Proving Theorems and Verifying Programs Automatically

Applied Logic for Computer Science

UWO – December 3, 2017

Plan

1. Introduction to SMT solving
2. Using Yices for checking assertions
3. Equality Reasoning
4. Theory Reasoning

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4. Theory Reasoning

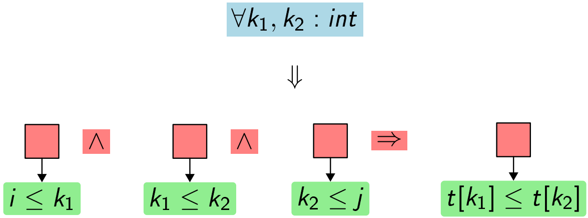
A logical formula . . .

$$sorted(t, i, j) =$$

$$\forall k_1, k_2 : int \;.\; i \leq k_1 \;\wedge\; k_1 \leq k_2 \;\wedge\; k_2 \leq j \Rightarrow t[k_1] \leq t[k_2]$$

...as seen by an SMT solver

$sorted(t, i, j) =$



- Instantiation
- Logic reasoning
- Theory reasoning (here: Arithmetic)

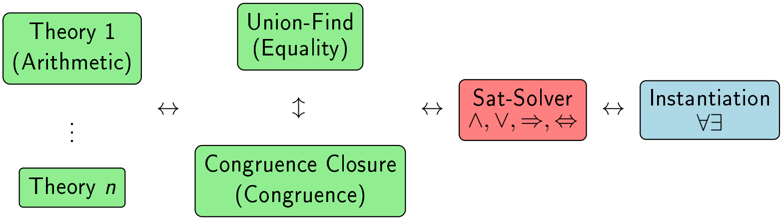
Satisfiability Modulo Theories

SMT provers divide the problem in three parts

- The theory part: equality reasoning, arithmetic reasoning, ...
- The satisfiability part: deals with logical connectors $\wedge$, $\vee$, $\Rightarrow$, $\neg$,...
- The instantiation of quantified axioms

We will look at each of the three parts in turn

The different parts of an SMT solver



A more detailed example

Hypotheses

- $H_1 : a > 0$
- $H_2 : \forall xy. x \geq y \rightarrow max(x, y) = x$

Goal

$G : f(max(a, 0)) = f(a)$

Solved by an SMT Solver (1)

Negate the Goal

$H_1 \wedge H_2 \rightarrow G$ becomes $H_1 \wedge H_2 \wedge \neg G$

Launch Sat-Solver

Assume H_1, H_2 and $\neg G$ and try to derive a contradiction

- Assume the inequality $a > 0$
- Register the lemma: $\forall xy . x \geq y \rightarrow max(x, y) = x$
- Assume the inequality $f(max(a, 0)) \neq f(a)$
- Currently no contradiction!

Instantiation

Specialize the lemma by applying it to a and 0 and replace $\rightarrow$:

$a \geq 0 \rightarrow max(a, 0) = a \quad \Leftrightarrow \quad a < 0 \vee max(a, 0) = a$

Solved by an SMT Solver (2)

Split the disjunction

First assume $a < 0$, then assume $\neg(a < 0)$, try to find a contradiction in both cases

Assuming $a < 0$

Direct contradiction with H_1 (using knowledge about the symbols $<$ and $\geq$)

Assuming $\neg(a < 0)$

- It follows $max(a, 0) = a$
- Deduce $f(max(a, 0)) = f(a)$
- Contradiction with $\neg G$

We have obtained a contradiction in all cases, the negated formula is unsatisfiable, that means the input formula is valid!

Plan

Using yices interactively

```
moreno@gorgosaurus:~$ yices -i
Yices (version 1.0.40). Copyright SRI International.
GMP (version 5.1.1). Copyright Free Software Foundation, Inc.
Build date: Wed Dec  4 09:42:16 PST 2013
Type `(exit)' with parentheses to exit.
Type `(help)' with parentheses for help.
yices > (define f::(-> int int))

yices > (define i::int)

yices > (define j::int)

yices > (assert (= (- i 1) (+ j 2)))

yices > (assert (/= (f (+ i 3)) (f (+ j 6))))
unsat

yices >
```

Using yices interactively

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moreno@gorgosaurus:~$ yices -i
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Build date: Wed Dec  4 09:42:16 PST 2013
Type '(exit)' with parentheses to exit.
Type '(help)' with parentheses for help.
yices > (define x::int)

yices > (define y::int)

yices > (define z::int)

yices > (assert (= (+ (* 3 x) (* 6 y) z) 1))

yices > (assert (= z 2))

yices > (check)
unsat
```

Using `yices` interactively

Input file smt.ys

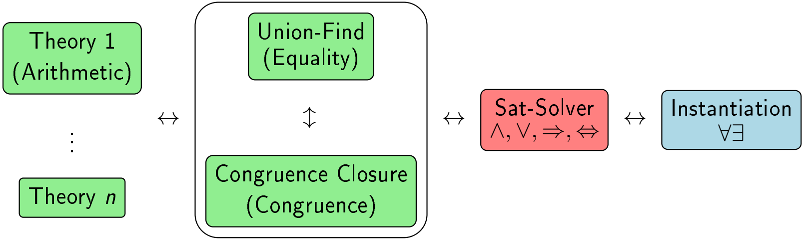
```
(define  x::int)
(define  y::int)
(define  f::(-> int int))
(assert  (/= (f (+ x 2)) (f (- y 1))))
(assert  (= x (- y 4)))
(check)
```

Call on the command line

