Regular Chains under Linear Changes of Coordinates and Applications

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September 17, 2015

Plan

Plan

Motivation (1/2)

- Polynomial system solving is an important problem in both science and engineering
- One method for solving such systems relies on triangular decompositions
- A triangular decomposition encodes the solutions of a polynomial system using special sub-systems called regular chains.
- Several encodings are possible.

Example

For the variable order $b < a < y < x$, with $F = \{a\,x + b, b\,x + y\}$, we have the following

$$V(F) = \overline{V(T_1) \setminus V(a\,b)}^{Z},$$

with $T_1 = \{b\,x + y, a\,y - b^2\}$ or

$$V(F) = (V(T_1) \setminus V(a\,b)) \;\cup\; V(T_2) \;\cup\; V(T_3)$$

with $T_2 = \{x, y, b\}$ and $T_3 = \{y, a, b\}$.

Motivation (2/2)

Example (Cont'd)

Recall $F = \{a\,x + b, b\,x + y\}$, $T_1 = \{b\,x + y, a\,y - b^2\}$, $T_2 = \{x, y, b\}$ and $T_3 = \{y, a, b\}$:

- $V(F) = \overline{V(T_1) \setminus V(a\,b)}^Z$ implicitly describes the lines $(0, 0, a, 0)$ and $(x, 0, 0, 0)$, whereas
- $V(F) = V(T_1) \setminus V(t) \cup V(T_2) \cup V(T_3)$ explicitly gives all points.

Observe that we have $V(T_1) \neq \overline{V(T_1) \setminus V(a\,b)}^Z = V(T_1 : (ab)^\infty)$.

Question

For $F \subset \mathbb{Q}[x_1, \ldots, x_n]$ and a regular chain $T \subset \mathbb{Q}[x_1, \ldots, x_n]$ with h_T as product of initials such that we have $V(F) = \overline{V(T) \setminus V(h_T)}^Z$ how to compute

$$V(F) \ \setminus \ (V(T) \setminus V(h_T))$$

if only T (thus not F) is known?

The problem: formal statement

Notations

- Let $T \subset \mathbb{C}[x_1 < \cdots < x_n]$ be a regular chain.
- Let h_T be the product of initials of polynomials of T.
- Let $W(T)$ be the quasi-component of T, that is $V(T) \setminus V(h_T)$.
- $\overline{W(T)}^Z$ is the intersection of all algebraic sets containing $W(T)$.

Problem statement

Compute the non-trivial limit points of $W(T)$, that is, the set

$$\lim(W(T)) = \overline{W(T)}^Z \setminus W(T).$$

Basic properties

- $\overline{W(T)}^Z = V(\mathrm{sat}(T))$ where $\mathrm{sat}(T) := \langle T \rangle : h_T^\infty$,
- $\lim(W(T)) = \overline{W(T)} \cap V(h_T)$,
- If $\dim(\mathrm{sat}(T)) = d$ then $\lim(W(T)) = \emptyset$ or $\dim(\lim(W(T))) = d - 1$.

Why is the problem difficult?

Remark

Given regular chain $T \subset \mathbb{C}[x_1 < \cdots < x_n]$, we have $\overline{W(T)}^Z \subseteq V(T)$ but

$$\overline{W(T)}^Z \neq V(T)$$

may hold, which implies that a command like Triangularize(T) may not compute $\overline{W(T)}^Z$, not even implicitly.

Example

Consider $T = \{z\,x - y^2, y^4 - z^5\}$. We have

- $V(T) \;=\; W(T) \,\cup\, V(y, z)$
- $\overline{W(T)}^Z \;=\; W(T) \,\cup\, V(y, z, x)$

The former can be computed by Triangularize(T) with `output=lazard` option while the latter requires to compute a generating set of $\mathrm{sat}(T) = T : h_T^{\infty}$ since we have $V(\mathrm{sat}(T)) \;=\; \overline{W(T)}^Z$.

