lines

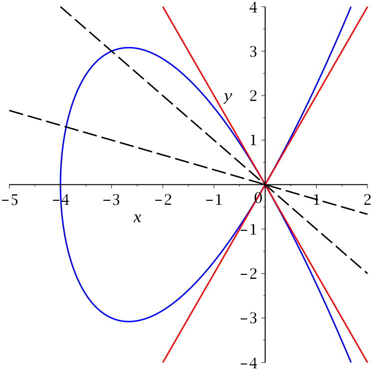
Tangent cones of space curves



Answer

The command `LimitPoints` for computing limit points corresponding to regular chains can be used to compute the limit of secant lines, as well.

Tangent cones of space curves



Answer

The command LimitPoints for computing limit points corresponding to regular chains can be used to compute the limit of secant lines, as well.

Tangent cones of space curves: example

- $\mathcal{C} = W(R)$ a curve with $R := \{2x_3^2 + x_1 - 1, 2x_2^2 + 2x_1^2 - x_1 - 1\}$
- Let $p = (x_1, x_2, x_3)$ be a singular point on C, e.g. $(1, 0, 0)$.

Compute the tangent cone of $\mathcal{C}$ at p

1. Let $q = (y_1, y_2, y_3)$ be a point on a secant line through p
2. When q is close enough to p, one of $x_1 - y_1$, $x_2 - y_2$ or $x_3 - y_3$ does not vanish, say $x_1 - y_1$
3. Hence, when q is close enough to p, $\vec{v} = (s_1, s_2, s_3)$ leads (pq) with
$$s_1 := 1, s_2 := \frac{x_2 - y_2}{x_1 - y_1}, s_3 := \frac{x_3 - y_3}{x_1 - y_1}$$
4. Viewing s_2, s_3 as new variables, consider $T := R \cup R'$ with
$$R' = \{(x_i - y_1)s_2 - (x_2 - y_2), (x_i - y_1)s_3 - (x_3 - y_3)\}$$
5. T is a regular chain for $s_2 > s_3 > x_3 > x_2 > x_1$
6. Computing the limit points of $W(T)$ around $x_1 - y_1 = 0$ yields the limits of the slopes s_2 and s_3, and thus the tangent cone.

Tangent cones of space curves: example

```
> R := PolynomialRing([x_3, x_2, x_1]) :
  Curve := [x_3^2 + x_2^2 + x_1^2 − 1, x_3^2 − x_2^2 − x_1*(x_1 − 1)] :
  rc := Chain([x_1 − 1, x_2, x_3], Empty(R), R) :
  tc := TangentCone(rc, Curve, R, equations); Display(tc[1][2], R);
```

$$tc := \{[[\_x\_1 - 1, -\_x\_2^2 + 3\ \_x\_3^2], regular\_chain]\}$$

$$\begin{cases} x\_3 = 0 \\ x\_2 = 0 \\ x\_1 - 1 = 0 \end{cases}$$

```
> tc := TangentCone(rc, Curve, R, slopes);
```

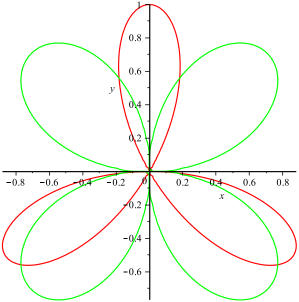
$$tc := \{[[\%x\_1, \%x\_2 - 1, 3\ \%x\_3^2 - 1], regular\_chain], [[\%x\_1, \%x\_2^2 - 3, \%x\_3 - 1], regular\_chain]\}$$

Plan

1. Motivating Examples
2. Polynomials over Power Series
   - The Ring of Puiseux Series
   - The Hensel-Sasaki Construction: Bivariate Case
   - Limit Points: Review and Complement
3. Applications
   - Limits of Multivariate Real Analytic Functions
   - Tangent Cones
   - Intersection Multiplicities