```
moreno@gorgosaurus:~$ yices -e smt.ys
sat
(= x 0)
(= y 4)
(= (f 2) 1)
(= (f 3) 5)
```

Plan

Equality Reasoning



Equality reasoning - The problem

Terms

$t ::= c \mid f(t_1, \cdots, t_n)$

Given

a list of equations $t = t'$

We want to know

Does the equation $t_1 \stackrel{?}{=} t_2$ follow?

Using the axioms

Reflexivity	$t = t$
Symmetry	$t_1 = t_2 \to t_2 = t_1$
Transitivity	$t_1 = t_2 \wedge t_2 = t_3 \to t_1 = t_3$
Congruence	$t_1 = t_2 \to f(t_1) = f(t_2)$

Example

Given

- $f^2(a) = f(f(a)) = a$
- $f^5(a) = f(f(f(f(f(a))))) = a$

We want to prove

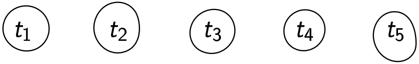
$f(a) = a$

Proof

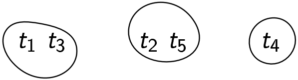
1. $f^5(a) = f^3(a)$ (Congruence)
2. $f^2(a) = f^3(a) = a$ (Transitivity, Symmetry)
3. $f^3(a) = f(f^2(a)) = f(a)$ (Congruence)
4. $f(a) = a$ (Transitivity of (2) and (3))

Disjoint Sets

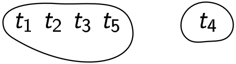
- Goal: deal with the first three axioms efficiently
- Idea: put all terms into disjoint sets
- When two terms are in the same set, they are equal
- Initial state: every term is in his own set:



- After treating $t_1 = t_3$ and $t_2 = t_5$:



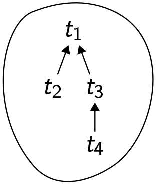
- After treating $t_1 = t_2$:



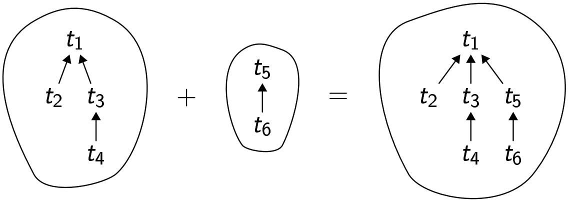
- Deciding $t \stackrel{?}{=} t'$ amounts to checking if t, t' are in the same set

Union-Find (1975)

- Represent each set by a tree with upward pointers:



- The root is the representative
- Operation find to find the representative of any term: just follow the arrows
- Operation union to treat an equality: simply point one root to the other



Two important optimizations

- Keep trees small: let point root of smaller tree to root of larger tree
- Path compression: "flatten" trees, each time we are searching for a root r starting from t, let t point directly to r afterwards
- Result: Algorithm is quasi-linear (optimal)
- Incrementality: we can add equations one by one, interleave equations $t_1 = t_2$ with queries $t_1 \stackrel{?}{=} t_2$

Inequalities $t_1 \neq t_2$

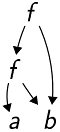
- Simply maintain the information that two sets of terms must be different
- Merging sets for which an inequality was registered leads to an inconsistency

Congruence Closure (1980)

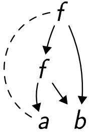
- Deal with the fourth axiom: Congruence

$$\forall xy.x = y \rightarrow f(x) = f(y)$$

  for any function symbol f
- Solution: represent a term by a directed acyclic graph (DAG) with sharing. Example: $f(f(a, b), b)$



- Add an equivalence relation to this graph (using union-find):



  represents $f(f(a, b), b) = a$

Finding new equalities

- Build a reverse dictionary mapping nodes to their fathers:

$$\begin{aligned} a &\mapsto f(a,b), g(a) \\ b &\mapsto f(a,b) \end{aligned}$$

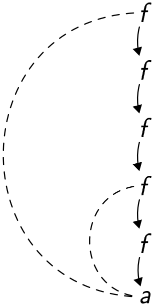
- Two new operations: find and merge.