Using Puiseux series

In our CASC 2013 paper, we compute $\lim(W(T))$ whenever T is a one-dimensional regular chain over $\mathbb{C}$:

- computations done w.r.t Euclidean topology (instead of Zariski topology) thanks to a theorem of D. Mumford.
- relies on Puiseux parametrizations
- not trivial to extend to a regular chain in higher dimension

Example

$$T := \begin{cases} x_1 x_3^2 + x_2 \\ x_1 x_2^2 + x_2 + x_1 \end{cases}$$

The regular chain T has four Puiseux expansions around $x_1 = 0$:

$$\begin{cases} x_3 &= 1 + O(x_1^2) \\ x_2 &= -x_1 + O(x_1^2) \end{cases} \qquad \begin{cases} x_3 &= -1 + O(x_1^2) \\ x_2 &= -x_1 + O(x_1^2) \end{cases}$$

$$\begin{cases} x_3 &= {x_1}^{-1} - \frac{1}{2}x_1 + O(x_1^2) \\ x_2 &= -{x_1}^{-1} + x_1 + O(x_1^2) \end{cases} \qquad \begin{cases} x_3 &= -{x_1}^{-1} + \frac{1}{2}x_1 + O(x_1^2) \\ x_2 &= -{x_1}^{-1} + x_1 + O(x_1^2) \end{cases}$$

Why using change of coordinate system?

Motivation

This is a fundamental technique to obtain a more convenient representation, and reveal properties, of the algebraic or differential representation of a geometrical object.

Applications of random linear changes of coordinates

- ▷ Obtaining a separating element, in computing rational univariate representation (RUR) of a zero-dimensional polynomial ideal.
- ▷ Getting rid off "vertical components" for instance in computing the tangent cone of a space curve (see yesterday's talk).
- ▷ Noether normalization of a polynomial ideal.

Our goals

- Compute $\lim(W(T))$, as stated after, but also
- Study Noether normalization for ideals of the form $\mathrm{sat}(T)$.

How to use change of coordinates for computing $\lim(W(T))$? (1/2)

First idea: *Lever l'indétermination*

Since $W(T) = V(T) \setminus V(h_T)$, the difficulty in computing $\lim(W(T))$ is to "approach" $V(h_T)$ while staying in $W(T)$. Hence:

- Find a linear change of coordinates A and a regular chain C such that $\overline{W^A(T)} = \overline{W(C)}$ and we can converge to $V^A(h_T)$ within $W(C)$ (thus staying away of $V(h_C)$)
- Then, we have $\lim(W(T)) = \left(V(C) \cap V^A(h_T)\right)^{A^{-1}}$

Example

- Consider $T := \{x_4, x_2x_3 + x_1^2\} \subset \mathbb{Q}[x_1 < x_2 < x_3 < x_4]$ and the linear change of coordinates $A : (x_1, x_2, x_3, x_4) \longmapsto (x_4, x_2 + x_3, x_2, x_1)$
- Using the `PALGIE` algorithm, we obtain $C := \{x_4, x_3^2 + x_2x_3 + x_1^2\}$.
- Since C is monic, we can converge to $V^A(h_T)$ within $W(C)$ and have:

$$\lim(W(T)) = \left(V(C) \cap V^A(h_T)\right)^{A^{-1}} = V(x_4, x_2, x_1).$$

How to use change of coordinates for computing $\lim(W(T))$? (2/2)

Second idea: *Aller à la pêche aux générateurs de* $\mathrm{sat}(T)$

- Recall $\lim(W(T)) = \overline{W(T)} \cap V(h_T) \subseteq V(T) \cap V(h_T)$
- Since $\overline{W(T)} = V(\mathrm{sat}(T))$, there exist polynomial sets $F \subseteq \mathcal{I}(V(\mathrm{sat}(T)))$ such that $V(T \cup F \cup h_T) = \lim(W(T))$ holds.
- One may obtain such F by applying a change of coordinates A to T.