$> F := \left[(x^2+y^2)^2 + 3x^2y - y^3, (x^2+y^2)^3 - 4x^2y^2\right] :$

$> \text{plots[implicitplot]}(Fs, x = -2..2, y = -2..2) :$



$> R := \text{PolynomialRing}\,([x, y], 101) :$

$> \text{TriangularizeWithMultiplicity}(F, R);$

$$\left[\left[1, \begin{cases} x - 1 = 0 \\ y + 14 = 0 \end{cases}\right]\right], \left[\left[1, \begin{cases} x + 1 = 0 \\ y + 14 = 0 \end{cases}\right]\right], \left[\left[1, \begin{cases} x - 47 = 0 \\ y - 14 = 0 \end{cases}\right]\right],$$
$$\left[\left[1, \begin{cases} x + 47 = 0 \\ y - 14 = 0 \end{cases}\right]\right], \left[\left[14, \begin{cases} x = 0 \\ y = 0 \end{cases}\right]\right] \tag{7}$$

The command RegularChains:-TriangularizeWithMultiplicity computes the

TriangularizeWithMultiplicity

We specify `TriangularizeWithMultiplicity`:

Input $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ such that $V(f_1, \ldots, f_n)$ is zero-dimensional.

Output Finitely many pairs $[(T_1, m_1), \ldots, (T_\ell, m_\ell)]$ where $T_1, \ldots, T_\ell$ are regular chains of $\mathbb{C}[x_1, \ldots, x_n]$ such that for all $p \in V(T_i)$

$$\mathcal{I}(p; f_1, \ldots, f_n) = m_i \text{ and } V(f_1, \ldots, f_n) = V(T_1) \uplus \cdots \uplus V(T_\ell)$$

`TriangularizeWithMultiplicity` works as follows

1. Apply `Triangularize` on $f_1, \ldots, f_n$,
2. Apply $\mathsf{IM}_n(T; f_1, \ldots, f_n)$ on each regular chain T.

$\mathsf{IM}_n(T; f_1, \ldots, f_n)$ works as follows

1. if $n = 2$ apply Fulton's algorithm extended for working at a regular chains instead of a point (S. Marcus, M., P. Vrbik; CASC 2013),
2. if $n > 2$ attempt a reduction from dimension n to $n - 1$ (P. Alvandi, M., É. Schost, P. Vrbik; CASC 2015),

Fulton's Properties

The intersection multiplicity of two plane curves at a point *satisfies and is uniquely determined by* the following.

(2-1) $I(p; f, g)$ is a non-negative integer for any C, D, and p such that C and D have no common component at p. We set $I(p; f, g) = \infty$ if C and D have a common component at p.

(2-2) $I(p; f, g) = 0$ if and only if $p \notin C \cap D$.

(2-3) $I(p; f, g)$ is invariant under affine change of coordinates on $\mathring{A}^2$.

(2-4) $I(p; f, g) = I(p; g, f)$

(2-5) $I(p; f, g)$ is greater or equal to the product of the multiplicity of p in f and g, with equality occurring if and only if C and D have no tangent lines in common at p.

(2-6) $I(p; f, gh) = I(p; f, g) + I(p; f, h)$ for all $h \in k[x, y]$.

(2-7) $I(p; f, g) = I(p; f, g + hf)$ for all $h \in k[x, y]$.

Algorithm 5: $\mathsf{IM}_2(p; f, g)$

Input: $p = (\alpha, \beta) \in \mathring{A}^2(\mathbb{C})$ and $f, g \in \mathbb{C}[y \succ x]$ such that $\gcd(f, g) \in \mathbb{C}$

Output: $I(p; f, g) \in \mathbb{N}$ satisfying (2-1)–(2-7)

if $f(p) \neq 0$ *or* $g(p) \neq 0$ **then**

  return 0;

$r, s = \deg(f(x, \beta)), \deg(g(x, \beta))$; assume $s \geq r$.

if $r = 0$ **then**

  write $f = (y - \beta) \cdot h$ and $g(x, \beta) = (x - \alpha)^m (a_0 + a_1(x - \alpha) + \cdots)$;

  return $m + \mathsf{IM}_2(p; h, g)$;

