What computer algebra systems can offer to tackle realizability problems of matroids? (survey talk)

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Plan

1. Oriented Matroids
   - Axioms and examples
   - The realizability problem
2. Realization computations
   - Solvability sequences and other certificates
   - Using polynomial optimization software
   - Using computer algebra
   - Conclusions

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Vector configurations

Notations

- Let $X = x_1, \ldots, x_n \subset (\mathbb{R}^d)^n$ be a full rank d matrix.
- Define $E = \{1, 2, \ldots, n\}$. Let $\mathcal{B} \subset 2^E$ consist of all column index sets of the bases of X and define the map:

$$\chi : \begin{array}{ccc} \Lambda(E,d) & \rightarrow & \{-,0,+\} \\ (i_1,\ldots,i_d) & \longmapsto & \mathrm{sign}([i_1,\ldots,i_d]) \end{array}$$

  where $[i_1, \ldots, i_d] := \det(x_{i_1}, \ldots, x_{i_d})$ and $\Lambda(E,d)$ consists of all d-sequences of pairwise distinct elements of E.

Notion of a matroid

- $\mathcal{B}$ satisfies the Steinitz exchange axiom: for all $B_1, B_2 \in \mathcal{B}$ and all $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$ such that $B_1 \setminus \{e\} + f \in \mathcal{B}$.
- The pair $M = (E, \mathcal{B})$ is called an ordinary matroid.
- The map χ not only encodes $\mathcal{B}$ (the "incidence structure" of M) but also orientation (positions of points relative to hyperplanes).

Affine point configurations

Notations

- $X = x_1, \ldots, x_n \subset (\mathbb{R}^d)^n$ a $d \times n$ matrix and $E = \{1, 2, \ldots, n\}$.
- For $y^t \in (\mathbb{R}^d)^*$, we define

$$\mathcal{C}(y) = (\mathrm{sign}(y^t x_1), \ldots, \mathrm{sign}(y^t x_d)) \quad \text{and} \quad \mathcal{L} = \{\mathcal{C}(y) \mid y \in \mathbb{R}^d\}.$$

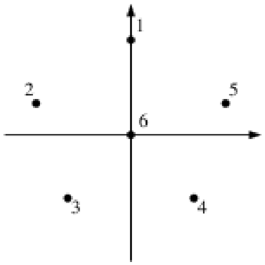
Notion of a covector

- For a hyperplane $H_y = \{x \in \mathbb{R}^d \mid y^t x = 0\}$, if $\mathcal{C}(y)^- = \emptyset$ then H_y determines a face of the positive cone

$$\mathrm{pos}(x_1, \ldots, x_n) = \{\lambda_1 x_1 + \cdots + \lambda_d x_d \mid 0 \le \lambda_i \in \mathbb{R}, 1 \le i \le n\}.$$

- The face lattice of $\mathrm{pos}(x_1, \ldots, x_n)$ can be recovered as $\mathcal{L} \cap (0, +)^E$.
- Assume $(x_i)_d = 1$ for all $i = 1 \cdots n$, then X gives the homogeneous coordinates of an affine point set $X' \subset \mathbb{R}^{d-1}$.
- The face lattice of the convex polytope $\mathrm{conv}(X')$ is then identical to the face lattice of $\mathrm{pos}(x_1, \ldots, x_n)$.

Example



(a)

$$\begin{pmatrix} 0 & -3 & -2 & 2 & 3 & 0 \\ 3 & 1 & -2 & -2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$
$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$$

$$\begin{array}{cccc}
\chi(1,2,3) = + & \chi(1,3,5) = + & \chi(2,3,4) = + & \chi(2,5,6) = \\
\chi(1,2,4) = + & \chi(1,3,6) = + & \chi(2,3,5) = + & \chi(3,4,5) = + \\
\chi(1,2,5) = + & \chi(1,4,5) = + & \chi(2,3,6) = + & \chi(3,4,6) = + \\
\chi(1,2,6) = + & \chi(1,4,6) = & \chi(2,4,5) = + & \chi(3,5,6) = + \\
\chi(1,3,4) = + & \chi(1,5,6) = & \chi(2,4,6) = + & \chi(4,5,6) = +.
\end{array}$$

Chirotope axioms

Notations

- Let $\chi : \Lambda(E,d) \to \{-,0,+\}$ with $E = \{1,2,\ldots,n\}$ and $0 \leq d \leq n$.
- Recall $\Lambda(E,d) = \{(i_1,\ldots,i_d) \mid \{i_1,\ldots,i_d\} \in \binom{E}{d}\}$.