Example

- Let $T := \{x_2^5 - x_1^4, x_1x_3 - x_2^2\}$ be a regular chain of $\mathbb{Q}[x_1 < x_2 < x_3]$.
- Let $C := \{x_3^5 - x_1^3, x_3^2x_2 - x_1^2\}$ be a regular chain of $\mathbb{Q}[x_1 < x_3 < x_2]$ for which we have $\mathrm{sat}(C) = \mathrm{sat}(T)$.
- We shall exhibit a theorem implying $\sqrt{\langle T, C\rangle} = \sqrt{\mathrm{sat}(T)}$ from which we shall deduce $\lim(W(T)) = V(x_1, x_2, x_3)$.

Plan

Linear change of coordinates

Notations

Let $\mathbf{k}$ be a field and $\mathbf{x} = x_1 < \cdots < x_n$ be n ordered variables.

Linear change of coordinates

We call *linear change of coordinates in* $\overline{\mathbf{k}}^n$ any bijective map A of the form

$$A : \begin{array}{ccc} \overline{\mathbf{k}}^n & \rightarrow & \overline{\mathbf{k}}^n \\ \mathbf{x} & \longmapsto & (A_1(\mathbf{x}), \ldots, A_n(\mathbf{x})) \end{array} \quad (1)$$

where $A_1, \ldots, A_n$ are linear forms over $\overline{\mathbf{k}}$.

Notation

- For $f \in \mathbf{k}[x_1, \ldots, x_n]$, we write $f^A(\mathbf{x}) := f(A_1(\mathbf{x}), \ldots, A_n(\mathbf{x}))$.
- $V^A(F) := V(\{f^A \mid f \in F\})$ and $W^A(T) := V^A(T) \setminus V^A(h_T)$.
- For $U := V(F)$ with $F \subset \mathbf{k}[x_1, \ldots, x_n]$, we define $U^A := V^A(F)$.
- For $\mathcal{I} := \langle F \rangle$, we define $\mathcal{I}^A := \langle f^A \mid f \in F \rangle$.

Change of Variable Order

Problem 1

Given

- two orderings $\mathcal{R}_1$ and $\mathcal{R}_2$ on $\{x_1, \ldots, x_n\}$, and
- $T \subset \mathbf{k}[\mathbf{x}]$ a regular chain **w.r.t** $\mathcal{R}_1$,

then compute finitely many regular chains $C_1, \ldots, C_e$ **w.r.t** $\mathcal{R}_2$ such that

$$\overline{W(T)}^Z = \overline{W(C_1)}^Z \ \cup \ \cdots \ \cup \ \overline{W(C_e)}^Z$$

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Example

Let $T = \{z\,x^2 + y^2, y^4 - z^3\}$ be a regular chain w.r.t $\mathcal{R} = z < y < x$. Let $\mathcal{R}' = z < x < y$, then

$$C = \texttt{PALGIE}(T, \mathcal{R}') = \{y^2 + x^2 z, x^4 - z\}.$$

In fact, we have $\mathrm{sat}(T)_{\mathcal{R}} = \mathrm{sat}(C)_{\mathcal{R}'}$.

Change of Variable Order

Problem 1

Given

- two orderings $\mathcal{R}_1$ and $\mathcal{R}_2$ on $\{x_1, \ldots, x_n\}$, and
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$$\overline{W(T)}^Z = \overline{W(C_1)}^Z \cup \cdots \cup \overline{W(C_e)}^Z$$

In the work of

- F. Boulier, F. Lemaire, and M. M. M.

the differential counterpart of this problem, assuming $\mathrm{sat}(T)$ is prime.

- An answer can be derived for the algebraic case and
- this algorithm is called `PALGIE` (Prime ALGebraic IdEal).

Change of Variable Order

Problem 1

Given

- two orderings $\mathcal{R}_1$ and $\mathcal{R}_2$ on $\{x_1, \ldots, x_n\}$, and
- $T \subset \mathbf{k}[\mathbf{x}]$ a regular chain **w.r.t** $\mathcal{R}_1$,

then compute finitely many regular chains $C_1, \ldots, C_e$ **w.r.t** $\mathcal{R}_2$ such that

$$\overline{W(T)}^Z = \overline{W(C_1)}^Z \ \cup \ \cdots \ \cup \ \overline{W(C_e)}^Z$$

Extending the `PALGIE` algorithm to a solution of the problem above can be achieved by standard methods from regular chains theory.