  $\mathsf{IM}_2(p; (y - \beta) \cdot h, g) = \mathsf{IM}_2(p; (y - \beta), g) + \mathsf{IM}_2(p; h, g)$

  $\mathsf{IM}_2(p; (y - \beta), g) = \mathsf{IM}_2(p; (y - \beta), g(x, \beta)) = \mathsf{IM}_2(p; (y - \beta), (x - \alpha)^m) = m$

if $r > 0$ **then**

  $h \leftarrow \mathsf{monic}(g) - (x - \alpha)^{s-r} \mathsf{monic}(f)$;

  return $\mathsf{IM}_2(p; f, h)$;

Reducing from dim n to dim $n-1$: using transversality

The theorem again:

Theorem

Assume that $h_n = V(f_n)$ is non-singular at p. Let v_n be its tangent hyperplane at p. Assume that h_n meets each component (through p) of the curve $\mathcal{C} = V(f_1, \ldots, f_{n-1})$ transversely (that is, the tangent cone $TC_p(\mathcal{C})$ intersects v_n only at the point p). Let $h \in k[x_1, \ldots, x_n]$ be the degree 1 polynomial defining v_n. Then, we have

$$I(p; f_1, \ldots, f_n) = I(p; f_1, \ldots, f_{n-1}, h).$$

How to use this theorem in practise?

Assume that the coefficient of x_n in h is non-zero, thus $h = x_n - h'$, where $h' \in k[x_1, \ldots, x_{n-1}]$. Hence, we can rewrite the ideal $\langle f_1, \ldots, f_{n-1}, h \rangle$ as $\langle g_1, \ldots, g_{n-1}, h \rangle$ where g_i is obtained from f_i by substituting x_n with h'. Then, we have

$$I(p; f_1, \ldots, f_n) = I(p|_{x_1, \ldots, x_{n-1}}; g_1, \ldots, g_{n-1}).$$

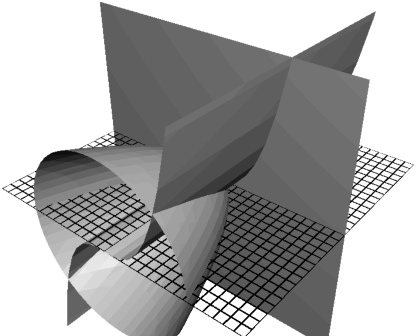
Reducing from dim n to dim $n-1$: a simple case (1/3)

Example

Consider the system

$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$

near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$



Reducing from dim n to dim $n-1$: a simple case (2/3)

Example

Recall the system

$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$

near the origin $o := (0, 0, 0) \in V(f_1, f_2, f_3)$.

Computing the IM using the definition

Let us compute a basis for $\mathcal{O}_{\mathring{A}^3,o}/ <f_1, f_2, f_3>$ as a vector space over $\overline{k}$.

Setting $x = 0$ and $y = z^3$, we must have $z^2(z^4 + 1) = 0$ in $\mathcal{O}_{\mathring{A}^3,o} = \overline{k}[x, y, z]_{(z,y,z)}$.

Since $z^4 + 1$ is a unit in this local ring, we see that

$$\mathcal{O}_{\mathring{A}^3,o}/ <f_1, f_2, f_3>=<1, z>$$

as a vector space, so $I(o; f_1, f_2, f_3) = 2$.

Reducing from dim n to dim $n-1$: a simple case (3/3)

Example

Recall the system again

$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$

near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$.

Computing the IM using the reduction

We have

$$\mathcal{C} := V(x, x + y^2 - z^2) = V(x, (y-z)(y+z)) = TC_o(\mathcal{C})$$

and we have

$$h = y.$$

Thus $\mathcal{C}$ and $V(f_3)$ intersect transversally at the origin. Therefore, we have

$$I_3(p; f_1, f_2, f_3) = I_2((0,0); x, x - z^2) = 2.$$