Definition of a chirotope

The map χ is a chirotope of rank d and (E,χ) is an oriented matroid if

1. $\{X \in \binom{E}{d} \mid \chi(X) \neq 0\}$ is the set of the bases of a matroid.
2. χ is alternating, that is, any transposition of two components changes the sign.
3. χ satisfies the three-term Grassmann-Plücker identity, that is, for all $\lambda \in E^{d-2}$ and all $a,b,c,d \in E \setminus \{\lambda\}$, the set
$$\{\chi(\lambda,a,b)\chi(\lambda,c,d), -\chi(\lambda,a,c)\chi(\lambda,b,d), \chi(\lambda,a,d)\chi(\lambda,b,c)\}$$
contains either $\{-1,+1\}$ or is identically zero $\{0\}$.

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Realization spaces

Definition

Let $\chi : \Lambda(E, d) \to \{-, 0, +\}$ be a chirotope with $\chi(1, \ldots, d) = +$. The realization space $\mathcal{R}(\chi)$ is the set of all matrices $X = (x_1, \ldots, x_n) \subset (\mathbb{R}^d)^n$

1. whose induced chirotope is χ, and such that
2. x_i is the i-th unit vector, for $i = 1 \cdots d$.

If $\mathcal{M}$ is the oriented matroid corresponding to χ we write $\mathcal{R}(\mathcal{M}) = \mathcal{R}(\chi)$.

Example

$$\begin{array}{lllll} \chi(123) = +1 & \chi(124) = 0 & \chi(125) = +1 & \chi(134) = 1 & \chi(135) = 1 \\ \chi(145) = +1 & \chi(234) = +1 & \chi(235) = +1 & \chi(245) = 1 & \chi(345) = 0 \end{array}$$

$$X = \begin{pmatrix} 1 & 0 & 0 & v_{14} & v_{15} \\ 0 & 1 & 0 & v_{24} & v_{25} \\ 0 & 0 & 1 & v_{34} & v_{35} \end{pmatrix}$$

$$\mathcal{R}(\chi) = \{(v_{14}, v_{15}, v_{24}, v_{25}, v_{34}, v_{35}) \in \mathbb{R}^6 \mid \mathrm{sign}[ijk] = \chi(i, j, k), i, j, k \in E\}.$$

Stable equivalence of semialgebraic sets

- Let U, V be semialgebraic sets, obtained as a disconnected union of connected semialgebraic sets $U = U_1 \coprod \cdots \coprod U_k$, $V = V_1 \coprod \cdots \coprod V_k$.
- We say that U and V are rationally equivalent if there exist homeomorphisms $U_i \overset{\phi_i}{\mapsto} V_i$ defined by rational maps.
- Let $U \subset \mathbb{R}^{n+d}, V \subset \mathbb{R}^n$ be semialgebraic sets, $U = U_1 \coprod \cdots \coprod U_k$, $V = V_1 \coprod \cdots \coprod V_k$ with U_i mapping to V_i under the natural projection π deleting last d coordinates. We say that $\pi : U \mapsto V$ is a stable projection if there exist integer polynomial maps $\phi_1, \ldots, \phi_l, \psi_1, \ldots, \psi_m : \mathbb{R}^n \mapsto (\mathbb{R}^d)^*$ such that
$$U_i = \{(v, v') \in \mathbb{R}^{n+d} \mid v \in V_i \text{ and } \langle \phi_a(v), v' \rangle > 0, \langle \psi_b(v), v' \rangle = 0,\ a = 1, \ldots, l, b = 1, \ldots, m\}.$$
- The stable equivalence is an equivalence relation on semialgebraic subsets generated by stable projections and rational equivalence.
- Stable equivalence preserves the number of connected components and the existence of rational points.

