CS2209A 2017
Applied Logic for Computer Science

Lecture 16

Resolution for Predicate Logic

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Revisit: main rules of inference in propositional logic

- **Valid argument**:
  AND of premises $\rightarrow$ **conclusion** is a **tautology**

- Modus ponens:
  $(p \rightarrow q) \wedge p \rightarrow q$ is a tautology

- Hypothetical syllogism:
  $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology

- Disjunctive syllogism:
  $(A \vee B) \wedge \neg A \rightarrow B$ is a tautology

- Resolution:
  $(A \vee C) \wedge (B \vee \neg C) \rightarrow (A \vee B)$ is a tautology

Rules of inference

- These patterns describe how new knowledge can be derived from existing knowledge, both in the form of propositional logic formulas (sentences).

- When describing an inference rule, the *premise* specifies the pattern that must match our knowledge base and the *conclusion* is the new knowledge inferred.

Modus ponens, modus tollens, AND elimination, AND introduction, and universal instantiation

- If the sentences P and $P \rightarrow Q$ are known to be true, then **modus ponens** lets us infer Q.
- Under the inference rule **modus tollens**, if $P \rightarrow Q$ is known to be true and Q is known to be false, we can infer P.
- **AND elimination** allows us to infer the truth of either of the conjuncts from the truth of a conjunctive sentence.
  E.g. $P \wedge Q$ lets conclude both P and Q are true.
- **AND introduction** lets us infer the truth of a conjunction from the truth of its conjuncts.
  E.g. if both P and Q are true, then $P \wedge Q$ are true.
- **Universal instantiation** states that if any universally quantified variable in a true sentence is replaced by any appropriate term from the domain, the result is a true sentence. Thus, if a is from the domain of X, $\forall X\ p(X)$ lets us infer $p(a)$.

Definition

- A predicate logic (or calculus) expression X **logically follows** from a set S of predicate calculus expressions if every interpretation and variable assignment that satisfies S also satisfies X.
  - An *interpretation* is an assignment of specific values to domains and predicates.
- An inference rule is **sound** if every predicate calculus expressions also logically follows from S.
- An inference rule is **complete** if, given a set S of predicate calculus expressions, the rule can infer every expression that logically follows from S.

Logic and finding a proof

- Given
  - a knowledge base represented as a set of propositional sentences.
  - a goal stated as a propositional sentence
  - list of inference rules

- We can write a program to repeatedly apply inference rules to the knowledge base in the hope of deriving the goal.

Developing a proof procedure

- Deriving (or refuting) a goal from a collection of logic facts corresponds to a very large search tree.

- A large number of *rules of inference* could be utilized.

- The selection of which rules to apply and when would itself be non-trivial.

Resolution and CNF

- ***Resolution*** is a single rule of inference that can operate efficiently on a special form of sentences.
- The special form is called *conjunctive normal form* (CNF) or ***clausal form***, and has these properties:
  - – Every sentence is a disjunction (OR) of literals (clauses)
  - – All sentences are implicitly conjuncted (ANDed).

Predicate Logic Resolution

- We have to worry about the arguments to predicates, so it is harder to know when two literals match and can be used by resolution.
  - For example, does the literal Father(Bill, Chelsea) match Father(x, y) ?

- The answer depends on how we substitute values for variables.

Proof procedure for predicate logic

- Same idea, but a few added complexities:
  - conversion to CNF is much more complex.
  - Matching of literals requires providing a matching of variables, constants and/or functions.

$$\neg \text{Skates}(x) \vee \text{LikesHockey}(x)$$

$$\neg \text{LikesHockey}(y)$$

We can resolve these only if we assume x and y refer to the same object.

Predicate Logic and CNF

- Converting to CNF is harder - we need to worry about variables and quantifiers.
  - Eliminate all implications $\rightarrow$
  - Reduce the scope of all $\neg$ to single term
  - Make all variable names unique
  - Move quantifiers left (prenex normal form)
  - Eliminate Existential Quantifiers
  - Eliminate Universal Quantifiers
  - Convert to conjunction of disjuncts
  - Create separate clause for each conjunct.

Eliminate Existential Quantifiers

- Any variable that is existentially quantified means that
  - *there is some value for that variable that makes the expression true.*

- To eliminate the quantifier, we can replace the variable with a **function**.

- We don't know what the function is, we just know it exists.

Skolem functions

- Named after the Norwegian logician Thoralf Skolem
- **Example:** $\exists\, y$ President(y)

  We replace y with a new function *func*:

  President(*func()*)

  func is called a Skolem function.

- In general the function must have the same number of arguments as the number of **universal** quantifiers in the current scope.

Skolemization Example

- In general the function must have the *same number of arguments* as the number of **universal** quantifiers in the current scope.

- **Example:** $\forall x \; \exists y$ Father(y, x)
  - create a new function named foo and replace y with the function.
  - $\forall x$ Father$(foo(x), x)$

Unification

- Two formulas are said to unify if there are legal instantiations (assignments of terms to variables) that make the formulas in question *identical*.