```
merge(t1,t2) =
  union(t1,t2);
  F1,F2 = fathers(t1), fathers(t2);
  for each x in F1, y in F2 do
    if congruent(x,y) then merge(x,y);
  done
```

Congruence Closure — Example

Given

- $f^2(a) = f(f(a)) = a$
- $f^5(a) = f(f(f(f(f(a))))) = a$



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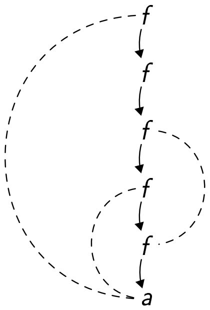

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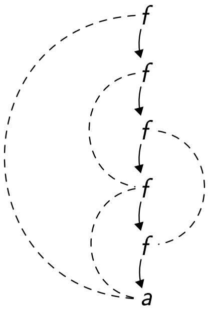

Congruence Closure — Example

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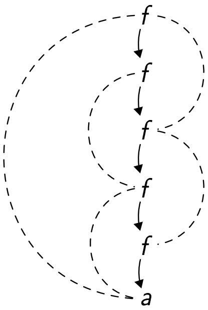

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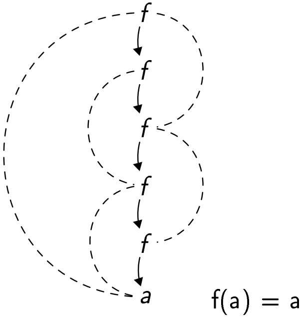
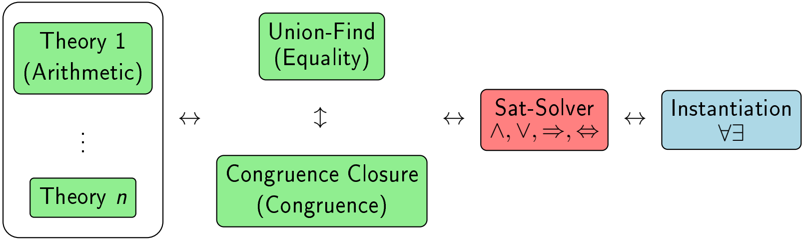


Plan

Theory Reasoning (Arithmetic)



Arithmetic reasoning

Arithmetic

- Interprets the function symbols $+$, $-$, $\times$, $\div$, and the arithmetic constants
- But also the relation symbols $\leq$, $<$, $\geq$, $>$

There are a few algorithms to deal with Linear Arithmetic

- Gauss Elimination (Equality only)
- Fourier-Motzkin
- Simplex Algorithm

We will look more closely at these methods

Gauss Elimination

Goal: deal with equalities in linear arithmetics

- Transform term into sums of monomials: $\sum_i^k c_i t_i$
- When treating an equality between such polynomes

$$\sum_i^k c_i t_i = \sum_j^k d_i s_i$$

isolate a monomial, say, t_1, and build the equation

$$t_1 = \sum_j^k \frac{d_i}{c_1} s_i - \sum_{i \neq 1}^k \frac{c_i}{c_1} t_i$$

Fourier-Motzkin Algorithm (1)

Goal: deal with inequalities in linear arithmetics

basic notions

- An inequality C in canonical form:

$$\sum_{i=1}^{n} a_i x_i \leq a_0 \qquad a_i \in \mathbb{Q}$$

- Note αC the multiplication of an inequation with a coefficient α:

$$\sum_{i=1}^{n} \alpha a_i x_i \leq \alpha a_0$$

- Note $C_1 + C_2$ the addition of two inequations :

$$\sum_{i=1}^{n} (a_i + b_i) x_i \leq a_0 + b_0$$

Fourier-Motzkin Algorithm (2)

Set $I = \{C_1 \cdots C_n\}$ the starting set of inequations. Each step of the algorithm will eliminate a variable from the set of the equations.

- Let I^+ (I^-) be the set of equations where x appears with positive (negative) coefficient
- Compute

$$I_x = \bigcup_{C \in I^-, D \in I^+} \beta C + \alpha D \qquad \alpha x \in C, -\beta x \in D$$

- Let I_0 the set of inequations in I without x
- Replace I par $I' = I_0 \cup I_x$
- In particular, if x appears only with coefficients of the same sign in I, suppress all inequations where x appears
- When I does not contain variables any more, either we have satisfiable inequalities (like $1 \leq 2$) or an inconsistency

Fourier-Motzkin Algorithm (3)

- Complexity: double exponential
- Not incremental
- Still behaves well in practice
- Can be easily extended to deduce equations between terms

References

- http://yices.csl.sri.com/old/download-yices1-full.html (The yices software)
- http://www.cs.cornell.edu/gomes/papers/SATSolvers-KR-Handbook.pdf SAT solvers handbook
- https://en.wikipedia.org/wiki/Boolean_satisfiability_problem (SAT)
- https://en.wikipedia.org/wiki/Satisfiability_modulo_theories (SMT)
- https://en.wikipedia.org/wiki/DPLL_algorithm (DPLL)
- http://0a.io/boolean-satisfiability-problem-or-sat-in-5-minutes/

This lecture follows partly a presentation by Hans Zantema (Eindhoven University of Technology), another by Luciano Serafini (Fondazione Bruno Kessler, Trento) and another by David L. Dill (Stanford University).