Mnev's universality theorem

Fact

If $\mathcal{M}$ is a rank 2 (or, by duality, a rank $n-2$) oriented matroid, then $\mathcal{M}$ is realizable and $\mathcal{R}(\mathcal{M})$ is stably equivalent to some $\mathbb{R}^m$.
However, for a rank 3 riented matroid $\mathcal{M}$, the realization space $\mathcal{R}(\mathcal{M})$ can be arbitrarily complicated:

- *For $d = 3$ and $n = 9$ there is an oriented matroid with no realization with rational coordinates (Perles).*
- *For $d = 3$ and $n = 14$ there is an oriented matroid with a disconnected realization space (Suvorov).*

Theorem (Mnev's Universality Theorem)

For every semi-algebraic set V defined over by polynomials over $\mathbb{Z}$ there is a chirotope a rank 3 matroid such that V and $\mathcal{R}(\mathcal{M})$ are stably equivalent.

Mnev's universality theorem: consequences

Corollary

- *The full field of real algebraic numbers is needed to realize all oriented matroids of rank 3.*
- *The realizability problem for oriented matroids is (polynomial time) equivalent to solving arbitrary finite systems of polynomial equations and strict inequalities with integer coefficients. (Mnev)*
- *The realizability problem for oriented matroids is NP-hard*
- *Realizability of rank 3 oriented matroids cannot be characterized by excluding a finite set of forbidden minors (Bokowski & Sturmfels).*

Theorem (Richter-Gebert & Ziegler bases on Basu, Pollack & Roy)

The number of bit operations needed to decide the realizability of a rank d oriented matroid on n points in the Turing machine model of complexity is bounded by

$$(S/K)^K \; S \; d^{O(K)}$$

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Certificates

Definition (certificates)

- certificates are as sufficient conditions of realizability or non-realizability.
- Solvability sequences and final polynomials belong to the former and latter categories, respectively.

Definition (final polynomial)

- The notion of final polynomials was first introduced by Bokowski, Sturmfels and others for proving the non-existence certain convex polytopes.
- The final polynomial of rank d and order n oriented matroid belongs to $\mathbb{R}[\Lambda(E,d)]$ (with $E = \{1, \ldots, n\}$) the polynomial algebra freely generated over $\mathbb{R}$ by all brackets $[\lambda]$ for $\lambda \in \Lambda(E,d)$.

Solvability sequences: idea

Principle

- Given an oriented matroid $\mathcal{M}$, a solvability sequence is a semi-algebraic system, together with an ordering of its variables, whose solutions define realizations of $\mathcal{M}$.
- This should be understood a heuristical method that, when it succeeds, yields a realizability certificate.
- This method was first proposed by Bokowski and Sturmfels.

Example (Recall)

$$\begin{array}{lllll} \chi(123) = +1 & \chi(124) = 0 & \chi(125) = +1 & \chi(134) = 1 & \chi(135) = 1 \\ \chi(145) = +1 & \chi(234) = +1 & \chi(235) = +1 & \chi(245) = 1 & \chi(345) = 0 \end{array}$$

$$X = \begin{pmatrix} 1 & 0 & 0 & v_{14} & v_{15} \\ 0 & 1 & 0 & v_{24} & v_{25} \\ 0 & 0 & 1 & v_{34} & v_{35} \end{pmatrix}$$

Solvability sequences: the method

1. Assume $\{1, \ldots, d\}$ is a basis β of the underlying ordinary matroid.
2. Assume a realization $V \in \mathbb{R}^{d^n}$ with

$$V = \begin{pmatrix} 1 & \cdots & 0 & v_{1,d+1} & \cdots & v_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & v_{d,d+1} & \cdots & v_{d,n} \end{pmatrix}$$

   where v_{ij} are unknown coefficients.
3. Considering $d-1$ columns among the first d and another among the last $n-d$ together with χ shows that each variable v_{ij} satisfies

$$v_{ij} < 0, \;\; v_{ij} = 0 \;\; \text{or} \;\; v_{ij} > 0.$$

4. Similarly we obtain degree two equations and inequalities that can be simplied with the degree one constraints, in particular using the tirck on the next slide.
5. Continuing in this manner, using an elimination à la Fourier-Motzkin might prove realizability together with a variable ordering.