Change of Coordinate System

Problem 2

Given a regular chain T and a linear change of coordinates A, compute finitely many regular chains $C_1, \ldots, C_e$ such that ,

$$\overline{W^A(T)}^Z = \overline{W(C_1)}^Z \cup \cdots \cup \overline{W(C_e)}^Z$$

Given

- A is a linear change of coordinate system and
- $T = \{t_1(x_1, \ldots, x_d), \ldots, t_{n-d}(x_1, \ldots, x_n)\}$,

Apply the extended version of `PALGIE` algorithm to

$$T^A \begin{cases} t_{n-d}^A(x_1, \ldots, x_n) = 0 \\ \vdots \\ t_1^A(x_1, \ldots, x_d) = 0 \\ h_T^A \neq 0 \end{cases}$$

Plan

Noether normalization: definition

Setting

- Let $\mathcal{P}$ be a prime ideal and G a lexicographical Gröbner basis of $\mathcal{P}$
- Let $T_v = \{v \in \mathbf{x} \mid \forall g \in G \ \mathrm{mvar}(g) \neq v\}$. W.l.o.g $T_v = \{x_1, \ldots, x_d\}$.

The variable x_s is *integral* over $\mathbf{k}[\mathbf{x}]$ modulo $\mathcal{P}$ if there exists $f \in \mathcal{P}$ s.t. $\mathrm{mvar}(f) = x_s$ and $\mathrm{init}(f) \in \mathbf{k}$.

Let

$$A = \left(\begin{array}{c|ccc} & a_{1,d+1} & \cdots & a_{1,n} \\ \mathrm{I}_{d\times d} & \vdots & \vdots & \vdots \\ & a_{d,d+1} & \cdots & a_{d,n} \\ \hline \mathbf{0} & & \mathrm{I}_{(n-d)\times(n-d)} & \end{array}\right)$$

Then for a generic choice of $a_{1,d+1}, \ldots, a_{d,n}$ the following properties hold:

- $x_1, \ldots, x_d$ are algebraically independent modulo $\mathcal{P}^A$,
- x_{d+i} is integral over $\mathbf{k}[x_1, \ldots, x_d]$ modulo $\mathcal{P}^A$ for all $i = 1, \ldots, n-d$.

In this case we say that $\mathcal{P}^A$ is in Noether position.

Noether normalization: example

Below, we use the Noether package from A. Hashemi:

```
> read "Noether.mpl" : F := [a·x + b, b·x + y]; LinearChange(F, [x, y, a, b]);
```

$$F := [a\,x + b,\ b\,x + y]$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

```
> G := eval(F, [x = x, y = y, a = 2· x + a, b = b + y]);
```

$$G := [(a + 2\,x)\,x + b + y,\ (b + y)\,x + y]$$

```
> G := Groebner:-Basis(G, plex(x, y, a, b));
```

$$G := [-a\,b\,y - a\,y^2 + b^3 + 3\,b^2\,y + 3\,b\,y^2 + y^3 + 2\,y^2,\ -a\,y + b^2 + 2\,b\,x + 2\,b\,y + y^2 + 2\,y,\ a\,y - b^2 - 2\,b\,y + 2\,x\,y - y^2,\ a\,x + 2\,x^2 + b + y]$$

We see that x and y are integral modulo $\langle F \rangle^A$ for
$A : (x, y, a, b) \longmapsto (x, y, 2x + a, b + y)$.

Noether normalization and regular chains

Theorem (P. Aubry, D. Lazard & M. M. M.; 1999)

For the prime ideal $\mathcal{P}$ and the lexicographical Gröbner basis G of $\mathcal{P}$, there exists a regular chain $T \subseteq G$ s.t we have $\overline{W(T)}^Z = V(\mathcal{P})$.

