CS2209A 2017
Applied Logic for Computer Science

Lecture 17

Relations, first-order formula and database application

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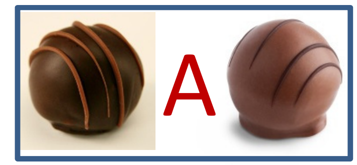
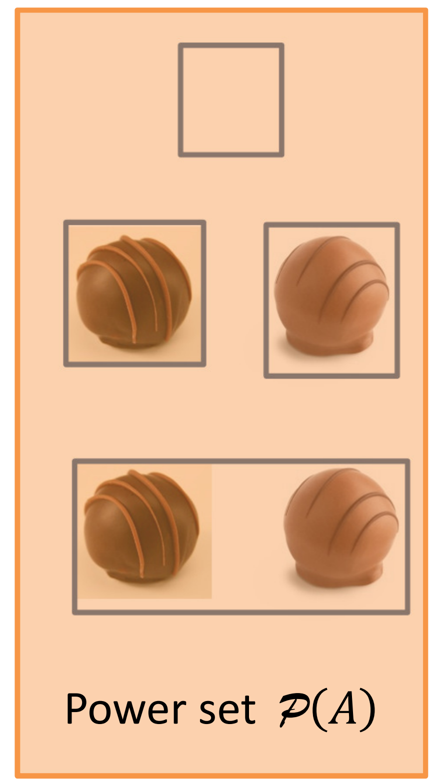
Power sets

- A **power set** of a set A, $\mathcal{P}(A)$, is a set of **all subsets** of A.
  - Think of sets as boxes of elements.
  - A subset of a set A is a box with elements of A (maybe all, maybe none, maybe some).
  - Then $\mathcal{P}(A)$ is a box containing boxes with elements of A.
  - When you open the box $\mathcal{P}(A)$, you don't see chocolates (elements of A), you see boxes.
  - A={1,2}, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
  - $A = \emptyset$, $\mathcal{P}(A) = \{\emptyset\}$.
    - They are not the same! There is nothing in A, and there is one element, an empty box, in $\mathcal{P}(A)$
- If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.



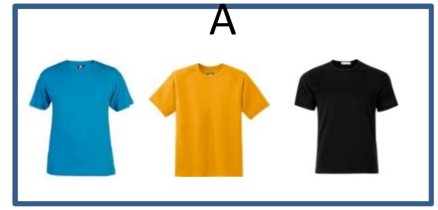
Subsets of A:

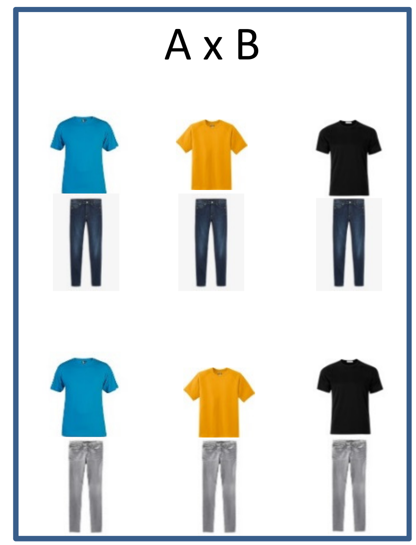


Cartesian products

- **Cartesian product** of A and B is a set of all pairs of elements with the first from A, and the second from B:
  - A x B = $\{(x, y) \mid x \in A,\ y \in B\}$
  - A={1,2,3}, B={a,b}
  - $A \times B =$ $\{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
  - A={1,2}, $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

	a	b
1	(1,a)	(1,b)
2	(2,a)	(2,b)
3	(3,a)	(3,b)





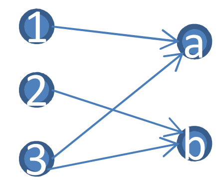
- Order of pairs does not matter, order within pairs does: $A \times B \neq B \times A$.
- Number of elements in $A \times B$ is $|A \times B| = |A| \cdot |B|$