Solvability sequences: tricks

Using gradients

Let Δ be a bracket of degreee $e \geq 2$ and v be a variable occuring in Δ. Writing the Taylor expandion of Δ at the origin, we have:

$$\Delta = v \, \frac{\partial \Delta}{\partial v} + R,$$

where $\frac{\partial \Delta}{\partial v}$ is itself a bracket Δ' of degreee $e - 1$ Thus, if $\Delta') \neq 0$, we deduce a constraint on v.

Other tricks

- The oriented matroid $\mathcal{M}$ is unchanged if each column or row of V is multiplied by a non-zero constant: this might reduce the degree of each constraint by 1 and the number of unknowns by $n - 1$.
- Choose the basis β so as to reduce the number of constraints of degree higher than 2 (PhD thesis of Nakayama). The motivation is is to solve the remaining reduced system with a LP-solver.

Solvability sequences: example (1/2)

Consider the chirotope of IC(8,4,157756) from Finschis classification.

```
1111211121121231112112123112123123411121121231121231234112123123412345
2223322332334442233233444233444555522332334442334445555233444555566666
3344434445555553444555555666666666634445555556666666666777777777777777
4555566666666667777777777777777777788888888888888888888888888888888888
0++++++++0++++0+++++0+-0++-0--+--++++++++00--+----------+-+--+--++-+++
```

- The basis β is chosen by Nakayama's algorithm.
- Unknowns are introduced such that they are all strictly positive.
- All the other constraints are shown in the semi-algebraic system below.

$$V = \begin{pmatrix} 1 & 0 & 0 & v_{14} & 0 & -v_{16} & -v_{17} & 0 \\ 0 & 1 & 0 & -v_{24} & 0 & v_{26} & 0 & v_{28} \\ 0 & 0 & 1 & v_{34} & 0 & 0 & -v_{37} & -v_{38} \\ 0 & 0 & 0 & 0 & 1 & v_{46} & v_{47} & v_{48} \end{pmatrix} \quad \begin{cases} v_{14}v_{26} = v_{16}v_{24}, \quad v_{14}v_{37} = v_{17}v_{34}, \\ v_{16}v_{47} = v_{17}v_{46}, \quad v_{24}v_{38} = v_{28}v_{34}, \\ v_{28}v_{46} < v_{26}v_{48}, \quad v_{38}v_{47} < v_{37}v_{48}, \\ v_{26}v_{37}v_{48} < v_{26}v_{38}v_{47} + v_{28}v_{37}v_{46}, \end{cases}$$

Solvability sequences: example (2/2)

Below, we normalize six positive unknowns to 1.

$$V = \begin{pmatrix} 1 & 0 & 0 & v_{14} & 0 & -v_{16} & -\mathbf{1} & 0 \\ 0 & 1 & 0 & -\mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 1 & v_{34} & 0 & 0 & -\mathbf{1} & -v_{38} \\ 0 & 0 & 0 & 0 & 1 & v_{46} & \mathbf{1} & v_{48} \end{pmatrix}$$

- This normalization simplifies the system to the one on the left.
- Using the variable ordering $v_{14} \prec v_{16} \prec v_{34} \prec v_{38} \prec v_{46} \prec v_{48}$ and substitution, we obtain the simpler system on the right, which is clearly consistent. Therefore we have obtained a solvability sequence.

$$\begin{cases} v_{14} = v_{16}, \quad v_{14} = v_{34}, \\ v_{16} = v_{46}, \quad v_{38} = v_{34}, \\ v_{46} < v_{48}, \quad v_{38} < v_{48}, \\ v_{48} < v_{38} + v_{46}, \end{cases}$$

$$\begin{cases} v_{46} < v_{48}, \\ v_{48} < 2v_{46}, \\ (v_{46} > 0, v_{48} > 0). \end{cases}$$

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Linear programming (LP)

Linear programming (LP) is the problem of minimizing a linear function, subject to linear inequality constraints. An LP in standard form writes as:

$$\min \; c^T x \;\; s.t. \; \begin{cases} Ax & = & b \\ x & \geq & 0 \end{cases}$$

Two types of algorithms solve LP efficiently in practice: There are essentially two families of algorithms for solving LP:

- basis exchange algorithms (simplex, cross-cross),
- interior point methods.

