On solving parametric polynomial systems

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joint work with
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Plan

1. Motivation and background
2. Main results
3. Conclusion and future work

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Related work

1. ...
2. Comprehensive Gröbner bases (Weispfenning, 1992)
3. Comprehensive Gröbner systems (Montes, 2004) (Kapur, Yao Sun, Wang 2010-2011)
4. Border polynomial (Yang, Xia and Hou, 1999)
5. Discriminant variety (Lazard and Rouillier, 2007)
6. Comprehensive triangular decomposition (Chen, Golubitsky, Lemaire, MMM, Pan 2007)
7. ...

Solving a parametric polynomial system?

Input: $f := a\,x^2 + x + 1$ with parameter a and $(a, x) \in \mathbb{R}^2$
Output:

$$\begin{cases} x = \frac{-1+\sqrt{1-4\,a}}{2\,a} \text{ or } x = \frac{-1-\sqrt{1-4\,a}}{2\,a}, & \text{when } a \neq 0 \\ x = -1, & \text{when } a = 0 \end{cases}$$

"Genererically", the properties of solutions depend on the parameter values continuously

- Generically: the points related to "discontinuity" are few
- Properties of the solutions: number, value, representation form (regular chains, Gröbner bases)

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Various notions of continuity

Let $f := a\,x^2 + x + 1$ with parameter a and α be a fix parameter value.

Example (Simple solutions)

Let $\alpha \notin \{0, \frac{1}{4}\}$. For all a near α, f has 2 complex simple roots.

Example (Real solutions)

Let $\alpha \in (0, \frac{1}{4})$. For all a near α, f has 2 simple real roots α_1, α_2, which are semi-algebraic functions of a:

$$\alpha_1 = \frac{-1 + \sqrt{1 - 4\,a}}{2\,a}, \alpha_2 = \frac{-1 - \sqrt{1 - 4\,a}}{2\,a}$$

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More examples

Let $f_2 := a\,y^2 + x\,y - 1$, $f_1 := (a-1)\,x^2 - 1$ and $S := \{f_1 = 0, f_2 = 0\}$ with parameter a

Example (Gröbner bases)

Let $\alpha \notin \{0,1\}$. For all $a = \beta$ near α, the reduced Gröbner basis of the polynomial ideal $\langle f_1, f_2\rangle|_{a=\beta}$ w.r.t. $x \prec y$ has the formula:

$$\{\,\frac{f_2}{a}, \frac{f_1}{a-1}\,\}_{a=\beta}.$$

Example (Triangular decomposition)

Let $\alpha \notin \{0,1\}$. For all $a = \beta$ near α,

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forms a triangular decomposition of $S|_{a=\beta}$.

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Notations

Parameters/values: $U = u_1, u_2, \ldots, u_d \;\; / \;\; \mathbb{C}^d$
Unknowns/values: $X = x_1, x_2, \ldots, x_s \;\; / \;\; \mathbb{C}^s$
Parametric algebraic system $S := [F, H]$:

$$\begin{cases} f_1 = f_2 = \cdots = f_\ell = 0 \\ h_1 \neq 0, h_2 \neq 0, \cdots, h_k \neq 0 \end{cases}$$

where

$$F = \{f_1, f_2, \ldots, f_\ell\},$$
$$H = \{h_1, h_2, \ldots, h_k\},$$
$$F, H \subset \mathbb{Q}[U, X].$$

$Z(S)$: zero set of S in $\mathbb{C}^{d+s}$
Assumption: S is *well-determinate*, i.e.

$$\mathcal{I} := \langle F \rangle : (\prod_{h \in H} h)^\infty$$

is of dimension d and U is maximally algebraically independent modulo $\mathcal{I}$.

Two notions of continuity

$$\Pi_{\mathrm{U}} : Z(S) \subset \mathbb{C}^{s+d} \mapsto \mathbb{C}^d$$
$$\Pi_{\mathrm{U}}(x_1, \ldots, x_s, u_1, \ldots, u_d) = (u_1, \ldots, u_d)$$

Definition

Let α in $\mathbb{C}^d$. We say that S is

1. *Z-continuous* at α: if there exists an open ball $\mathcal{O}_\alpha$ centered at α s.t. for any $\beta \in \mathcal{O}_\alpha$ we have $\#(Z(S(\beta)) = \#(Z(S(\alpha)))$.
2. Π_{U}*-continuous* at α: if there exists an open ball $\mathcal{O}_\alpha$ centered at α and a finite partition, say $\{C_1, \ldots, C_k\}$ of $\Pi_{\mathrm{U}}^{-1}(\mathcal{O}_\alpha) \cap Z(S)$ such that for each $j \in \{1, \ldots, k\}$

$$\Pi_{\mathrm{U}}|_{C_j} : C_j \xrightarrow{\Pi_{\mathrm{U}}} \mathcal{O}_\alpha$$

is a diffeomorphism.

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2. *Π_{U}-continuous* at α: if there exists an open ball $\mathcal{O}_\alpha$ centered at α and a finite partition, say $\{C_1, \ldots, C_k\}$ of $\Pi_{\mathrm{U}}^{-1}(\mathcal{O}_\alpha) \cap Z(S)$ such that for each $j \in \{1, \ldots, k\}$

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Border polynomial and discriminant variety

We reformulate these two well-known concepts using the previous continuity notions.

Definition (Border polynomial)

A non-zero polynomial b in $\mathbb{Q}[U]$ is called a *border polynomial* (BP) of the parametric polynomial system S if the zero set $V(b)$ of b in $\mathbb{C}^d$ contains all the points at which S is not Z-continuous.