Cartesian products

- We can define the Cartesian product for any number of sets:
  - $A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \ldots x_k) | \, x_1 \in A_1 \ldots x_k \in A_k\}$
  - $A = \{1,2,3\}$, $B =$ {a,b}, C={3,4}
  - $A \times B \times C = \{(1, a, 3), (1, a, 4), \ (1, b, 3), (1, b, 4),$
    $(2, a, 3), (2, a, 4), \ (2, b, 3), (2, b, 4),$
    $(3, a, 3), (3, a, 4), \ (3, b, 3), (3, b, 4)\}$

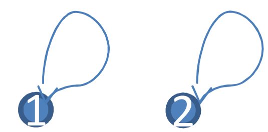
Relations

- **A relation** is a subset of a Cartesian product of sets.
  - If of two sets (set of pairs), call it a **binary** relation.
  - Of 3 sets (set of triples), **ternary**. Of k sets (set of tuples), **k-nary**
  - A={1,2,3}, B={a,b}
    - $A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$
    - R = {(1,a), (2,b),(3,a), (3,b)} is a relation. So is R={(1,b)}.



Graph of R (bipartite)

  - A={1,2},
    - $A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$
    - R={(1,1), (2,2)} (all pairs (x,y) where x=y)
    - R={(1,1),(1,2),(2,2)} (all pairs (x,y) where $x \leq y$).

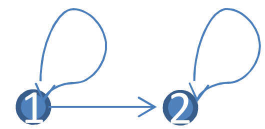


Graph of {(1,1),(2,2)}

  - A=PEOPLE
    - COUPLES ={(x,y) | Loves(x,y)}
    - PARENTS ={(x,y) | Parent(x,y)}
  - A=PEOPLE, B=DOGS, C=PLACES
    - WALKS = {(x,y,z) | x walks y in z}
      - Jane walks Buddy in spring bank park.

Types of binary relations

- A binary relation $R \subseteq A \times A$ is
  - **Reflexive** if $\forall x \in A, R(x,x)$



R_1

    - Every x is related to itself.
    - E.g. A={1,2}, $R_1 = \{ (1,1), (2,2), (1,2)\}$
    - On A = $\mathbb{Z}$, $R_2 = \{(x,y)|x = y\}$ is reflexive
    - But not $R_3 = \{(x,y)|\ x < y\}$
  - **Symmetric** if $\forall x, y \in A,\ (x,y) \in R \leftrightarrow (y,x) \in R$
    - R_1 and R_3 above are not symmetric. R_2 is.
    - A = $\mathbb{Z}$, $R_4 = \{(x,y)|x \equiv y \ mod\ 3 \}$ is symmetric.

Types of binary relations

- A binary relation $R \subseteq A \times A$ is
  - **Transitive**
    if $\forall x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \rightarrow\ (x, z) \in R$
    - R_1, R_2, R_3, R_4 are all transitive.
    - $R_5 = \{(x, y) | x, y \in \mathbb{Z} \wedge x + 1 = y\}$ is not transitive.
    - PARENT = $\{(x, y) | x, y \in PEOPLE \wedge x\ \ is\ a\ parent\ of\ y\}$ is not.
    - A **transitive closure** of a relation R is a relation $R^* = \{(x, z)\ |\ \exists k \in \mathbb{N}\ \ \exists y_0, \ldots, y_k \in A\ \ (x = y_0 \wedge z = y_k) \wedge \forall i \in \{0, \ldots, k-1\}\ R(y_i, y_{i+1})\}$
      - That is, can get from x to z following R arrows.

Types of binary relations

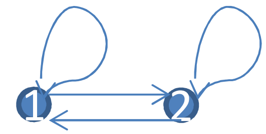
- A binary relation $R \subseteq A \times A$ is
  - **Anti-reflexive** if $\forall x \in A, \neg R(x,x)$
    - R can be neither reflexive nor anti-reflexive.
    - E.g. A={1,2}, $R_6 = \{(1,2)\}$
      - but not $R_1 = \{(1,1), (2,2), (1,2)\}$ (reflexive)
      - nor $R_7 = \{(1,1), (1,2)\}$ (neither)
    - For $A = \mathbb{Z}$, not $R_2 = \{(x,y) | x = y\}$
      - Nor $R_4 = \{(x,y) | x \equiv y \ mod\ 3\}$
    - But $R_3 = \{(x,y) |\ x < y\}$ is anti-reflexive.
      - So are $R_5 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x + 1 = y\}$
      - And PARENT = $\{(x,y) \in PEOPLE \times PEOPLE\ | x\ \ is\ a\ parent\ of\ y\}$