Notations

- Let A be a linear change of coordinates such that $\mathcal{P}^A$ is in Noether position.
- Let C be the regular chain extracted (i.e. contained) from the lexicographical Groebner basis of $\mathcal{P}^A$.

Theorem

If T generates $\mathrm{sat}(T)$, then the regular chain C is **monic***, that is, for each $f \in C$ the initial $\mathrm{init}(f)$ lies in $\mathbf{k}$.*

Noether normalization and regular chains

Theorem (P. Aubry, D. Lazard & M. M. M.; 1999)

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- Let A be a linear change of coordinates such that $\mathcal{P}^A$ is in Noether position.
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Theorem

If T generates $\mathrm{sat}(T)$, then the regular chain C is **monic**, *that is, for each $f \in C$ the initial $\mathrm{init}(f)$ lies in $\mathbf{k}$.*

What happens when T does not generate $\mathrm{sat}(T)$?

What happens when T does not generate $\mathrm{sat}(T)$?

Recall that we saw $\langle T \rangle \neq \mathrm{sat}(T)$ for T defined below.

```
R := PolynomialRing([x, y, a, b]) : F := [a*x + b, b*x + y] :
dec := Triangularize(F, R) :
T := dec[1] :
Display(T, R);
```

$$\begin{cases} b\,x + y = 0 \\ a\,y - b^2 = 0 \\ a \neq 0 \\ b \neq 0 \end{cases}$$

```
> S := Saturate(⟨op(Equations(T, R))⟩, a*b) : G := Generators(S);
```

$$G := \{x\,a + b,\ a\,y - b^2,\ b\,x + y\}$$

```
> read "Noether.mpl" : LinearChange(G, [x, y, a, b]);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

```
> GA := eval(G, [x = x, y = y, a = a - x, b = b + 2 y]) : SH := Saturate(⟨op(GA)⟩, (a-x)*(b+2*y)) :
  H := Generators(SH);
```

$$H := \{b\,x + 2\,x\,y + y,\ -a\,x + x^2 - b - 2\,y,\ -2\,a\,y + 2\,b^2 - b\,x + 8\,b\,y + 8\,y^2 - y\}$$