Definition (Discriminant variety)

An algebraic set $\mathcal{W} \subsetneq \mathbb{C}^d$ is a *discriminant variety* of the parametric polynomial system S if $\mathcal{W}$ contains all the points at which S is not Π_U-continuous.

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Remarks

1. Π_{U}-*continuity* implies Z-continuity
2. The minimal DV: the intersection of all DV
3. The "minimal" BP?: does not exist in general; see the case the minimal DV is not a hypersurface.

Motivation and contributions

- Are those different notions of continuity (and discontinuity) computatable?
- How are they related to each other?

Our contributions:

- *Z-continuity* and Π_{U}-*continuity* are equivalent for "triangular" systems
- An explicit form of BP/minimal DV for "triangular" systems
- A new characterization for the non-properness locus of the mnimal DV of sat(T)
- The difference between the minimal DV of T and that of sat(T)
- Given T, characterize T' among all T' satisfying sat(T') = sat(T) such that T' poccessing the minimal BP.

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Plan

Equivalence BP-DV for triangular systems (1/2)

Let $S := [T, H]$ be a squarefree triangular algebraic system (STAS). Denote by $B_{sep}(T)$, $B_{ini}(T)$, $B_{ie}([T, H])$ respectively the set of the irreducible factors of

$$\prod_{t \in T} \mathrm{ires}(\mathrm{discrim}(t, \mathrm{mvar}(t)), T), \prod_{t \in T} \mathrm{ires}(\mathrm{init}(t), T), \text{ and } \prod_{f \in H} \mathrm{ires}(f, T).$$

Denote $\mathbf{BPS}(T) := B_{sep}(T) \cup B_{ini}(T) \cup B_{ie}([T, H])$.

Equivalence BP-DV for triangular systems (2/2)

Lemma

Let $b = \prod_{f \in B_{sep}(T) \cup B_{ini}(T) \cup B_{ie}([T,H])} f$*; let* $N := \prod_{f \in T} \mathrm{mdeg}(f)$*. Then for each parameter value* $\alpha \in \mathbb{C}^d$*:*

1. *if* $b(\alpha) \neq 0$*, then* $\# Z(S(\alpha)) = N$ *holds;*
2. *if* $b(\alpha) = 0$*, then* $\# Z(S(\alpha))$ *is either infinite or less than* N*.*

(This means b is a border polynomial of S)

Proposition

The minimal discriminant variety of $S := [T, H]$ *is*

$$V\left(\prod_{f \in B_{ini}(T) \cup B_{sep}(T) \cup B_{ie}([T,H])} f\right).$$

clearly BP $\equiv$ DV for STASes

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$\mathcal{O}_\infty$: set of non-properness of Π_{U}

$\mathcal{O}_\infty(S)$: the set of points where Π_{U} is not proper
($\mathcal{O}_\infty(S)$ is related to the number of solutions counting multiplicities)

Proposition

We have $\mathcal{O}_\infty(T) = V(\prod_{f \in B_{ini}(T)} f)$.

Proposition

For each $i = 1, \ldots, s$, *let* g_i *be a polynomial generating the principal ideal* $\mathrm{sat}(T) \cap \mathbb{Q}[U, x_i]$. *Then we have*

$$\mathcal{O}_\infty(\mathrm{sat}(T)) = \cup_{i=1}^{s} V(\mathrm{init}(g_i)).$$

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Discriminant variety: T vs $\mathrm{sat}(T)$

Proposition

We have

$$DV_T \setminus DV_{\mathrm{sat}(T)} \subseteq \mathcal{O}_\infty(T) \setminus \mathcal{O}_\infty(\mathrm{sat}(T)).$$

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Let T_1 and T_2 be two regular chains satisfying $\mathrm{sat}(T_1) = \mathrm{sat}(T_2)$. If $B_{ini}(T_1) \subseteq B_{ini}(T_2)$ holds, then $B_{ini}(T_1) \cup B_{sep}(T_1) \subseteq B_{ini}(T_2) \cup B_{sep}(T_2)$ holds.

Theorem

Let T^* be another regular chain satisfying $\mathrm{sat}(T) = \mathrm{sat}(T^*)$. If T^* is canonical, then we have $B_{ini}(T^*) \subseteq B_{ini}(T)$.

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Plan

Summary and future work

Propose the framework of "continuity" to unify the notions of BP and DV

For an STAS $S := [T, H]$

- $b := \prod_{f \in B_{sep}(T) \cup B_{ini}(T) \cup B_{ie}([T,H])} f$ is a border polynomial of S
- $V(b = 0)$ is the minimal DV of S
- New characterization of $\mathcal{O}_\infty$:

$$\mathcal{O}_\infty(T) = \prod_{f \in B_{ini}(T)} f \ = 0,$$
$$\mathcal{O}_\infty(\text{sat}(T)) = \prod_{i \in \{1,2,\ldots,s\}} g_i \ = 0$$

- $$DV_T \setminus DV_{\text{sat}(T)} \subseteq \mathcal{O}_\infty(T) \setminus \mathcal{O}_\infty(\text{sat}(T)).$$

- The regular chains in canonical form possess the smallest BPes.

Summary and future work

Future work:

- Show that *Z-continuity* and Π_{U}-*continuity* are equivalent for equidimensional systems
- Investigate the relation of other notions: comprehensive GB, comprehensive TD and the notion of BP or DV
- Design better algorithm to compute BP