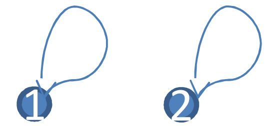
Graph of {(1,2)}

Types of binary relations

- A binary relation $R \subseteq A \times A$ is
  - **Anti-symmetric**
    if $\forall x, y \in A, (x, y) \in R \land (y, x) \in R \rightarrow x = y$
    - $R_1, R_3, R_5, R_6, R_7, PARENT$ are anti-symmetric. R_4 is not.
    - R_2 is both symmetric and anti-symmetric.
    - $R_8 = \{(1,2), (2,1), (1,3)\}$ is neither symmetric nor anti-symmetric.

Equivalence





- A binary relation $R \subseteq A \times A$ is an **equivalence** if R is reflexive, symmetric and transitive.
  - E.g. **A={1,2},** $R = \{(1,1),(2,2)\}$ or $R = A \times A$
  - Not $R_1 = \{ (1,1), (2,2), (1,2)\}$
    nor $R_3 = \{(x, y) |\ x < y\}$
  - On A = $\mathbb{Z}$, $R_2 = \{(x, y) | x = y\}$ is an equivalence
  - So is $R_4 = \{(x, y) | x \equiv y\ mod\ 3\ \}$
    - Reflexive: $\forall x \in \mathbb{Z},\ x \equiv x\ mod\ 3$
    - Symmetric: $\forall x, y \in \mathbb{Z},\ x \equiv y \bmod 3 \rightarrow y \equiv x\ \ mod\ 3$
    - Transitive: $\forall x, y, z \in \mathbb{Z},\ x \equiv y \bmod 3 \wedge y \equiv z\ \ mod\ 3 \rightarrow x \equiv z\ \ mod\ 3$

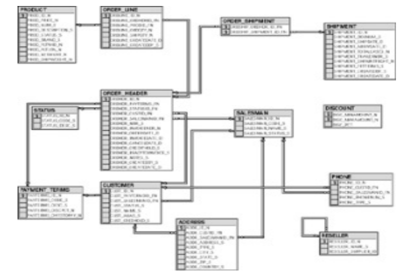
Equivalence

- An equivalence relation partitions A into **equivalence classes**:
  - Intersection of any two equivalence classes is ∅
  - Union of all equivalence classes is A.
  - $R_4 = \{(x, y) | x \equiv y \; mod \; 3 \}$:

$$\mathbb{Z} = \{x \mid x \equiv 0 \; mod \; 3\} \cup \{x \mid x \equiv 1 \; mod \; 3\} \cup \{x \mid x \equiv 2 \; mod \; 3\}$$

  - $R = A \times A$ gives rise to a single equivalence class.
  - $R = \{(1,1), (2,2)\}$ to two.

Databases and predicates



Relation R	Predicate P
A set of tuples	True/false on a given tuple
R=$\{(x_1, \ldots, x_k) \mid P(x_1, \ldots, x_k)$ is true$\}$	$P(x_1, \ldots, x_k) \equiv (x_1, \ldots, x_k) \in R$

– In a database, store relations as tables.

– Then ask queries as predicate logic formulas

- Return the set of all database elements satisfying the formula.

Well-Formed Formula for First Order Predicate Logic

- Not all strings can represent propositions of the predicate logic. Those which produce a proposition when their symbols are interpreted must follow the rules given below, and they are called **wffs** (well-formed formulas) of the first order predicate logic.

- A predicate name followed by a list of variables such as $P(x, y)$, where P is a predicate name, and x and y are variables, is called an **atomic formula**.