```
> dec := Triangularize(H, R) : Display(dec, R);
```

$$\left[\begin{cases} b\,x - 8\,y^2 + (2\,a - 8\,b + 1)\,y - 2\,b^2 = 0 \\ 8\,y^3 + (-2\,a + 12\,b - 1)\,y^2 + \left(-a\,b + 6\,b^2\right)\,y + b^3 = 0 \\ b \neq 0 \end{cases},\ \begin{cases} 2\,x + 1 = 0 \\ 8\,y - 2\,a - 1 = 0 \\ b = 0 \end{cases},\ \begin{cases} x - a = 0 \\ y = 0 \\ b = 0 \end{cases},\ \begin{cases} x = 0 \\ y = 0 \\ b = 0 \end{cases} \right]$$

The C is the leftmost one above: it is not monic

$\langle T \rangle \neq \mathrm{sat}(T)$: another example

Example

Consider $T = \{x_2^5 - x_1^4, x_1 x_3 - x_2^2\} \subset \mathbb{Q}[x_1 < x_2 < x_3]$

- for which $\mathrm{sat}(T)$ is prime, $\overline{W(T)}^Z \neq V(T)$ holds and
- $\mathrm{sat}(T)^A$ is in Noether position for $A = \left(\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} . \right)$.

Using the extension version of `PALGIE`, we compute

$$C = \left\{ \begin{array}{l} c_2 = \left(-x_1^3 + 2x_2^2 x_1\right) x_3 + x_1^2 x_2^2 - x_2^4 + x_2^3 \\ c_1 = x_2^5 - 2x_2^4 + x_2^3 + 4x_1^2 x_2^2 - x_1^4 \end{array} \right.$$

such that $\mathrm{sat}(C) = \mathrm{sat}(T)^A$ and observe that C is not monic.

Why is monicity interesting?

Notations

- Let (again) T be a regular chain with $\mathcal{P} := \mathrm{sat}(T)$ prime
- Let A be a linear change of coordinates
- Let C be a regular chain such that $\mathrm{sat}(C) = \mathcal{P}^A$.

Proposition

(i) if $\mathrm{sat}(T)$ is radical and $\langle h_T, (h_C^{A^{-1}}) \rangle = \mathbf{k}[\mathbf{x}]$ holds, then $T \cup C^{A^{-1}}$ generates $\mathrm{sat}(T)$,

(ii) if the regular chain C is **monic**, then $C^{A^{-1}}$ generates $\mathrm{sat}(T)$.

Lever l'indétermination !

Notations

- Let h_T (resp. h_C) be the product of the initials of T (resp. C)
- Let r_T the (resp. r_C) be the iterated resultant of h_T (resp. h_C) w.r.t. T (resp. C).

Theorem

If $V(r_T^A, r_C)$ is empty, then we have

$$\lim(W(T)) \quad = \quad \{A^{-1}(\mathbf{y}) \mid \mathbf{y} \in V(h_T^A) \cap W(C)\}. \tag{2}$$

Example

- Consider $T := \{x_4, x_2x_3 + x_1^2\} \subset \mathbb{Q}[x_1 < x_2 < x_3 < x_4]$ and $A : (x_1, x_2, x_3, x_4) \longmapsto (x_4, x_2 + x_3, x_2, x_1)$
- Using the `PALGIE` algorithm, we obtain $C := \{x_4, x_3^2 + x_2x_3 + x_1^2\}$.
- Since C is monic, then $r_C \in \mathbb{Q}$ and the theorem applies:
$$V^{A^{-1}}(C, h_T^A) \quad = \quad V(x_4, x_2, x_1) \quad = \quad \lim(W(T)).$$

Plan

Idea

- Recall $\lim(W(T)) = \overline{W(T)} \cap V(h_T) \subseteq V(T) \cap V(h_T)$
- Since $\overline{W(T)} = V(\mathrm{sat}(T))$, there exist polynomial sets $F \subseteq \mathcal{I}(V(\mathrm{sat}(T)))$ such that $V(T \cup F \cup h_T) = \lim(W(T))$ holds.
- One may obtain such F by applying a change of coordinates A to T.

Lemma

We have $\sqrt{\langle T \rangle} = \sqrt{\mathrm{sat}(T)}$ *if and only if* $V(T, h_T)$ *is empty or* $\dim(V(T, h_T)) < \dim(\mathrm{sat}(T))$.

Idea and a theorem (2/3)

Notations

- Assume $x_1, \ldots, x_d$ are the free variables of $\operatorname{sat}(T)$, thus, $\operatorname{sat}(T) \cap \mathbf{k}[x_1, \ldots, x_d] = \langle 0 \rangle$.
- For $i = 1 \cdots s$, let $C_i \subset \mathbf{k}[\mathbf{x}]$ be a regular chain s. t. $\langle C_i \rangle \subseteq \sqrt{\operatorname{sat}(T)}$.
- Let $\mathcal{I} = \langle T, C_1, \ldots, C_s \rangle$.

Theorem

Then $\sqrt{\operatorname{sat}(T)} = \sqrt{\mathcal{I}}$ *if and only if there exist regular chains* T_i, $i = 1, \ldots, t$, *such that each of the following properties hold:*

(i) $\sqrt{\mathcal{I}} = \cap_{i=1}^{t} \sqrt{\operatorname{sat}(T_i)}$,