Among the many LP solvers, the ones below support exact computations:

- CDD by Fukuda,
- CGAL's LP solver by Fischer, G¨rtner, Schönherr, F. Wessendorp,
- EXLP by Kiyomi,
- LRS by Avis.

Semidefinite programming (SDP)

- The set of real symmetric $n \times n$ matrices is denoted $\mathcal{S}^n$.
- A matrix $A \in \mathcal{S}^n$ is calle positive semidefinite if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$, and positive definite if $x^T Ax > 0$ for all nonzero $x \in \mathbb{R}^n$.
- The set of positive semidefinite matrices is denoted $\mathcal{S}^n_+$, which is a proper cone (i.e., closed, convex, pointed, and solid).

Semidefinite programming (SDP) is a specific kind of convex optimization problem. An SDP in standard form writes as:

$$\min \ \operatorname{Tr}(CX) \ \ s.t. \ \left\{ \begin{array}{ccc} \operatorname{Tr}(A_i X) & = & b_i \quad i = 1 \cdots m \\ X & \succcurlyeq & 0 \end{array} \right.$$

where $C, A_1, \ldots, A_m \in \mathcal{S}^n_+$ and where $x \succcurlyeq 0$ means $X \in \mathcal{S}^n_+$.

Among the many SDP solvers, let us mention SeDuMi by Sturm's team at McMaster and SDPA by Kojima's research group.

Polynomial optimization using Lasserre's relaxation method

- Let $f(x), g_1(x), \ldots, g_m(x)$ be polynomials over $\mathbb{R}$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Consider the problem (POP):
$$\min \ f(x) \ \ s.t. \ \ g_1(x) \geq 0, \ldots, g_m(x) \geq 0.$$
Let $d := \max\{d_1, \ldots, d_m\}$ with $d_i := \lceil \deg(g_i)/2 \rceil$ and let $N \geq d$ and $e \leq N$ be non-negative integers.
- Let $u_e(x)$ be the vector consisting of all monomials or degree less or equal to e degree-lex ordered for $x_1 < \cdots < x_n$.
- Let $M_e(x) := u_e(x)u_e(x)^T$ be the moment matrix. Its order is $\binom{n+e}{n}$.
- (POP) is equivalent to $(\mathrm{POP})_\mathrm{N}$
$$\min \ f(x) \ \ s.t. \ \ g_1(x)M_{N-d_1}(x) \geq 0, \ldots, g_m(x)M_{N-d_m}(x) \geq 0$$
- For $k = 1 \cdots m$, define symmetric matrices $A_\alpha^{(N.k)}$ such that
$$g_k(x)M_{N-d_k}(x) = \sum_{|\alpha| \leq 2N} x^\alpha A_\alpha^{(N.k)}$$
- Replace each monomial x^α by a variable y_α we obtain an SDP:
$$\min \ \sum_\alpha f_\alpha y_\alpha \ \ s.t. \ \ \sum_\alpha y_\alpha A_\alpha^{(N.k)} \succcurlyeq 0 \ \text{ for } \ k = 1 \cdots m.$$

Lasserre's relaxation method: basic example

Example

Consider the quadratic optimization problem:

$$\min \; x_1 x_2 \; \; s.t. \; 1 - x_1^2 - x_2^2 \geq 0$$

Lasserre's relaxation leads to:

$$\min \; y_{11} \; \; s.t. \; \left\{ \begin{array}{rcl} 1 - y_{20} - y_{02} & \geq & 0 \\ \begin{bmatrix} 1 & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix} & \succcurlyeq & 0 \end{array} \right.$$

The optimal value of the above SDP is 1/2, which is equal to the minimum of the QP.