Well-Formed Formula for First Order Predicate Logic

- **wffs are constructed using the following rules:**
  - *True* and *False* are wffs.
  - Each propositional constant (i.e. specific proposition), and each propositional variable (i.e. a variable representing propositions) are wffs.
  - Each atomic formula (i.e. a specific predicate with variables) is a wff.
  - If A, B, C are wffs, then so are $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$
  - If x is a variable (representing objects of the universe of discourse), and A is a wff, then so is $\exists x\ A$ and $\forall x\ A$.

Relational database and query

Example: three tables, *student*, *course*, and *enroll*

COURSE	
CNO	TITLE
CSc2310	Java
CSc2510	Theory
CSc4710	Database

STUDENT	
SID	NAME
1111	Jones
2222	Smith
3333	Blake
4444	Damon
5555	Bell

ENROLL		
SID	CNO	GRADE
1111	CSc2310	F
1111	CSc2510	F
2222	CSc2510	A
2222	CSc4710	B
3333	CSc2310	A
3333	CSc2510	A
3333	CSc4710	F

- A query using predicate logic is of the form { X1,...,Xn | P(X1,...,Xn) } , where P(X1,...,Xn) is a **wff** in predicate logic with free variables X1,...,Xn.

Relational database and query

COURSE	
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2222	CSc2510	A
2222	CSc4710	B
3333	CSc2310	A
3333	CSc2510	A
3333	CSc4710	F

- **Now, some query examples:**
  - Get names of students enrolled in CSc2510.
    { N | (exists S,G)(student(S,N) and enroll(S,'CSc2510',G)) }
  - Get CNO and TITLE of courses in which Smith is enrolled.
    { C,T | (exists S,G)(course(C,T) and enroll(S,C,G) and student(S,'Smith')) }
  - Get names of students who have enrolled in at least 2 courses.
    { N | (exists S,C1,G1,C2,G2)(student(S,N) and enroll(S,C1,G1) and enroll(S,C2,G2) and C1 <> C2) }
  - Get names of students who have an "A" grade in CSc2510.
    { N | (exists S)(student(S,N) and enroll(S,'CSc2510','A')) }

Limitations of first-order logic

- With predicate logic, can we express everything?
- The fact is that some very natural properties cannot be expressed in predicate logic, because here the notion of “*expressing*" is quite different.
- A **propositional formula** only talks about **finitely many** things, and if worst comes to worst it can just list all the cases, describing a truth table.
- Whereas in the **predicate logic** case we have the ability to talk about **infinitely many** things at once.
- And it is not possible to list all infinitely many cases of relationships among, say, natural numbers, at least not with a finite-length formula.

Limitations of first-order logic

- Here is an example of a very natural property which is not *expressible* in first-order logic.
  - **Transitivity in databases**: Consider an airline database such as the one that travel agents use to find and book flights.

- Now ask the database to "*give me a way to fly from London Ontario to London UK, and I don't care how many times I need to change planes*".

- This kind of query, called "**transitivity**", is not expressible in first-order logic.

Limitations of first-order logic

- A very natural property which is not *expressible* in first-order logic.
  - **Transitivity in databases**: Consider an airline database such as the one that travel agents use to find and book flights.
- For any fixed number of plane changes you can, indeed, ask if there is a way to get from London ON to London UK, however you always need to set a limit on the number of intermediate locations.
- For example, to ask if there is a way to get to London UK via four intermediate cities you would write
  - $\exists y_1, y_2, y_3, y_4 \ Flight(London\ ON, y_1) \wedge Flight(y_1, y_2) \wedge Flight(y_2, y_3) \wedge Flight(y_3, y_4) \wedge Flight(y_4, London\ UK)$

Transitivity in databases

- Some databases do have special ways of computing this kind of relation, but it is an addition to the language of first-order logic and can be quite computationally intensive, especially in large databases.

- More often, it seems, the database system just tries the first several queries, until they reach a number of intermediate steps that seems too large for them

Transitivity in databases

- You might have noticed that you pretty much never see a route with more than three intermediate cities, and sometimes get an answer “there is no way“

- You can also ask a combination of such queries: for example, “Is there a direct flight from London ON to London UK, or a flight with one intermediate city”?
  - $\exists y\ Flight(London\ ON, London\ UK) \vee Flight(London\ ON, y) \wedge\ Flight(y, London\ UK)$