(ii) $|T_1| = \cdots = |T_t| = n - d$,

(iii) h_T *is regular modulo all* $\sqrt{\operatorname{sat}(T_i)}$.

Remark

This theorem yields an algorithmic criterion to test $\sqrt{\operatorname{sat}(T)} = \sqrt{\mathcal{I}}$.

Idea and a theorem (3/3)

Theorem (same as before)

Then $\sqrt{\mathrm{sat}(T)} = \sqrt{\mathcal{I}}$ *if and only if there exist regular chains* T_i, $i = 1, \ldots, t$, *such that each of the following properties hold:*

(i) $\sqrt{\mathcal{I}} = \cap_{i=1}^{t} \sqrt{\mathrm{sat}(T_i)}$,

(ii) $|T_1| = \cdots = |T_t| = n - d$,

(iii) h_T *is regular modulo all* $\sqrt{\mathrm{sat}(T_i)}$.

Example

- Let $T := \{x_2^5 - x_1^4, x_1 x_3 - x_2^2\}$ be a regular chain of $\mathbb{Q}[x_1 < x_2 < x_3]$.
- Let $C := \{x_3^5 - x_1^3, x_3^2 x_2 - x_1^2\}$ be a regular chain of $\mathbb{Q}[x_1 < x_3 < x_2]$ for which we have $\mathrm{sat}(C) = \mathrm{sat}(T)$.
- Triangularize($T \cup C$) returns T, D with $D := \{x_1, x_2, x_3\}$.
- Clearly $\mathrm{sat}(D)$ is a redundant component: we have $\mathrm{sat}(T) \subseteq \mathrm{sat}(D)$.
- Hence the theorem applies and $\sqrt{T \cup C} = \sqrt{\mathrm{sat}(T)}$ holds.

Algorithm Closure($W(T)$)

▷ Let $T \subset \mathbf{k}[x_1 < \cdots < x_n]$ be a regular chain s. t. $\mathrm{sat}(T)$ is prime

1. Let $i := 1$,
2. Let $\mathcal{R} := x_i < x_{i+1} < \cdots < x_n < x_1 < \cdots < x_{i-1}$,
3. $\mathcal{D} := \mathtt{PALGIE}(T, R)$,
4. Let C be the only regular chain in $\mathcal{D}$,
5. If $V(C) = \overline{W(T)}$ then output C and exit, otherwise $G := G \ \cup \ C$,
6. $\mathcal{D} :=$ `triangular decomposition of` $V(G)$,
7. if h_T is regular w.r.t each regular chain in $\mathcal{D}$ then output G and exit,
8. If $i < n$ then $i := i + 1$ and go to bullet 2, otherwise output `Failed`.

Remarks

- $V(C) = \overline{W(C)}$ can be tested by the previous lemma.
- Since $\overline{W(C)} = \overline{W(T)}$, one can test $V(C) = \overline{W(T)}$.

Plan

Summary

- ▷ We have presented algorithmic criteria to compute $\lim(W(T))$ for an arbitrary regular chain
- ▷ This extends our previous work based on Puiseux series where T is required to have dimension one
- ▷ Our algorithmic criteria make use of linear changes of coordinates.
- ▷ We first look for a random linear change of coordinates A and a regular chain C such that $\overline{W^A(T)} = \overline{W(C)}$ and $\lim(W(T)) = \left(V(C) \cap V^A(h_T)\right)^{A^{-1}}$ holds.
- ▷ If T generates $\mathrm{sat}(T)$, this criterion always works.
- ▷ Second, we try to discover more generators of $\mathrm{sat}(T)$ by applying change of variable orders on T.
- ▷ The procedure **Closure**$(W(T))$ implements that idea. Note that this procedure might fail, but it appears to be practically effective.

Take away and work in progress

- ▷ We have exhibited relations between Noether normalization of saturated ideals and regular chains T generating their saturated ideals.
- ▷ We have enhanced the `RegularChains` library with a new command `ChangeOfCoordinates` implementing the map $(T, A) \longmapsto C$ such that $\mathrm{sat}(C) = \mathrm{sat}(T)^A$ holds, when $\mathrm{sat}(T)$
- ▷ We have presented new algorithmic criteria to compute $\lim(W(T))$, without restrictions on the dimension of $\mathrm{sat}(T)$.
- ▷ Nevertheless, obtaining $\lim(W(T))$ (or, equivalently, computing a generating system of $\mathrm{sat}(T)$) without Gröbner basis computation still does not have a complete algorithmic solution.
- ▷ We are currently extending our approach based on Puiseux series from dimension 1 to higher dimension.