Lasserre's relaxation method applied to a realizability problem (1/2)

Example

Assume that after applying Nakayama's solvability sequence algorithm, the reduced system is given:

$$X = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & x_{22} & 1 & 1 & 0 & -x_{26} \\ 0 & -x_{32} & 0 & 1 & 1 & x_{36} \end{pmatrix}.$$

where $x_{22}, x_{26}, x_{32}, x_{36}$ are positive unknowns satisfying:

$$1 - x_{32} > 0, \; x_{26} - 1 > 0, \quad \text{and} \quad x_{22}x_{36} - x_{26}x_{32} > 0.$$

Considering $x_{22} + x_{26} + x_{32} + x_{36}$ as the objective function our problem becomes:

$$\min \;\; x_{22} + x_{26} + x_{32} + x_{36} \;\; s.t. \;\; M_1(x)(1 - x_{32}) > 0, M_1(x)(x_{26} - 1) > 0, \quad \text{and} \quad x_{22}x_{36} - x_{26}x_{32} > 0.$$

Lasserre's relaxation method applied to a realizability problem (2/2)

$$\langle LSDP_2\rangle \begin{cases} \max & y_{1000}+y_{0100}+y_{0010}+y_{0001} \\ \text{s.t.} & \tilde{L}_1(\mathbf{y}), \tilde{L}_2(\mathbf{y}), \tilde{L}_3(\mathbf{y}) \succeq O \\ & \tilde{M}_2(\mathbf{y}) \succeq O \end{cases}$$

where each $\tilde{L}_i(\mathbf{y})$ is linear constraints. For example, $\tilde{L}_1(\mathbf{y}) \succeq O$ is represented as

$$\begin{pmatrix} 1-y_{0010} & y_{1000}-y_{1010} & y_{0100}-y_{0110} & y_{0010}-y_{0020} & y_{0001}-y_{0011} \\ y_{1000}-y_{1010} & y_{2000}-y_{2010} & y_{1100}-y_{1110} & y_{1010}-y_{1020} & y_{1001}-y_{1011} \\ y_{0100}-y_{0110} & y_{1100}-y_{1110} & y_{0200}-y_{0210} & y_{0110}-y_{0120} & y_{0101}-y_{0111} \\ y_{0010}-y_{0020} & y_{1010}-y_{1020} & y_{0110}-y_{0120} & y_{0020}-y_{0030} & y_{0011}-y_{0021} \\ y_{0001}-y_{0011} & y_{1001}-y_{1011} & y_{0101}-y_{0111} & y_{0011}-y_{0021} & y_{0002}-y_{0012} \end{pmatrix} \succeq O.$$

By solving (4.1) using SparsePOP, we obtain the solution $(x_{22}, x_{26}, x_{32}, x_{36}) = (3826, 5422, 0.770, 3825)$, which satisfies all constraints and shows the realizability of χ.

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Testing realizability via computer algebra

Testing the realizability of an oriented matroid reduces to check whether a system of polynomial equations and strict inequalities is consistent or not.

```
> R := PolynomialRing([x22, x36, x26, x32]);
                              R := polynomial_ring

> F := [x22>0, x32>0, x36 >0, 1-x32>0, x26-1>0, x22*x36-x26*x32>0];
   F := [0 < x22, 0 < x32, 0 < x36, 1 < x32, 1 < x26, 0 < x22 x36 - x26 x32]

> boxes := SamplePoints(F,R);
                                    boxes := [box]

> Display(boxes, R);
                                 { x22 = 5/2
                                 {
                                 { x36 = 1/2
                                [{                  ]
                                 { x26 = 3/2
                                 {
                                 { x32 = 1/2
```

Complexity of semi-algebraic set sampling

MAPLE's `RegularChains:-SamplePoints` command

Consider a semi-algebraic set S given by

$$f_1(x) = \cdots = f_m(x) = 0, \ g_1(x) > 0, \ldots, g_s(x) > 0$$

where $f_1(x), \ldots, f_m(x), g_1(x), \ldots, g_s(x) \in \mathbb{R}[x]$, each of degree at most d, with $x = (x_1, \ldots, x_n)$. The command `RegularChains:-SamplePoints` computes at least one point per connected component of S.