- The act of unifying is called **unification**. The instantiation that unifies the formulas in question is called a **unifier**.

- There is a simple algorithm called the ***unification algorithm*** that does this.

Unification

- **Example**: Unify the formulas $Q(a, y, z)$ and $Q(y, b, c)$
- **Solution**:
  - Since y in $Q(a, y, z)$ is a different variable than y in $Q(y, b, c)$, rename y in the second formula to become $y1$.
  - This means that one must unify $Q(a, y, z)$ with $Q(y1, b, c)$.
  - An instance of $Q(a, y, z)$ is $Q(a, b, c)$ and an instance of $Q(y1, b, c)$ is $Q(a, b, c)$.
  - Since these two instances are identical, $Q(a, y, z)$ and $Q(y, b, c)$ unify.
  - The unifier is $y1 = a$, $y = b$, $z = c$.

Unification

- **Unification**: matching literals and doing substitutions that resolution can be applied.
- **Substitution**: when a variable name is replaced by another variable or element of the domain.
  - Notation [a/x] means replacing all occurrences of x with a in the formula
  - Example: substitution [5/x] in $p(x) \vee Q(x,y)$ results in $p(5) \vee Q(5,y)$

Unification

- It is an algorithm for determining the substitutions needed to make two predicate logic expressions match.
- A variable cannot be unified with a term containing that variable. The test for it is called the **occurs check**.
  - Example: cannot substitute x for $x + y$ in $\mathrm{p}(x + y)$
  - Most applicable when rather than having variables we have whole expressions (terms) evaluating to elements of the domain.
  - Example: $x + y$ is a term; when $x, y \in \mathbb{Z}$ and $x + y \in \mathbb{Z}$, with terms we can write formulas such as $p(x + y) \vee Q(y - 2)$

Algorithm to convert to clausal form (1)

(1) Eliminate conditionals $\rightarrow$, using the equivalence

$p \rightarrow q \equiv \neg p \vee q$

e.g. $(\exists x)(p(x) \wedge (\forall y)\big(f(y) \rightarrow h(x,y)\big))$ becomes

$(\exists x)(p(x) \wedge (\forall y)\big(\neg f(y) \vee h(x,y)\big))$

(2) Eliminate negations or reduce the scope of negation to one atom.

e.g. $\neg\neg p \equiv p$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg\big(\forall x \in S, F(x)\big) \equiv \exists x \in S,\ \neg F(x)$

$\neg\big(\exists x \in S, F(x)\big) \equiv \forall x \in S,\ \neg F(x)$

(3) Standardize variables within a well-formed formula so that the bound or free variables of each quantifier have unique names. e.g. $(\exists x)\neg p(x) \vee (\forall x)p(x)$ is replaced by $(\exists x)\neg p(x) \vee (\forall y)p(y)$

Algorithm to convert to clausal form (2)

(4) Advanced step: if there are existential quantifiers, eliminate them by using Skolem functions

e.g. $(\exists x)p(x)$ is replaced by $p(a)$

$(\forall x)(\exists y)k(x,y)$ is replaced by $(\forall x)\ k(x, f(x))$

(5) Convert the formula to prenex form

e.g. $(\exists x)(p(x) \wedge (\forall y)\ (\neg f(y) \vee h(x,y)))$ becomes

$(\forall y)\ (p(a) \wedge (\neg f(y) \vee h(a,y)))$

(6) Convert the formulas to CNF, which is a conjunctive of clauses. Each clause is a disjunction.

e.g. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(7) Drop the universal quantifiers

e.g. the formula in (5) becomes $p(a) \wedge (\neg f(y) \vee h(a,y))$

Algorithm to convert to clausal form (3)

(8) Eliminate the conjunctive signs by writing the formula as a set of clauses

e.g. $p(a) \land (\neg f(y) \lor h(a,y))$ becomes $p(a),$ $(\neg f(y) \lor h(a,y))$

(9) Rename variables in clauses, if necessary, so that the same variable name is only used in one clause.

e.g. $p(x) \lor q(y) \lor k(x,y)$ and $\neg p(x) \lor q(y)$ becomes $p(x) \lor q(y) \lor k(x,y)$ and $\neg p(x1) \lor q(y1)$

Example: Resolution for predicate logic

Anyone passing his history exams and winning the lottery is happy.

$\forall$ **X (pass (X,history)** $\wedge$ **win (X,lottery)** $\rightarrow$ **happy (X))**

Anyone who studies or is lucky can pass all his exams.

$\forall$ **X** $\forall$ **Y (study (X)** $\vee$ **lucky (X)** $\rightarrow$ **pass (X,Y))**

John did not study but he is lucky.

$\neg$ **study (john)** $\wedge$ **lucky (john)**

Anyone who is lucky wins the lottery.

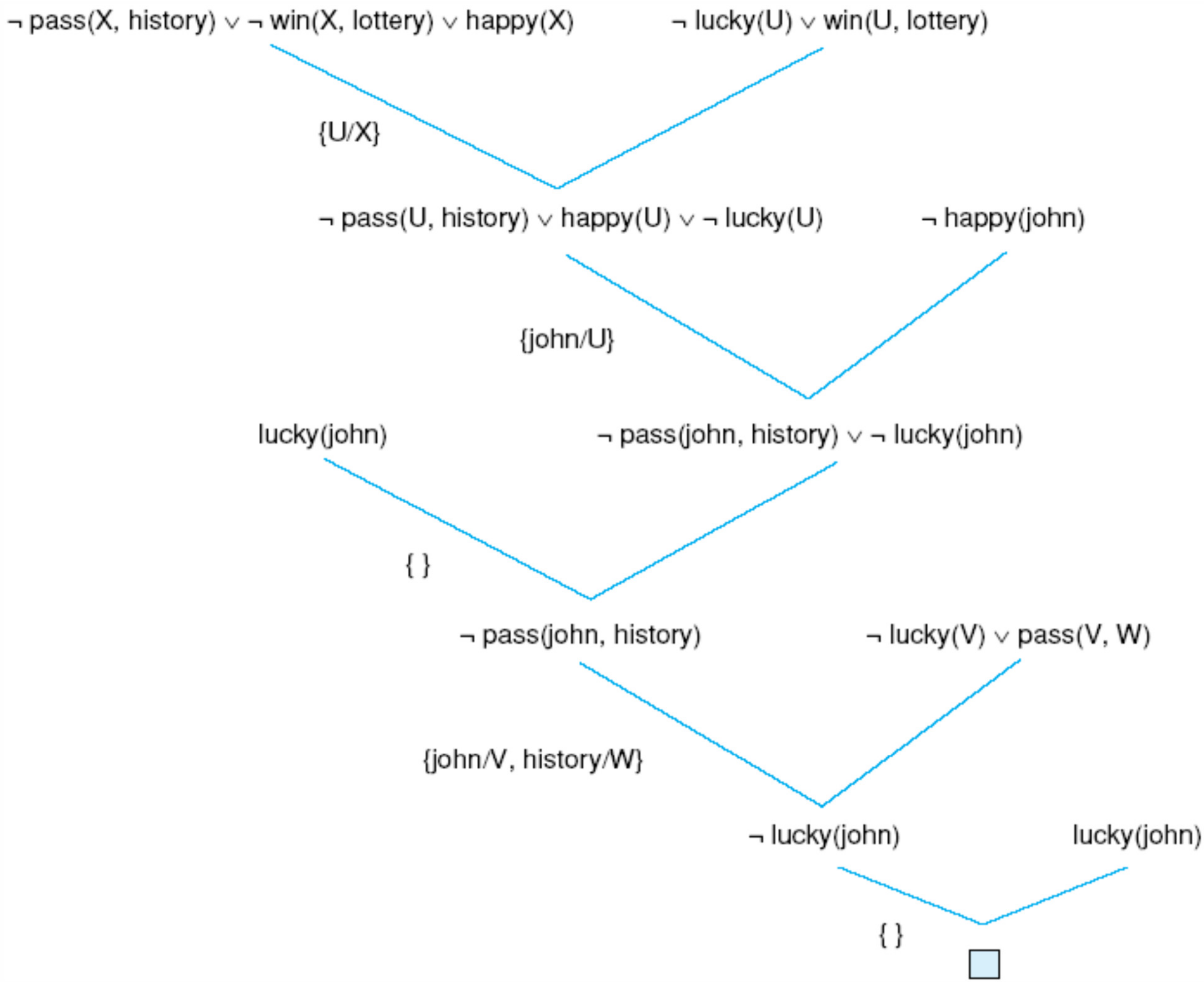
$\forall$ **X (lucky (X)** $\rightarrow$ **win (X,lottery))**

These four predicate statements are now changed to clause form (Section 12.2.2):

1. $\neg$ **pass (X, history)** $\vee$ $\neg$ **win (X, lottery)** $\vee$ **happy (X)**
2. $\neg$ **study (Y)** $\vee$ **pass (Y, Z)**
3. $\neg$ **lucky (W)** $\vee$ **pass (W, V)**
4. $\neg$ **study (john)**
5. **lucky (john)**
6. $\neg$ **lucky (U)** $\vee$ **win (U, lottery)**

Into these clauses is entered, in clause form, the negation of the conclusion:

7. $\neg$ **happy (john)**


¬ pass(X, history) ∨ ¬ win(X, lottery) ∨ happy(X)
¬ lucky(U) ∨ win(U, lottery)
{U/X}
¬ pass(U, history) ∨ happy(U) ∨ ¬ lucky(U)
¬ happy(john)
{john/U}
lucky(john)
¬ pass(john, history) ∨ ¬ lucky(john)
{ }
¬ pass(john, history)
¬ lucky(V) ∨ pass(V, W)
{john/V, history/W}
¬ lucky(john)
lucky(john)
{ }