In theory:

The total number of operations in $\mathbb{R}$ for such computation amounts to $O((m+s)^{n+1} d^{O(n)})$ (Basu, Pollack & Roy).

Complexity of semi-algebraic set sampling

MAPLE's RegularChains:-SamplePoints command

Consider a semi-algebraic set S given by

$$f_1(x) = \cdots = f_m(x) = 0,\ g_1(x) > 0, \ldots, g_s(x) > 0$$

where $f_1(x), \ldots, f_m(x), g_1(x), \ldots, g_s(x) \in \mathbb{R}[x]$, each of degree at most d, with $x = (x_1, \ldots, x_n)$. The command `RegularChains:-SamplePoints` computes at least one point per connected component of S.

In practice:

- $O((m+s)^{n+1} d^{O(n)})$ still holds whenever $d = 1$ or $V(f_1, \ldots, f_m) \subset \mathbb{C}^n$ has dimension zero.
- If $\dim(V(f_1, \ldots, f_m) = \delta$ then a 2^{2^δ} appears.
- However, if $V(f_1, \ldots, f_m)$ is strongly equidimensional and S has dimension δ, `RegularChains:-RealTriangularize` certifies $S \neq \emptyset$ in singly-exponential time (Chen, Davenport, M.M.M., Xia & Xiao).

Is polynomial optimization really applicable? (1/2)

If we use apply an exact minimization algorithm based on quantifier elimination (QE) to the original problem, we obtain no solutions as one can expect.

```
> H := [x22>0, x32>0, x36 >0, 1>x32, x26>1, x22*x36-x26*x32 > 0];
   H := [0 < x22, 0 < x32, 0 < x36, x32 < 1, 1 < x26, 0 < x22 x36 - 

> f := x22+x36+x26+x32;
                            f := x22 + x36 + x26 + x32

> MinimizeWithConstraints (f,H,z);
                                        []
```

Is polynomial optimization really applicable? (2/2)

If we relax the inequalities, we obatin an non-feasible point.

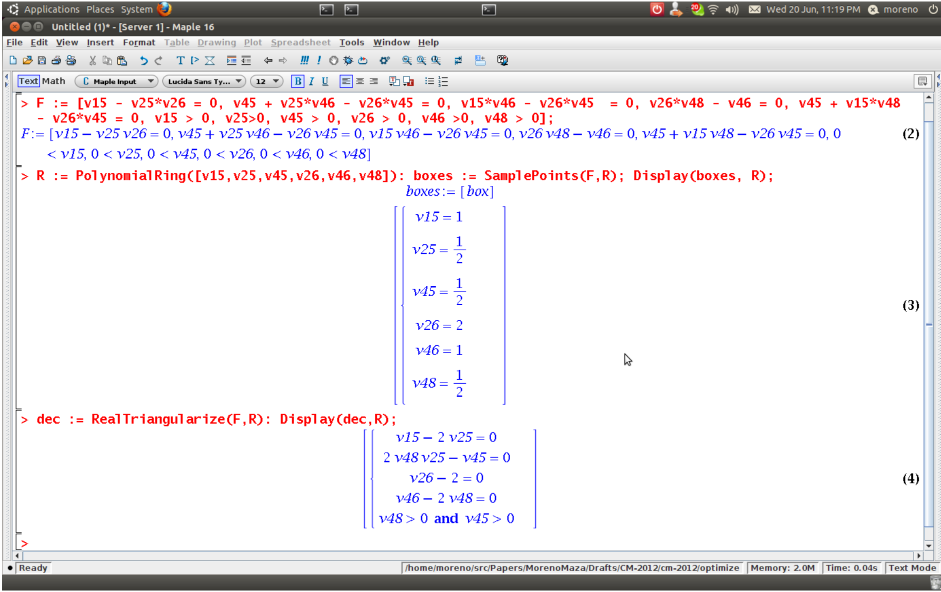
```
> H := [x22>=0, x32>=0, x36 >=0, 1>=x32, x26>=1, x22*x36-x26*x32>=0];
H :=

    [0 <= x22, 0 <= x32, 0 <= x36, x32 <= 1, 1 <= x26, 0 <= x22 x36 - x26 x32]

> f := x22+x36+x26+x32;
                         f := x22 + x36 + x26 + x32

> MinimizeWithConstraints (f,H,z);
                              { x22 = 0
                              {
                              { x26 = 1
                              {
                              { x32 = 0
                              {
                              { x36 = 0
                              {
                              {  z = 1
```

Testing realizability via computer algebra: 8_4_156392



Testing realizability via computer algebra: 8_4_157568

Untitled (1)* - [Server 1] - Maple 16

$$F := [0 < v14 - v45,\ 0 < v14 - v34,\ 0 < v44 - v14,\ 0 < v34 - v14\,v32,\ v45 - v32\,v45 - v32 = 0,\ v34\,v45 + v45 - v44 + v34 = 0,\ 0 < v32,\ 0 < v14,\ 0 < v34,\ 0 < v44,\ 0 < v45] \quad (2)$$

```
> R := PolynomialRing([v32, v14, v34, v44, v45]): boxes := SamplePoints(F,R); Display(boxes,R);
```

$$boxes := [box, box, box, box, box, box, box, box]$$

$$\left[\left\{\begin{array}{l} v32 = \left[\frac{819}{4096}, \frac{1639}{8192}\right] \\ v14 = \frac{17}{64} \\ v34 = \left[\frac{7}{128}, \frac{29}{512}\right] \\ v44 = \frac{41}{128} \\ v45 = \frac{1}{4} \end{array}\right.,
\left\{\begin{array}{l} v32 = \left[\frac{819}{4096}, \frac{1639}{8192}\right] \\ v14 = \frac{5}{16} \\ v34 = \left[\frac{35}{256}, \frac{71}{512}\right] \\ v44 = \frac{27}{64} \\ v45 = \frac{1}{4} \end{array}\right.,
\left\{\begin{array}{l} v32 = \left[\frac{819}{4096}, \frac{1639}{8192}\right] \\ v14 = \frac{5}{4} \\ v34 = 1 \\ v44 = \frac{3}{2} \\ v45 = \frac{1}{4} \end{array}\right.,
\left\{\begin{array}{l} v32 = \left[\frac{1911}{4096}, \frac{3823}{8192}\right] \\ v14 = \frac{15}{16} \\ v34 = \left[\frac{273}{512}, \frac{137}{256}\right] \\ v44 = \frac{15}{8} \\ v45 = \frac{7}{8} \end{array}\right.,\right.$$

$$\left.\left\{\begin{array}{l} v32 = \left[\frac{1911}{4096}, \frac{3823}{8192}\right] \\ v14 = \frac{9}{4} \\ v34 = \left[\frac{179}{128}, \frac{717}{512}\right] \\ v44 = \frac{7}{2} \\ v45 = \frac{7}{8} \end{array}\right.,
\left\{\begin{array}{l} v32 = \left[\frac{1911}{4096}, \frac{3823}{8192}\right] \\ v14 = \frac{33}{4} \\ v34 = \left[\frac{1041}{256}, \frac{2083}{512}\right] \\ v44 = \frac{17}{2} \\ v45 = \frac{7}{8} \end{array}\right.,
\left\{\begin{array}{l} v32 = \left[\frac{4915}{8192}, \frac{1229}{2048}\right] \\ v14 = \frac{13}{8} \\ v34 = \left[\frac{147}{128}, \frac{589}{512}\right] \\ v44 = \frac{35}{8} \\ v45 = \frac{3}{2} \end{array}\right.,
\left\{\begin{array}{l} v32 = \left[\frac{19}{32}, \frac{39}{64}\right] \\ v14 = \frac{5}{2} \\ v34 = 2 \\ v44 = \frac{13}{2} \\ v45 = \frac{3}{2} \end{array}\right.\right] \quad (3)$$

Ready | /home/moreno/src/Papers/MorenoMaza/Drafts/CM-2012/cm-2012/optimize | Memory: 2.0M | Time: 0.04s | Text Mode

Plan

1. Oriented Matroids
   - Axioms and examples
   - The realizability problem
2. Realization computations
   - Solvability sequences and other certificates
   - Using polynomial optimization software
   - Using computer algebra
   - Conclusions